Proof the Skewes' number is not an integer using lattice points and tangent line

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Abstract

Skewes' number was discovered in 1933 by South African mathematician Stanley Skewes as upper bound for the first sign change of the difference $\pi(x) - \mathrm{li}(x)$. Whether a Skewes' number is an integer is an open problem of Number Theory. Assuming Schanuel's conjecture, it can be shown that Skewes' number is transcendental. In our paper we have chosen a different approach to prove Skewes' number is an integer, using lattice points and tangent line. In the paper we acquaint the reader also with prime numbers and their use in RSA coding, we present the primary algorithms Lehmann test and Rabin-Miller test for determining the prime numbers, we introduce the Prime Number Theorem and define the prime-counting function and logarithmic integral function and show their relation.

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1. INTRODUCTION

The prime numbers play an essential role in mathematics, e.g. they are the cornerstone of modern cryptographic algorithms and protocols, such as digital signatures, public key encryption, etc. The basic attribute of prime numbers is that they are no longer decomposable into the product of other numbers, and this very attribute is important for modern cryptography. In number theory we know several algorithms, by means of which we can decompose composite numbers into the product of prime numbers while we will use the same algorithm to decompose multi-digit numbers as for example to decompose small numbers. The results of the mathematical research to date show that there is no general rule according to which it is possible to quickly decompose large numbers into prime numbers universally and that the process of decomposition of large numbers takes a long time even for modern computer technology.

Thus, we can easily algorithmically decompose only numbers of a certain "small" size into the product of prime numbers and this fact is the basis of virtual security. If we take two very large prime numbers, where both the first and the second

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represent some information, it is easy to multiply them with each other, giving a very large composite number practically decomposable (unless we know the so-called private key, i.e. some necessary information how the large number came up). Algorithms that would look for the factors of a product by trying all possibilities, would have an NP complete time complexity [1].

This security, using prime number attributes, allows for public key encryption [2]. It is an encryption in which each of its principles is known and is public, but no one decrypts the encrypted information, because the public key only (i.e. knowledge of the encryption principle) is not enough. To decipher the cipher and read the message, it is necessary to obtain prime number elements of prime factorization that can only be obtained by the recipient of the message who also knows the private key (that is, some necessary information on how the product originated).

There are currently several encryption algorithms that use prime factorization, but historically the most important is RSA [3]. One of the first standards was RSA-768 [4], which represents a 232-digit number. The standard has been broken by scientists by bringing together hundreds of computers that have been working for a period representing 2000 years of one computer work. Later, the RSA-1024, RSA-2048 or RSA-4096 standards have been developed which are used today.

All virtual security builds on exceptional properties of prime numbers, and that is the reason we are still studying prime numbers with great importance. Prime numbers are the basic building blocks of all numbers, and much of the modern knowledge comes only from discovery of other prime number properties.

2. GENERATING PRIME NUMBERS

Many modern cryptographic algorithms and protocols require prime numbers, so it is important to be able to properly generate them and be able to effectively decide whether a given number n is a prime. Probability of randomly selected number near the number n being the prime number is approximately $\frac{1}{\ln n}$ [5]. Then the total number of primes less than n is approximately $\frac{n}{\ln n}$ (more in Section 3).

When generating prime numbers, a number n is randomly selected and one of the known prime number tests is used to determine if it is prime. If not, another random number n can be selected and repeat the test. Number n must be selected appropriately (e.g. in the form of $2^p - 1$, where p is a prime number [6]) to increase the probability of choosing a prime number.

One of the known tests, e.g. Lehmann test [7] or Rabin-Miller test [8] can be used for prime number testing. Lehmann's prime number test of n is an algorithm consisting of the following steps:

1. we choose a random number m less than n
2. we calculate $m^{\frac{n-1}{2}} \bmod n$
3. if $m^{\frac{n-1}{2}} \not\equiv 1(\bmod n)$ or $m^{\frac{n-1}{2}} \not\equiv 1(\bmod n)$, the number n is not a prime number
4. if $m^{\frac{n-1}{2}} \equiv 1(\bmod n)$ or $m^{\frac{n-1}{2}} \equiv -1(\bmod n)$, the probability of n not being a prime is maximum 50%

The algorithm is repeated k-times with another randomly selected number m, and if the test succeeds k-times, the probability of n not being prime, will be $m\ \frac{1}{2^k}$.

Rabin-Miller's prime number test is a very fast probability algorithm which, if it finds that a given number is composite, then it is really composite. If the algorithm in response returns that number n is a prime number, it's only true with a certain probability. Rabin-Miller's prime number test of n is an algorithm consisting of the following steps:

1. first for a random number n we find such a number u, so that $2^u | n - 1$, thus $n - 1 = 2^u \cdot l$, where l is an odd number
2. then we select a random number m , so that $1 < m < n$ applies
3. for each $i = 0, \cdots, u - 1$ we count $v_i = m^{2^{i} \cdot l} \bmod n$, and if i exists for which does not apply that $(v_i = -1 \vee m^l \bmod n = 1)$, then n is composite
4. if n is not composite, we repeat steps 2-3 k times
5. if after repeating the test k times n is not composite, it can be a prime number

The probability of passing the test by a composite number as a prime decreases faster in this test and is equal to $\frac{1}{4^k}$. Thus, by repeating the test multiple times, we can reduce the probability of an error to arbitrarily small one, though never zero, and we cannot say with certainty that the algorithm has returned a prime number.

Sometimes, e.g. in RSA cryptographic algorithms using the product of two prime numbers p, q, prime numbers are required to be strong prime numbers [9]. These prime numbers have certain properties that make it difficult to decompose a number into prime factors using standard procedures. Recommended properties include:

1. the largest common divisor of numbers $(p-1)$ and $(q-1)$ should be small
2. both numbers $(p-1)$ and $(q-1)$ should have great prime factors r, s
3. both numbers $(r-1)$ and $(s-1)$ should have great prime factors
4. both numbers $(r+1)$ and $(s+1)$ should have great prime factors
5. both numbers $\frac{(p-1)}{2}$ and $\frac{(q-1)}{2}$ should be prime numbers

3. PRIME-COUNTING FUNCTION AND THE PRIME NUMBER THEOREM

The prime-counting function, denoted as $\pi(x)$, is the function that counts the number of prime numbers less than or equal to a given real number x [10]:

$$\pi(x) = \sum_{p \leq x} 1$$

In the history of mathematics, there have been many attempts to find the exact formula for prime-counting function [11-12]. In the end of the 18th century French mathematician Adrien-Marie Legendre conjectured that

$$\pi(x) \sim \frac{x}{A \cdot \log(x) + B},$$

for constants A and B ($\log(x)$ denotes the natural logarithm) in his work [13] which was published in 1798 and summarized the number theory results of the 18th century. Legendre improved his conjecture in 1808 in his work [14] to

$$\pi(x) = \frac{x}{\log(x) - A(x)},$$

where $\lim_{x\to\infty} A(x) = 1.08366$ is Legendre's constant [15]. Carl Friedrich Gauss followed up on the research of Legendre from the 18th century and in 1849 he brought a better estimate for $\pi(x)$ based on empirical proofs obtained from the tables of prime numbers as:

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1, \text{ resp. } \pi(x) \sim \frac{x}{\log(x)}.$$

In 1854, a professor at St. Petersburg University, Pafnuty Lvovich Chebyshev, found approximation $0.92129 < \frac{\pi(x)}{\frac{x}{\log(x)}} < 1.10555$ for function $\pi(x)$. When studying function $\pi(x)$, Chebyshev used a real function which in the complex domain in its form $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is known under the name Riemann function (zeta function) defined in the whole complex plane except from point 1. This function had already been known to Euler in the 18th century but it was only Bernhard Riemann who fully discover its potential. Riemann tried to find nontrivial zero values of function $\zeta(z)$ during his attempt to use function ζ to prove formula $\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$. In 1859 he formulated a conjecture, the Riemann Hypothesis, [16] that in a plane of complex numbers $z = x + \mathrm{i}y$ in a planar strip defined by inequality $0 \leq x \leq 1$, all these zero points lie on line $x = \frac{1}{2}$. A decision on the validity of Riemann hypothesis would solve a large number of problems from various areas of mathematics, especially from the number theory domain, such as the question of prime number distribution. Riemann zeta function contains an infinite number of zeros and, by using available numerical methods, by 1986 it was proven that 1,500,000,000 zeros of Riemann zeta function lies on line $x = \frac{1}{2}$. Based on current calculations, this holds for up to 10 quintillions of zeros.

The theorem $\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$ in known as the Prime Number Theorem and it is one of the most remarkable results of modern mathematics. The theorem was proved simultaneously in 1896 by Hadamard [17] and de la Vallée-Poussin [18]. Hence the Prime Number Theorem shows that around any number x the density of prime numbers is about $\frac{1}{\log(x)}$. Several values for $\pi(x)$ are shown in the Table 1 together with the actual number of prime numbers up to x [19].

x	number of primes less than or equal to x	$\pi(x)$
1000	168	178
10000	1229	1246
100000	9592	9630
1000000	78498	78628
10000000	664579	664918
100000000	5761455	5762209
1000000000	50847534	50849235
10000000000	455052511	455055614

Table 1. Several values for $\pi(x)$

The Prime Number Theorem is equivalent to statement:

$$\pi(x) \sim \mathrm{li}(x),$$

where $\mathrm{li}(x)$ li is the logarithmic integral function [20] defined for all positive real numbers $x > 1$ as:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}.$$

The function $\frac{1}{\ln t}$ has a singularity at $t = 1$, and the function $\mathrm{li}(x)$ is interpreted as a Cauchy principal value [21]:

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0^+} \left(\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^x \frac{dt}{\ln t} \right).$$

It was thought for a long time that $\pi(x)$ is always an overestimate for all x:

$$\pi(x) < \mathrm{li}(x),$$

until John Edensor Littlewood, a British mathematician, showed in 1914 [22] that the logarithmic integral function underestimates the prime counting function for some large x $(\pi(x) > \mathrm{li}(x))$. Moreover that the difference $\pi(x) - \mathrm{li}(x)$ changes sign infinitely often. Littlewood's proof did not state a concrete such number x, it was just a proof of existence.

The first crossover point (as upper bound for the first sign change) was discovered by South African mathematician Stanley Skewes (Skewes was supervised for his PhD. under the Littlewood) and this number was named after him. In 1933 [23] Skewes showed there exists a number x below:

$$e^{e^{e^{79}}} < 10^{10^{10^{34}}}.$$

This bound assumed that the Riemann Hypothesis is true. In 1955 [24] Skewes showed a bound

$$e^{e^{e^{e^{7.705}}}} < 10^{10^{10^{964}}}$$

without assuming the Riemann hypothesis known as Skewes Second Number.

4. MAIN RESULTS

Consider a space with a rectangular coordinate system $[O, x, y]$. Coordinate axes x, y as numeric axes have integer values highlighted. If by these highlighted points we draw lines perpendicular to the considered axis we get a grid in which the node point is defined as $A[x_0, y_0], x_0, y_0 \in \mathbb{Z}$. In the work [25] we have shown a method to find solutions for a certain type of Diophantine equations using a tangent line (plane). We use the slope of a tangent line and its shift on the relevant axis to determine the properties of the tangent point and whether it is a grid point or not.

We now apply this procedure to determine whether the considered number b belongs to the set of integers or not. Consider a function $y = f(x)$ under the condition $y = f(x_0) = b$ and $x_0 \in \mathbb{Z}$. Then if the tangent line at the point $E[x_0, b]$ has a rational slope and at the same time a shift on the relevant axis s_x at the point $B[s_x, 0]$ is rational, number $b \in \mathbb{Z}$. In the case of a more complex number where we cannot provide the condition $x_0 \in \mathbb{Z}$, we consider $x_0 = \alpha \cdot \delta$ where $\alpha \in \mathbb{R}$ and $\delta \in \mathbb{Z}$. Then the slope will be in the form $k = \frac{1}{\alpha} \cdot q$ and $q \in \mathbb{Q}$. By such compensation of the grid, we can abolish the irrationality of the slope, which arose on the basis of the number we have chosen. Hence we get to the previous condition. In more computationally difficult tasks we also use the intersection with the y-axis. Then we use a line on which we have three points $A\left[0, s_y\right], B[s_x, 0]$ and $E[x_0, b]$.

We also use the gradient of field in our approach. The derivative of a scalar field $U = U(\vec{r})$ according to the vector $\vec{c}$ at a given point $\vec{r}$ is called the limit

$$\frac{\partial U}{\partial \vec{c}} = \lim_{\varepsilon \to 0} \frac{U(\vec{r}+\varepsilon\vec{c}) - U(\vec{r})}{\varepsilon}.$$

If we consider the unit vector $\vec{c}_0$, equality holds

$$\frac{\partial U}{\partial \vec{c}} = |\vec{c}| \cdot \frac{\partial U}{\partial \overrightarrow{c_0}},$$

where $\frac{\partial U}{\partial \overrightarrow{c_0}}$ means the rate at which the function U increases in the direction of the vector $\overrightarrow{c_0}$ at each point. The gradient of the field $\mathrm{grad}(U)$ is the vector defined at each point of the field and that has the direction of the normal line to the equiscular plane. In a right-angled coordinate system

$$\mathrm{grad}(U) = \frac{\partial U}{\partial x}\vec{\imath} + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial \vec{z}}\vec{k}.$$

From the above properties we can determine based on the $\mathrm{grad}(U)$ equations of the tangent plane to the area determined by the equation of the scalar field U, or in the case of a situation in space E_2 we have a tangent line to the curve. More information on the theory can be found in [26-27].

Now, we’ll show through lattice points and tangent line that Skewes' number is not an integer.

PROBLEM. $e^{e^{e^{79}}} \in \mathbb{N}$

PROOF. To prove that the Skewes' number is an integer, we use the property of a tangent line, which we place in the grid system. In this case, the slope of the tangent line depends on the properties of the function, with which we can create a mapping of our considered number to the number $e^{e^{e^{79}}}$. Consider a coordinate system xy where we have defined the grid by $y = k_1, x = k_2 \cdot e$; $k_1, k_2 \in \mathbb{Z}$. It is important for us that the grid is integer graduated on the y-axis, and due to the chosen function, we consider multiples of the number e on the x-axis. We choose this form of the line for the calculation:

$$(O - X) \cdot \text{grad}\,(F[A]) = D,$$

where O is the origin of the coordinate system, X is any point of the line, E is the tangent point with the coordinates $E[x_0, y_0]$ and D is the result of an inner product which depends on the distance of the line from the origin of the coordinate system.

For a line that does not pass through the origin of the coordinate system, we choose points that lie on the coordinate axes. Then the point of the line which lies on the coordinate axis y has the coordinates $A[0, s_y]$ and the point which lies on the coordinate axis x has the coordinates $B[s_x, 0]$. Now we solve the vector equation

$$(O - A) \cdot \text{grad}(F[A]) = D$$
$$(0 - 0, 0 - s_y) \cdot \text{grad}(F[A]) = D$$

Next we multiply the equation by the vector $\text{grad}(F[A])^{-1}$

$$(0, -s_y) = D$$
$$(O - B) \cdot \text{grad}(E) = D \cdot \text{grad}(F[A])^{-1}$$

We need to determine the intersection with the y-axis by multiplying the equation by the unit vector $1_y = (0,1)$ and we get

$$-s_y = D \cdot 1_y \cdot \text{grad}(F[A])^{-1}.$$

Next consider the curve given by the equation

$$y = x^{x^{x^{79}}}.$$

We simplify the equation to the form $F(x, y) = 0$. With a suitably chosen number for x, we are able to get to the required number.

Now consider a tangent line at a point $A\left[e, e^{e^{e^{79}}}\right]$, which leads us to the number $e^{e^{e^{79}}}$, whose properties we are determining. Hence $\frac{y-y_0}{x-x_0} = \frac{e^{e^{e^{79}}}}{D\cdot 1_x \cdot \mathrm{grad}(F[A])^{-1}+e}$ and

$$\frac{y-y_0}{x-x_0} = \frac{D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + e^{e^{e^{79}}}}{e}$$

$$\frac{y-y_0}{x-x_0} = e^{e^{e^{79}}}\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)$$

$$D = e^{e^{e^{79}}}\left(e^{-1+e^{79}} - 1 + e^{e^{79}}(e^{78} + 79e^{78})\right)$$

Let's determine the tangent line based on the intersections with the coordinate system and the tangent point

$$\frac{s_y - y_0}{0 - x_0} = \frac{0 - y_0}{s_x - x_0}$$

$$\mathrm{grad}(F[A])^{-1} = \frac{\mathrm{grad}(F[A])}{\sqrt{\mathrm{grad}(F[A]).\,\mathrm{grad}(F[A])}}$$

$$\mathrm{grad}(F[A]) = \left(e^{e^{e^{79}}}\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right), 1\right)$$

From that

$$\mathrm{grad}(F[A])^{-1} = \frac{\left(e^{e^{e^{79}}}\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right), 1\right)}{\sqrt{\left(\left(e^{e^{e^{79}}}\right)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)}}$$

$$\frac{e^{e^{e^{79}}}}{D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} + e} = \frac{D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + e^{e^{e^{79}}}}{e}$$

After simplification it holds

$$e^{e^{e^{79}}+1} = (D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} + e) \cdot \left(D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + e^{e^{e^{79}}}\right)$$

$$e^{e^{e^{79}}} = \frac{(D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} + e) \cdot \left(D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + e^{e^{e^{79}}}\right)}{e}$$

If $e^{e^{e^{79}}} \in \mathbb{N}$ then suppose that $e^{e^{e^{79}}} = k_3 \in \mathbb{N}$. Then

$$(D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} + e) \cdot \left(D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + k_3\right) = k_3 \cdot e.$$

Hence

$$k_3 \cdot e = D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} \cdot D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} k_3 + \\ + e \cdot D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + e \cdot k_3$$

From that

$$D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} \cdot D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} + D \cdot 1_x \cdot \mathrm{grad}(F[A])^{-1} k_3 + \\ + e \cdot D \cdot 1_y \cdot \mathrm{grad}(F[A])^{-1} = 0$$

After substitution we get

$$\frac{\left(\mathrm{k}_3\left(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78})\right)\right)^3}{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)} + \\ + \frac{\left(\mathrm{k}_3\left(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78})\right)\right)^2 k_3 + e\left(\mathrm{k}_3\left(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78})\right)\right)}{\sqrt{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)}} \\ = 0$$

And finally we have a form

$$\left(\mathrm{k}_3\left(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78})\right)\right) \cdot \\ \cdot \left(\frac{\left(\mathrm{k}_3(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78}))\right)^2}{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)} \right. \\ \left. + \frac{\left(\mathrm{k}_3(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78}))\right)k_3 + e}{\sqrt{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)}}\right) = 0$$

The trivial solution is $k_3 = 0$ which we do not consider, because $e^{e^{e^{79}}} = k_3$. Further we solve the equation

$$\frac{\left(\mathrm{k}_3\left(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78})\right)\right)^2}{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)} +$$
$$+ \frac{\left(\mathrm{k}_3(e^{-1+e^{79}} - 1 + \mathrm{k}_3(e^{78} + 79e^{78}))\right)k_3 + e}{\sqrt{\left((\mathrm{k}_3)^2\left(e^{-1+e^{79}} + e^{e^{79}}(e^{78} + 79e^{78})\right)^2 + 1\right)}} = 0$$

By simplification we get a polynomial that has no integer or rational coefficients and therefore not even a root for k_3 will not be a rational number, which means that $e^{e^{e^{79}}}$ is not a rational number, and it follows that it is not an integer.

5. CONCLUSION

The paper focused on the proof the Skewes' number is not an integer using lattice points and tangent line and on the problematics of prime numbers. In introduction section, we described the importance and special role of prime numbers between numbers and their importance in RSA coding. In the second section we presented algorithms Lehmann test and Rabin-Miller test which are used to determine whether a given number is a prime number and which are part of the algorithms for generating prime numbers. In this section we refer to the Prime Number Theorem, which is together with prime-counting function and logarithmic integral function described in section three. We also introduced the Skewes number in part three. The fourth section is a main contribution and showed a geometric way to prove the Skewes' number is not an integer.

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Vector valued nonuniform nonstationary wavelets and associated MRA on local fields

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Abstract

In this paper we study nonstationary wavelets associated with vector valued nonuniform multiresolution analysis on local fields. By virtue of dimension function a complete characterization of vector valued nonuniform nonstationary wavelets is obtained.

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1. INTRODUCTION

In order to systematically construct orthonormal wavelet bases Mallat and Meyer introduced in 1986 the multiresolution analysis (or multiscale approximation) as a general tool in ap- proximation theory and signal analysis. Thus they provided a natural framework for the understanding of wavelet bases and provided a well structured scheme which describes the various refinement steps clearly, such that this technique became accessible to engineers for practical implementation [36]. The concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2\left(\mathbb{R}^d\right)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. All these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [22] considered a generalization of Mallat's [35] celebrated theory of MRA based on spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Based on one-dimensional spectral pairs, Gabardo and Yu [23] considered sets of nonuniform wavelets in $L^2(\mathbb{R})$. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of these types of signals by a stable

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mathematical tool. Gabardo and Nashed [22] and Gabardo and Yu [23] filled this gap by the concept of nonuniform multiresolution analysis. The notion of nonstationary wavelet system is introduced independently by M. Z. Berkolayko, I. Y. Novikov [15] and by C. de Boor, R. DeVore, A. Ron [16]. In [16], the nonstationary system (called almost-wavelets) is used to construct an orthonormal shift invariant basis consisting of infinitely differ- entiable compactly supported functions. It is well known that it is impossible to construct stationary wavelet basis satisfying these properties. Further, nonstationary wavelets are studied in [14; 19]

During the last two decades, there is a substantial body of work that has been concerned with the construction of wavelets on local fields. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and MRA (multiresolution analysis) theories are quite different. For example, R. L. Benedetto and J. J. Benedetto [13] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Khrennikov, Shelkovich and Skopina [26] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [10], Albeverio, Evdokimov and Skopina proved that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA except those described in [26]. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [9; 18] . These wavelet bases were obtained by relaxing the basic condition in the definition of an MRA and form Riesz bases without any dual wavelet systems. For some related works on wavelets and frames on $\mathbb{Q}_p$, we refer to [11; 25; 29; 30]. On the other hand, Lang [31; 32; 33] constructed several examples of compactly supported wavelets for the Cantor dyadic group. Farkov [20; 21] has constructed many examples of wavelets for the Vilenkin p-groups. Jiang et al. [24] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$.

Recently, Shah and Abdullah [39] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of $\mathscr{Z}$ and a translate of $\mathscr{Z}$, where $\mathscr{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact Abelian group $\mathbb{K}^+$. The notion of nonuniform wavelet frames on non-Archimedean local fields was introduced by Ahmad and Sheikh [7] and established a complete characterization of tight nonuniform wavelet frames on non-Archimedean local fields. More results in this direction can also be found in [1; 2; 3; 4; 5; 6; 8; 34; 37; 38] and the references therein. Drawing the inspiration from the above work, we introduce the notion of nonuniform nonstationary wavelets, their characterization and the associated multiresolution analysis on local fields.

The remainder of the paper is as follows. In Section 2, we discuss preliminary results on local fields and some basic definitions. In section 3, we obtain the characterization of orthonormal vector valued nonuniform nonstationary wavelets and the associated multiresolution analysis is established.

2. PRELIMINARIES ON NON-ARCHIMEDEAN LOCAL FIELDS

2.1. Non-Archimedean Local Fields

A non-Archimedean local field $\mathbb{K}$ is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If $\mathbb{K}$ is of positive characteristic, then $\mathbb{K}$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let $\mathbb{K}$ be a fixed non-Archimedean local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$;

(c) $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in $\mathbb{K}$. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathscr{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \, \mathfrak{p}^\ell$ with $c_\ell \in \mathscr{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [40]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y,x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [40], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \;\; n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \;\; 0 \leq a_k < p, \;\; \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1 \zeta_1 + \cdots + a_{c-1} \zeta_{c-1}) \, \mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \leq b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(s) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$

for a fixed $s \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geq 0$.

Let the local field $\mathbb{K}$ be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.3}$$

2.2. Fourier Transforms on Non-Archimedean Local Fields

The Fourier transform of $f \in L^1(\mathbb{K})$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathscr{F}\{f(x)\} = \hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}\,dx. \tag{2.4}$$

It is noted that

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\,\overline{\chi_\xi(x)}dx = \int_{\mathbb{K}} f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on non-Archimedean local field $\mathbb{K}$ are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on non-Archimedean local fields of positive characteristic have the following properties:

—The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(\mathbb{K})$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.

—If $f \in L^1(\mathbb{K})$, then $\hat{f}$ is uniformly continuous.

—If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x| \leq q^k} f(x)\overline{\chi_\xi(x)}\,dx, \tag{2.5}$$

where $f_k = f\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f

converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\hat{f}\big(u(n)\big)\right|^2. \tag{2.7}$$

We also denote the test function space on K by Ω, i.e., each function f in Ω is a finite linear combination of functions of the form $\mathbf{1}_k(x-h), h\in K, k\in\mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. Then, it is clear that Ω is dense in $L^p(K), 1\le p<\infty$, and each function in Ω is of compact support and so is its Fourier transform. Since Ω is dense in $L^2(K)$ and closed under the Fourier transform, the set

$$\Omega^0 = \left\{f\in\Omega : \text{supp}\hat{f} \subset K\backslash\{0\}\right\}$$

is also dense in $L^2(\mathbb{K})$.

2.3. Uniform Stationary MRA on Local Fields

In order to be able to define the concepts of uniform MRA and wavelets on non-Archimedean local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = \mathbb{K}$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n) : n\in\mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\mathscr{Z} = \{u(n) : n\in\mathbb{N}_0\}$ can be treated as the translation set. Note that Λ is a subgroup of $\mathbb{K}^+$ and unlike the standard wavelet theory on the real line, the translation set is not a group. Let us recall the definition of a uniform MRA on non-Archimedean local fields of positive characteristic introduced by Jiang et al. in [24].

DEFINITION 2.1. Let $\mathbb{K}$ be a non-Archimedean local field of positive characteristic $p>0$ and $\mathfrak{p}$ be a prime element of $\mathbb{K}$. An MRA of $L^2(\mathbb{K})$ is a sequence of closed subspaces $\{V_j : j\in\mathbb{Z}\}$ of $L^2(\mathbb{K})$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j\in\mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{K})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(x)\in V_j$ if and only if $f(\mathfrak{p}^{-1}x)\in V_{j+1}$ for all $j\in\mathbb{Z}$;

(e) There exists a function $\phi\in V_0$, such that $\left\{\phi\big(x-u(k)\big) : k\in\mathbb{N}_0\right\}$ forms an orthonormal basis for V_0.

According to the standard scheme for construction of MRA-based wavelets, for each j, we define a wavelet space W_j as the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f(x) \in W_j \quad \text{if and only if} \quad f(\mathfrak{p}^{-1}x) \in W_{j+1}, \quad j \in \mathbb{Z}. \tag{2.7}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{K}) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j\geq 0} W_j \right). \tag{2.8}$$

As in the case of $\mathbb{R}^n$, we expect the existence of $q-1$ number of functions $\psi_1, \psi_2, \ldots, \psi_{q-1}$ to form a set of basic wavelets. In view of (2.7) and (2.8), it is clear that if $\{\psi_1, \psi_2, \ldots, \psi_{q-1}\}$ is a set of function such that the system $\{\psi_\ell(x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for W_0, then $\{q^{j/2}\psi_\ell(\mathfrak{p}^{-j}x - u(k)) : 1 \leq \ell \leq q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(\mathbb{K})$.

2.4. Nonuniform MRA on Non-Archimedean Local Fields

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{u(r)}{N}\right\} + \mathscr{Z}.$$

where $\mathscr{Z} = \{u(n) : n \in \mathbb{N}_0\}$. It is easy to verify that Λ is not a group on non-Archimedean local field $\mathbb{K}$, but is the union of $\mathscr{Z}$ and a translate of $\mathscr{Z}$. Following is the definition of nonuniform stationary multiresolution analysis (NUSMRA) on non-Archimedean local fields of positive characteristic given by Shah and Abdullah [39].

DEFINITION 2.2. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN-1$ such that r and N are relatively prime, an associated NUMRA on non-Archimedean local field $\mathbb{K}$ of positive characteristic is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ such that the following properties hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{K})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) There exists a function ϕ in V_0 such that $\{\phi(\cdot-\lambda) : \lambda \in \Lambda\}$, is a complete orthonormal basis for V_0.

It is worth noticing that, when $N = 1$, one recovers from the definition above the definition of an MRA on non-Archimedean local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathscr{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'. \tag{2.7}$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}\,, \tag{2.8}$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 2.2, this implies

$$L^2(\mathbb{K}) = \bigoplus_{j\in\mathbb{Z}} W_j, \tag{2.9}$$

a decomposition of $L^2(\mathbb{K})$ into mutually orthogonal subspaces.

Now we state the concept of vector-valued nonuniform multiresolution analysis (VNUMRA) on local field K of positive characteristic and establish a necessary and sufficient condition for the existence of associated wavelets.

Let M be a constant and $2 \le M \in \mathbb{Z}$. By $L^2\big(K, \mathbb{C}^M\big)$, we denote the set of all vector-valued functions $\mathbf{f}$ i.e.,

$$L^2\big(K, \mathbb{C}^M\big) = \Big\{\mathbf{f}(x) = \big(f_1(x), f_2(x), \ldots, f_M(x)\big)^T : x \in K, f_t(x) \in L^2(K)\Big\},$$

where $t = 1, 2, \ldots, M$ and T means the transpose of a vector. The space $L^2\big(K, \mathbb{C}^M\big)$ is called *vector-valued function space* on local field K of positive characteristic. For

$\mathbf{f} \in L^2(K, \mathbb{C}^M)$, $\|\mathbf{f}\|$ denotes the norm of vector-valued function $\mathbf{f}$ and is defined as:

$$\|\mathbf{f}\|_2 = \left(\sum_{t=1}^{M} \int_K |f_t(x)|^2 dx \right)^{1/2}.$$

For a vector-valued function $\mathbf{f} \in L^2(K, \mathbb{C}^M)$, the integration of $\mathbf{f}$ is defined as:

$$\int_K \mathbf{f}(x) dx = \left(\int_K f_1(x) dx, \int_K f_2(x) dx, \ldots, \int_K f_M(x) dx \right)^T.$$

Moreover, the Fourier transform of $\mathbf{f}$ is defined by

$$\hat{\mathbf{f}}(\xi) = \int_K \mathbf{f}(x) \overline{\chi_\xi(x)}\, dx.$$

For any two vector-valued functions $\mathbf{f}, \mathbf{g} \in L^2(K, \mathbb{C}^M)$, their vector-valued inner product $\langle \mathbf{f}, \mathbf{g} \rangle$ is defined as:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_K \mathbf{f}(x) \overline{\mathbf{g}(x)}\, dx.$$

With $\Lambda = \{0, r/N\} + \mathscr{Z}$ as defined above, we define the *vector-valued nonuniform multiresolution analysis* (VNUMRA) on local fields of positive characteristic as follows:

DEFINITION 2.3. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN - 1$ such that r and N are relatively prime, a VNUMRA on local field K of positive characteristic is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K, \mathbb{C}^M)$ such that the following properties hold:

(1) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(2) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K, \mathbb{C}^M)$;

(3) $\bigcap_{j \in \mathbb{Z}} V_j = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero vector of $L^2(K, \mathbb{C}^M)$;

(4) $\Phi(x) \in V_j$ if and only if $\Phi(\mathfrak{p}^{-1}Nx) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(5) There exists a function Φ in V_0 such that $\{\Phi(x - \lambda) : \lambda \in \Lambda\}$, is a complete orthonormal basis for V_0. The vector valued function Φ is called a *vector-valued scaling function* of the VNUMRA.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'.$$

It follows that for $j > J$,

$$V_j = V_J \bigoplus \left(\bigoplus_{\ell=0}^{j-J-1} W_{j-\ell} \right)$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 2.3, this implies

$$L^2\left(K, \mathbb{C}^M\right) = \bigoplus_{j\in\mathbb{Z}} W_j,$$

a decomposition of $L^2\left(K, \mathbb{C}^M\right)$ into mutually orthogonal subspaces.

As in the standard case, one expects the existence of $qN-1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2\left(K, \mathbb{C}^M\right)$.

DEFINITION 2.4. A set of functions $\left\{\Psi_1, \Psi_2, \ldots, \Psi_{qN-1}\right\}$ in $L^2\left(K, \mathbb{C}^M\right)$ will be called a *set of basic wavelets* associated with a given VNUMRA if the family of functions $\{\Psi_\ell(x-\lambda) : 1 \leq \ell \leq qN-1, \lambda \in \Lambda\}$ forms an orthonormal basis for W_0.

3. MAIN RESULTS

We start this section with the following definition

DEFINITION 3.1. Let $\Psi^{(j)} \in L^2\left(K, \mathbb{C}^M\right)$ for all $j \in \mathbb{Z}$, then the family of functions

$$\Psi_{j,\lambda} = \left\{ (qN)^{\frac{j}{2}} \Psi^{(j)} (\mathfrak{p}^{-1}N)^j x - \lambda \right\}_{j\in\mathbb{Z}, \lambda\in\Lambda}$$

is called a nonuniform nonstationary wavelet system for $L^2\left(K, \mathbb{C}^M\right)$.

LEMMA 3.1. If $f \in S$ and $\Psi^{(j)} \in L^2\left(K, \mathbb{C}^M\right)$, then

$$\sum_{\lambda\in\Lambda} \left| \langle f, \Psi_{j,\lambda} \rangle \right|^2 = \int_{\mathbb{K}} \overline{\widehat{f}(\xi)} \widehat{\Psi}^{(j)} \left((\mathfrak{p}^{-1}N)^{-j} \xi \right) \left\{ \sum_{s\in\mathbb{N}_0} \widehat{f}\left(\xi + (\mathfrak{p}^{-1}N)^{-j} u(s) \right) \overline{\widehat{\Psi}^{(j)}} \left((\mathfrak{p}^{-1}N)^{-j} \xi + u(s) \right) \right\} d\xi. \tag{3.1}$$

PROOF. For $\Psi \in L^2(K, \mathbb{C}^M)$, let

$$\Psi_{j,\lambda}(x) = (qN)^{j/2}\Psi^{(j)}\left((\mathfrak{p}^{-1}N)^j x - \lambda\right),\ j \in \mathbb{Z}, \lambda \in \Lambda.$$

Then, we have

$$\widehat{\Psi}_{j,\lambda}(\xi) = (qN)^{-j/2}\widehat{\Psi}^{(j)}\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\overline{\chi_\lambda\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)}. \tag{3.2}$$

By Parseval Identity and equation (3.2), we have

$$\sum_{\lambda\in\Lambda}\left|\left\langle f, \Psi_{j,\lambda}\right\rangle\right|^2 = \sum_{\lambda\in\Lambda}(qN)^j\int_{\mathbb{K}}\left\{\sum_{s\in\mathbb{N}_0}\int_{N\mathfrak{D}}\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\left(x+u(s)\right)\right)\chi_\lambda\left(x+u(s)\right)\overline{\widehat{\Psi}^{(j)}}\left(x+u(s)\right)\right\}$$

$$\times\overline{\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}x\right)}\,\overline{\chi_\lambda(x)}\widehat{\Psi}^{(j)}(x)dx.$$

Since $\sum_{s\in\mathbb{N}_0}\int_{N\mathfrak{D}}\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\left(x+u(s)\right)\right)\chi_\lambda\left(x+u(s)\right)\overline{\widehat{\Psi}^{(j)}}\left(x+u(s)\right)dx$ contains only finite non-zero terms for $f \in S$ and $\chi_\lambda\left(u(s)\right) = 1$, for all $\lambda \in \Lambda, s \in \mathbb{N}_0$, we have

$$\sum_{\lambda\in\Lambda}\left|\left\langle f, \Psi_{j,\lambda}\right\rangle\right|^2 = \sum_{\lambda\in\Lambda}(qN)^j\int_{\mathbb{K}}\left(\int_{N\mathfrak{D}}\left\{\sum_{s\in\mathbb{N}_0}\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\left(x+u(s)\right)\right)\overline{\widehat{\Psi}^{(j)}}\left(x+u(s)\right)\right\}\chi_\lambda(x)dx\right)$$

$$\times\overline{\chi_\lambda(y)}\,\overline{\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}y\right)}\widehat{\Psi}^{(j)}(y)dy.$$

By invoking Convergence theorem of Fourier series on $\mathfrak{D}$, we obtain (3.1). This completes the proof.

LEMMA 3.2. Let $f \in \Omega$ and $\Psi^{(j)} \in L^2(K, \mathbb{C}^M)$. If ess $\sup\left\{\sum_{j\in\mathbb{Z}}\left|\Psi^{(j)}\left((\mathfrak{p}^{-1}N)^j\xi\right)\right|^2\right\} < \infty$ for $\xi \in \mathfrak{B}^{-1}\setminus N\mathfrak{D}$, then

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\left\langle f, \Psi_{j,\lambda}\right\rangle\right|^2 = \int_{\mathbb{K}}|\widehat{f}(\xi)|^2\sum_{j\in\mathbb{Z}}\left|\Psi^{(j)}\left((\mathfrak{p}^{-1}N)^j\xi\right)\right|^2 + R_0(f), \tag{3.3}$$

where

$$\begin{aligned} R_0(f) &= \sum_{j\in\mathbb{Z}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\left\{\sum_{s\in\mathbb{N}}\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+u(s)\big)\Big)\overline{\widehat{\Psi}^{(j)}\big(\xi+u(s)\big)}\right\}d\xi \\ &= \sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{N}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+u(s)\big)\Big)\overline{\widehat{\Psi}^{(j)}\big(\xi+u(s)\big)}d\xi. \quad (3.4) \end{aligned}$$

Moreover if $\left\|\widehat{\Psi}^{(j)}\right\|_{L^2(\mathbb{K})}=1$, then the series in (3.4) converges absolutely on $\mathbb{K}$.

PROOF. For $R_0(f)$, we use the fact that for $f\in\Omega$, $\sum_{s\in\mathbb{N}_0}\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\big)\overline{\widehat{\Psi}^{(j)}\big(x+u(s)\big)}$ contains only finite non-zero terms, so we have

$$\begin{aligned} &\sum_{j\in\mathbb{Z}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\left\{\sum_{s\in\mathbb{N}}\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\Big)\overline{\widehat{\Psi}^{(j)}\big(x+u(s)\big)}\right\}d\xi \\ &\quad=\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{N}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\Big)\overline{\widehat{\Psi}^{(j)}\big(x+u(s)\big)}d\xi. \end{aligned}$$

We claim that for all $f\in\Omega^0$, (3.3) holds. Moreover, by using Lemma 3.1, we have

$$\begin{aligned} \sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\Psi_{j,\lambda}\rangle\right|^2 &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\left\{\left|\widehat{f}(\xi)\right|^2\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2\right. \\ &\quad\left.+(qN)^j\left(\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\sum_{s\in\mathbb{N}}\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\big)\overline{\widehat{\Psi}^{(j)}\big(x+u(s)\big)}\right)\right\}d\xi \\ &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{K}}\left|\widehat{f}(\xi)\right|^2\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2+R_0(f). \end{aligned}$$

Hence, our claim is true for $f\in\Omega^0$. Further by applying Levi Lemma and the given assumption, we obtain (3.3). We now show that the series (3.3) is absolutely convergent. Since

$$\left|\widehat{\Psi}^{(j)}(x)\widehat{\Psi}^{(j)}\big(x+u(s)\big)\right|\leq\frac{1}{2}\left(\left|\widehat{\Psi}^{(j)}(x)\right|^2+\left|\widehat{\Psi}^{(j)}\big(x+u(s)\big)\right|^2\right),$$

it suffices to verify that the series

$$\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{N}}(qN)^{-j}\int_{\mathbb{K}}\left|\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}x\big)\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\big)\right|\left|\widehat{\Psi}^{(j)}(x)\right|^2dx \qquad (3.5)$$

is convergent. As $u(s) \neq 0$ for $s \in \mathbb{N}$ and $f \in \Omega^0$, there exists a constant $J > 0$ such that

$$\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}x\big)\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\right)=0 \quad \forall |j|>J.$$

On the other hand, for each fixed $|j| \leq J$, there is a constant L such that

$$\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\right)=0 \quad \forall s>L.$$

Hence, it means that there are only finite non-zero terms in the series (3.5). Thus, there exists a constant C such that

$$\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{N}}(qN)^{-j}\int_{\mathbb{K}}\left|\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}x\big)\widehat{f}\left((\mathfrak{p}^{-1}N)^{-j}\big(x+u(s)\big)\right)\widehat{\Psi}^{(j)}(x)\widehat{\Psi}^{(j)}\big(x+u(s)\big)\right|dx \leq C\left\|\widehat{f}\right\|_{\infty}^{2}\left\|\widehat{\Psi}^{(j)}\right\|_{2}^{2}.$$

Thus, it follows that the series $\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{N}}\left|\left\langle f,\Psi_{j,\lambda}\right\rangle\right|^2$ is also convergent.

For given $s \in \mathbb{N}$, there is a unique pair (λ, m) with $\lambda \in \Lambda$ and $m \in q\Lambda + \mathscr{Q}$, where $q\Lambda = \{q\lambda : \lambda \in \Lambda\}$ and $\mathscr{Q} = \{1, 2, \ldots, qN-1\}$, such that $s = (qN)^{\lambda}m$. Therefore, we have $\{u(s)\}_{s\in\mathbb{N}} = \left\{(\mathfrak{p}^{-1}N)^{-\lambda}u(m)\right\}_{(\lambda,m)\in\Lambda\times(q\Lambda+\mathscr{Q})}$. Since the series (3.4) is absolutely convergent, we can estimate $R_0(f)$ by rearranging the series, changing the order of summation and integration by Levi Lemma as follows

$$\begin{aligned}
R_0(f) &= \sum_{j\in\mathbb{Z}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\Psi^{(j)}(\xi)\left\{\sum_{s\in\mathbb{N}}\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+u(s)\big)\big)\overline{\widehat{\Psi}^{(j)}\big(\xi+u(s)\big)}\right\}d\xi \\
&= \sum_{j\in\mathbb{Z}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\left\{\sum_{\lambda\in\Lambda}\sum_{m\in q\Lambda+\mathscr{Q}}\Psi^{(j)}(\xi)\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)\Big)\right. \\
&\qquad\left.\times\overline{\widehat{\Psi}^{(j)}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)}\right\}d\xi \\
&= \int_{\mathbb{K}}\sum_{j\in\mathbb{Z}}(qN)^j\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\left\{\sum_{\lambda\in\Lambda}\sum_{m\in q\Lambda+\mathscr{Q}}\Psi^{(j)}\big((\mathfrak{p}^{-1}N)^{-k}\xi\big)\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)\Big)\right. \\
&\qquad\left.\times\overline{\widehat{\Psi}^{(j)}(\mathfrak{p}^{-1}N)^{-k}\big(\big(\xi+u(m)\big)\big)}\right\}d\xi \\
&= \int_{\mathbb{K}}\sum_{j\in\mathbb{Z}}(qN)^j\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\left\{\sum_{m\in q\mathbb{N}_0+\mathscr{Q}}\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)\Big)\right. \\
&\qquad\left.\times\sum_{\lambda\in\Lambda}\Psi^{(j)}\big((\mathfrak{p}^{-1}N)^{-k}\xi\big)\overline{\widehat{\Psi}^{(j)}(\mathfrak{p}^{-1}N)^{-k}\big(\big(\xi+u(m)\big)\big)}\right\}d\xi \\
&= \int_{\mathbb{K}}\sum_{j\in\mathbb{Z}}(qN)^j\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\left\{\sum_{m\in q\mathbb{N}_0+\mathscr{Q}}\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)\Big)t_{\Psi^{(j)}}\big(u(m),\xi\big)\right\}d\xi \\
&= \sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{N}_0+\mathscr{Q}}(qN)^j\int_{\mathbb{K}}\overline{\widehat{f}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)}\widehat{f}\Big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+(\mathfrak{p}^{-1}N)^{-k}u(m)\big)\Big)t_{\Psi^{(j)}}\big(u(m),\xi\big)d\xi,
\end{aligned}$$

where

$$t_{\psi^{(j)}}\big(u(m),\xi\big)=\sum_{k\in\mathbb{N}_0}\Psi^{(j)}\big((\mathfrak{p}^{-1}N)^{-k}\xi\big)\overline{\widehat{\Psi}^{(j)}(\mathfrak{p}^{-1}N)^{-k}\big(\big(\xi+u(m)\big)\big)}.$$

THEOREM 3.3. Assume that $\Psi^{(j)}\in L^2\big(K,\mathbb{C}^M\big)$ and $\Big\|\widehat{\Psi}^{(j)}\Big\|_{L^2(\mathbb{K})}=1$ for $j\in\mathbb{Z}$. Then

$$\sum_{j\in\mathbb{Z}}\Big|\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\Big|^2=1 \quad \text{a.e } \xi\in K \tag{3.6}$$

and

$$\sum_{j=0}^{\infty}\widehat{\Psi}^{(n-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\overline{\widehat{\Psi}^{(n-j)}\big((\mathfrak{p}^{-1}N)^{-j}\big(\xi+u(m)\big)\big)}=0 \quad \text{a.e}\, n\in\mathbb{N}_0 \,\text{and}\, m\in q\Lambda+\mathscr{Q} \tag{3.7}$$

if and only if

$$\big\{(qN)^{j/2}\Psi^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}x-\lambda\big),\, j\in\mathbb{Z},\lambda\in\Lambda\big\}$$

is an orthonormal basis of $L^2(\mathbb{K})$.

PROOF. *Sufficiency part*: As $\left\|\widehat{\Psi}^{(j)}\right\|_{L^2(\mathbb{K})}=1$, it is clear that the system $\{\Psi_{j,\lambda} : j\in\mathbb{Z},\lambda\in\Lambda\}$ is an orthonormal basis if and only if for any $f\in L^2\big(K,\mathbb{C}^M\big)$, the Parseval identity holds.

Assume that the conditions (3.6) and (3.7) hold. Then for $n\in\mathbb{N}_0, m\in q\Lambda+\mathscr{Q}$, we have by Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|\langle f,\Psi_{j,\lambda}\rangle|^2 &= \int_{\mathbb{K}}\left|\widehat{f}(\xi)\right|^2\sum_{j\in\mathbb{Z}}\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2 d\xi\\ &= \|f\|^2_{L^2\left(K,\mathbb{C}^M\right)} \quad \forall f\in S.\end{aligned}$$

Necessary condition: We assume that $\big\{(qN)^{j/2}\Psi^{(j)}\big((\mathfrak{p}^{-1}N)^{j}x-\lambda\big),\, j\in\mathbb{Z},\lambda\in\Lambda\big\}$ is an orthonormal basis of $L^2\big(K,\mathbb{C}^M\big)$ and will prove the conditions (3.6) and (3.7). We assume Δ_j to be the set of regular points of $\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2$, so that for each $x\in\Delta_j$,

$$(qN)^n\int_{\xi-x\in\mathfrak{B}^n}\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2 d\xi\to\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}x\big)\right|^2,\quad \text{as}\, n\to\infty$$

Then $\left|\Delta_j^c\right|=0$, so that $\left|\bigcup_{j\in\mathbb{Z}}\Delta_j^c\right|=0$. Let $\xi_0\in\mathbb{K}\setminus\bigcup_{j\in\mathbb{Z}}\Delta_j^c$. For each fixed positive integer M, set

$$\widehat{f}(\xi)=(qN)^{m/2}\Phi_m(\xi-\xi_0),\quad m\geq M,$$

where $\Phi_m(\xi-\xi_0)$ is the characteristic function of $\xi_0+\mathfrak{B}^m$. Then it follows that for $s\in\mathbb{N}$ and $j\geq -M$, $\overline{\widehat{f}(\xi)}\widehat{f}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(s)\big)=0$, and hence $\|f\|_2^2=1$. Furthermore, we have

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\Psi_{j,\lambda}\rangle\right|^2=\sum_{j\geq -M}\int_{\xi_0+\mathfrak{B}^m}(qN)^m\left|\widehat{f}(\xi)\right|^2\left|\widehat{\Psi}^{(j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2 d\xi\leq B.$$

Therefore, in the limiting case, we have

$$\sum_{j\in\mathbb{Z}}\left|\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi_0\big)\right|^2=1\quad \text{a.e}$$

To prove (3.7), we let

$$\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}\left|\langle f,\Psi_{j,\lambda}\rangle\right|^2=I_1+I_2,$$

where

$$I_1=\sum_{j>-M}\sum_{\lambda\in\Lambda}\left|\langle f,\Psi_{j,\lambda}\rangle\right|^2\quad\text{and}\quad I_2=\sum_{j\leq -M}\sum_{\lambda\in\Lambda}\left|\langle f,\Psi_{j,\lambda}\rangle\right|^2.$$

Since, it has been already verified that $I_1=\sum_{j>-M}\left|\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi_0\big)\right|^2$, so to prove the result, it is enough to show that $\lim_{M\to+\infty}I_2=0$.
Using Lemma 3.1 and Schwartz Inequality, we have

$$0\leq I_2\leq\sum_{j\leq -M}\sum_{r\in\mathbb{N}_0}\left\{\int_{\mathbb{K}}\left|\overline{\widehat{f}(\xi)}\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2 d\xi\right\}^{\frac{1}{2}}$$
$$\times\left\{\int_{\mathbb{K}}\left|\widehat{f}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)\overline{\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi+u(r)\big)}\right|^2 d\xi\right\}^{\frac{1}{2}}.$$

If $\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\in\xi_0+\mathfrak{B}^m$ for a fixed $j\leq -M$, then it follows that $\left|(\mathfrak{p}^{-1}N)^{-j}u(r)\right|\leq(qN)^{-m}$, so $|u(r)|\leq(qN)^{-m-j}$. Therefore

$$I_2\leq\sum_{j\leq -M}\int_{\mathbb{K}}\left|\overline{\widehat{f}(\xi)}\widehat{\Psi}^{(-j)}\big((\mathfrak{p}^{-1}N)^{-j}\xi\big)\right|^2 d\xi\leq\sum_{j\leq -M}\int_{(\mathfrak{p}^{-1}N)^{-j}\xi_0+\mathfrak{B}^{-j+m}}\left|\widehat{\Psi}^{(-j)}(\xi)\right|^2 d\xi.$$

If $\xi_0\neq 0$, then for given $\varepsilon>0$, we choose M so that

$$(qN)^{-M}<|\xi_0|=(qN)^s\quad\text{and}\quad\int_{\mathfrak{B}^{M-s}}\left|\widehat{\Psi}^{(-j)}(\xi)\right|^2 d\xi<\varepsilon.$$

Therefore for all $j \leq -M$, we have

$$(\mathfrak{p}^{-1}N)^{-j}\xi_0 + \mathfrak{B}^{-j+m} \subset \mathfrak{B}^{M-s}. \tag{3.8}$$

Moreover for any $j_1 < j_2 \leq -M$, it can be easily verified that

$$\left\{(\mathfrak{p}^{-1}N)^{-j_1}\xi_0 + \mathfrak{B}^{-j_1+m}\right\} \cap \left\{(\mathfrak{p}^{-1}N)^{-j_2}\xi_0 + \mathfrak{B}^{-j_2+m}\right\} = \Phi. \tag{3.9}$$

Using (3.8) and (3.9), we have

$$I_2 \leq \int_{\mathfrak{B}^{M-s}} \left|\widehat{\Psi}^{(-j)}(x)\right|^2 dx < \varepsilon,$$

from which the result follows.

DEFINITION 3.4. Let $\mathbb{K}$ be a local field of positive characteristic $p \geq 0$ and $\mathfrak{p}$ be a prime element of $\mathbb{K}$. A collection of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2\big(K, \mathbb{C}^M\big)$ is called a *vector valued nonuniform nonstationary multiresolution analysis* (VVNUNMRA) if the following conditions hold:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2\big(K, \mathbb{C}^M\big)$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) for any $j \in \mathbb{Z}$ there is a function $\Phi^{(j)} \in V_j$ such that the sequence $\left\{\Phi^{(j)}\big(\cdot + (\mathfrak{p}^{-1}N)^{-j}\lambda\big) : \lambda \in \Lambda\right\}$ forms a Riesz basis (or orthonormal basis) for V_j.

The sequence $\left\{\Phi^{(j)}\right\}_{j\in\mathbb{Z}}$ is called a *scaling sequence* for the given VVNUNMRA. If we denote by P_j, the orthogonal projector on V_j, then condition (b) of the Definition 3.4 implies that $\lim_{j\to\infty} P_j f = f$ for any $f \in L^2\big(K, \mathbb{C}^M\big)$. It then follows from the condition (d) that for any $f \in V_j$, the function $f\big(x + (\mathfrak{p}^{-1}N)^j\lambda\big)$ also belong to V_j for any $\lambda \in \Lambda$. Without loss of generality, we assume that $\left\{\Phi^{(j)}\big(x + (\mathfrak{p}^{-1}N)^j\lambda\big)\right\}_{\lambda\in\Lambda}$ constitutes an orthonormal basis in V_j.

PROPOSITION 3.5. If $\{V_j\}_{j\in\mathbb{Z}}$ is a NUNMRA, then there exists a vector valued nonstationary orthonormal wavelet bases $\{\Psi_{j,\lambda}\}_{j\in\mathbb{Z},\lambda\in\Lambda}$, such that for any $f\in L^2(K,\mathbb{C}^M)$,

$$P_{j+1}f = P_j f = \sum_{\lambda\in\Lambda} \langle f,\Psi_{j,\lambda}\rangle \Psi_{j,\lambda} \tag{3.10}$$

PROOF. Let W_j be an orthogonal complement of V_j in V_{j+1}. Then

$$W_j \perp W_{j'}, \quad \text{for} \quad j\neq j' \tag{3.11}$$

and for $j_0 < j$,

$$V_j = V_{j_0} \oplus \left(\bigoplus_{\ell=j_0}^{j-1} W_\ell \right). \tag{3.12}$$

It then follows from the conditions (b) and (d) of the definition 3.4 that

$$L^2(K,\mathbb{C}^M) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{3.13}$$

Equation (3.10) is equivalent to the fact that for fixed j, the sequence $\{\Psi_{j,\lambda}\}_{\lambda\in\Lambda}$ forms an orthonormal basis in W_j. From (3.13), it follows that $\{\Psi_{j,\lambda}\}_{j\in\mathbb{Z},\lambda\in\Lambda}$ is an orthonormal basis in $L^2(K,\mathbb{C}^M)$. Hence the problem of construction of nonstationary wavelet bases satisfying (3.10) is to find $\Psi^{(j)}$ such that $\left\{\Psi^{(j)}\left(x+(\mathfrak{p}^{-1}N)^j\lambda\right)\right\}_{\lambda\in\Lambda}$ constitutes an orthonormal basis in W_j.

For the construction of the function $\Psi^{(j)}$, we use the following properties of $\varphi^{(j)}$ and W_j.

Since $\Phi^{(j)}\subset V_j\subset V_{j+1}$ and $\left\{\Phi_\lambda^{(j+1)}\right\}_{\lambda\in\mathbb{N}_0}$ is an orthonormal basis in V_{j+1}, it follows that

$$\Phi^{(j)}(x) = \sum_{\lambda\in\Lambda} h_{j+1,\lambda}\, \Phi_\lambda^{(j+1)}(x), \tag{3.14}$$

where

$$h_{j+1,\lambda} = \left\langle \Phi^{(j)},\Phi^{(j+1)} \right\rangle, \quad \sum_{\lambda\in\Lambda} \left|h_{j+1,\lambda}\right|^2 = 1. \tag{3.15}$$

Equation (3.14) can be written in the frequency domain as

$$\widehat{\Phi}^{(j)}(\xi) = m_{j+1}\left((\mathfrak{p}^{-1}N)^{j+1}\xi\right)\widehat{\varphi}^{(j+1)}(\xi), \tag{3.16}$$

where

$$m_{j+1}(\xi) = \sum_{\lambda \in \mathbb{N}_0} h_{j+1,\lambda}\, \chi_\lambda(\xi),$$

are called *vector valued nonuniform nonstationary masks*. It can be easily verified that

$$\sum_{\lambda \in \Lambda} \left| \Phi^{(j)}\big(\xi + (\mathfrak{p}^{-1}N)^{-j}\lambda\big) \right|^2 = (qN)^{-j} \quad \text{for a.e } \xi \in K. \tag{3.17}$$

From (3.16) and (3.17), we have

$$\sum_{\lambda \in \Lambda} \left| m_{j+1}\big((\mathfrak{p}^{-1}N)^{j+1}\xi + \mathfrak{p}\lambda\big) \Phi^{(j+1)}\big(\xi + (\mathfrak{p}^{-1}N)^{j}\lambda\big) \right|^2 = (qN)^{-j}.$$

Partitioning the sum into two parts and taking into account the integral periodicity of m_{j+1}, we get

$$\big|m_{j+1}(\xi)\big|^2 + \big|m_{j+1}\big(\xi + \mathfrak{p}u(N)\big)\big|^2 = qN. \tag{3.18}$$

We now characterize the subspaces W_j. Let $f \in W_j$. Then f is in V_{j+1} and is orthogonal to V_j. Then

$$f(x) = \sum_{k \in \mathbb{N}_0} f_k \Phi_k^{(j+1)}(x), \tag{3.19}$$

where $f_\lambda = \langle f, \Phi_\lambda^{(j+1)} \rangle$. Applying Fourier transform to equation (3.19), we have

$$\widehat{f}(\xi) = m_f\big((\mathfrak{p}^{-1}N)^{j+1}\xi\big)\, \widehat{\Phi}^{(j+1)}(\xi), \tag{3.20}$$

where

$$m_f(\xi) = \sum_{\lambda \in \Lambda} f_\lambda\, \chi_\lambda(\xi),$$

are integral periodic from $L^2(\mathfrak{D})$. Since f is orthogonal to V_j, we have for $\lambda \in \Lambda$,

$$\int_{\mathbb{K}} \widehat{f}(\xi)\, \overline{\widehat{\Phi}^{(j)}(\xi)}\, \chi_\lambda\big((\mathfrak{p}^{-1}N)^{j-1}\xi\big) d\xi = 0.$$

Moreover,

$$\int_{\mathbb{K}} \widehat{f}(\xi)\overline{\widehat{\Phi}^{(j)}(\xi)}\chi_\lambda\big((\mathfrak{p}^{-1}N)^{j-1}\xi\big)d\xi$$

$$= \int_{(\mathfrak{p}^{-1}N)^j\mathfrak{D}} \sum_{r\in\mathbb{N}_0} \widehat{f}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)\overline{\widehat{\Phi}^{(j)}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)}\chi_\lambda\big((\mathfrak{p}^{-1}N)^{j-1}\xi\big)d\xi$$

$$= 0. \tag{3.21}$$

Since (3.21) holds for all $\lambda \in \Lambda$, we have

$$\sum_{r\in\mathbb{N}_0} \widehat{f}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)\overline{\widehat{\Phi}^{(j)}\big(\xi+(\mathfrak{p}^{-1}N)^{j}u(r)\big)} = 0. \tag{3.22}$$

The series in (3.22) converges in $L^2(\mathfrak{D})$. Keeping in view (3.18) and using equations (3.20) and (3.21) in (3.22), we get

$$(qN)^{j+1} \sum_{r\in\mathbb{N}_0} \widehat{f}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)\overline{\widehat{\Phi}^{(j)}\big(\xi+(\mathfrak{p}^{-1}N)^{-j}u(r)\big)}$$

$$= m_f\big((\mathfrak{p}^{-1}N)^{j+1}\xi\big)\overline{m_{j+1}\big((\mathfrak{p}^{-1}N)^{j+1}\xi\big)}m_f\big((\mathfrak{p}^{-1}N)^{j+1}\xi+\mathfrak{p}u(N)\big)\overline{m_{j+1}\big((\mathfrak{p}^{-1}N)^{j+1}\xi+\mathfrak{p}u(N)\big)}$$

$$= 0.$$

It is evident from (3.18) that $\overline{m_{j+1}(\xi)}$ and $\overline{m_{j+1}\big(\xi+\mathfrak{p}u(N)\big)}$ can not vanish simultaneously. Hence, there exist integral periodic function $\lambda(\xi)$ such that

$$m_f(\xi) = \lambda(\xi)\overline{m_{j+1}\big(\xi+\mathfrak{p}u(N)\big)} \quad \text{a.e.} \tag{3.23}$$

and

$$\lambda(\xi)+\lambda\big(\xi+\mathfrak{p}u(N)\big) = 0. \tag{3.24}$$

Equation (3.24) can be rewritten as

$$\lambda(\xi) = \nu\big(\mathfrak{p}^{-1}N\xi\big)\overline{\chi(\xi)},$$

where ν is an integral periodic function. Therefore the Fourier transform of any function of W_j yields

$$\widehat{f}(\xi) = \overline{m_{j+1}\big(\xi+\mathfrak{p}u(N)\big)}\nu\big((\mathfrak{p}^{-1}N)^j\xi\big)\widehat{\Phi}^{(j+1)}(\xi)\overline{\chi\big(\mathfrak{p}^j\xi\big)}. \tag{3.25}$$

Moreover, it can be seen that ν is square integrable. Having system (3.25) in hand, it will not be difficult to find functions $\Psi^{(j)}$ in the W_j space such that

$\left\{\Psi^{(j)}\left(x+(\mathfrak{p}^{-1}N)^{-j}\lambda\right)\right\}_{\lambda\in\Lambda}$ constitutes an orthonormal basis in W_j. Therefore, we have

$$\widehat{\Psi}^{(j)}(\xi)=\overline{m_{j+1}\left(\xi+\mathfrak{p}u(N)\right)}\nu_{\Psi^{(j)}}\left((\mathfrak{p}^{-1}N)^{j}\xi\right)\widehat{\Phi}^{(j+1)}(\xi)\overline{\chi\left((\mathfrak{p}^{-1}N)^{j}\xi\right)}.$$

Therefore, substituting above expression in (3.17) and using (3.18), we have

$$\left|\nu_{\Psi^{(j)}}\right|^2=1 \quad \text{a.e.}$$

From (3.25), it follows that the integer shifts of $\Psi^{(j)}$ defined by

$$\widehat{\Psi}^{(j)}(\xi)=\overline{m_{j+1}\left(\xi+\mathfrak{p}u(N)\right)}\widehat{\Phi}^{(j+1)}(\xi)\overline{\chi\left((\mathfrak{p}^{-1}N)^{j}\xi\right)}. \tag{3.26}$$

forms a basis of W_j. Thus having a nonstationary multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ generated by a scaling function $\left\{\Phi^{(j)}\right\}$, one can construct a nonstationary orthonormal wavelet basis $\left\{\Psi_{j,\lambda}\right\}_{j\in\mathbb{Z},\,\lambda\in\Lambda}$ in $L^2\left(K,\mathbb{C}^M\right)$ satisfying (3.10).

DEFINITION 3.6. Suppose $\Psi^{(j)}\in L^2\left(K,\mathbb{C}^M\right)$ for $j\in\mathbb{Z}$. Then *dimension function* is defined by

$$D_{\Psi^{(j)}}(\xi)=\sum_{n=1}^{\infty}\sum_{\lambda\in\Lambda}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right)\right|^2 \quad \text{a.e}\,\xi\in K$$

Since

$$\int_{\mathfrak{D}}\sum_{n=1}^{\infty}\sum_{\lambda\in\Lambda}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right)\right|^2 d\xi=\sum_{n=1}^{\infty}(qN)^{-n}\int_{\mathbb{K}}\left|\widehat{\psi}^{(j)}(\xi)\right|^2 d\xi.$$

Hence $D_{\Psi^{(j)}}$ is well defined for a.e. $\xi\in K$.

PROPOSITION 3.7. For all $j\in\mathbb{Z}$ and for a.e. $\xi\in K$, we have

$$\left|\widehat{\Phi}^{(j)}\right|^2=\sum_{n=1}^{\infty}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\xi\right)\right|^2. \tag{3.27}$$

PROOF. If $\Psi^{(j)} \in L^2\left(K, \mathbb{C}^M\right)$, then equation (3.26) holds. Therefore using (3.18), we have from equations (3.16) and (3.26),

$$\begin{aligned}\left|\widehat{\Phi}^{(j)}(\xi)\right|^2+\left|\widehat{\Psi}^{(j)}(\xi)\right|^2 &= \left|m_{j+1}\left((\mathfrak{p}^{-1}N)^{j+1}\xi\right)\widehat{\Phi}^{(j+1)}(\xi)\right|^2 \\ &\quad+\left|\overline{m_{j+1}\left(\xi+\mathfrak{p}u(N)\right)}\widehat{\Phi}^{(j+1)}(\xi)\overline{\chi\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)}\right|^2 \\ &= qN\left|\widehat{\Phi}^{(j+1)}(\xi)\right|^2 \\ &= \left|\widehat{\Phi}^{(j+1)}(\mathfrak{p}^{-1}N\xi)\right|^2.\end{aligned}$$

Since the equality holds for a.e. $\xi \in K$, we have

$$\left|\widehat{\Phi}^{(j)}(\xi)\right|^2=\left|\widehat{\Phi}^{(j-1)}\left(\mathfrak{p}^{-1}N\xi\right)\right|^2+\left|\widehat{\Psi}^{(j-1)}\left(\mathfrak{p}^{-1}N\xi\right)\right|^2.$$

Iterating for any integer $L \geq 1$, we get

$$\left|\widehat{\Phi}^{(j)}(\xi)\right|^2=\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2+\sum_{n=1}^{L}\left|\widehat{\Psi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-n}\xi\right)\right|^2.$$

Since $\left|\widehat{\Phi}^{(j-L)}(\xi)\right| \leq 1$, the sequence

$$\left\{\sum_{n=1}^{L}\left|\widehat{\Psi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-n}\xi\right)\right|^2 : L \geq 1\right\}$$

of real numbers is bounded by 1, hence it converges. Therefore, $\lim_{L\to\infty}\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2$ also exists. Moreover

$$\int_{\mathbb{K}}\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2 d\xi=(qN)^{-L}\int_{\mathbb{K}}\left|\widehat{\Phi}^{(j-L)}(\xi)\right|^2 d\xi \to 0 \quad \text{as } L \to \infty.$$

Therefore, the application of the Fatou's Lemma yields

$$\int_{\mathbb{K}}\lim_{L\to\infty}\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2 d\xi \leq \lim_{L\to\infty}\int_{\mathbb{K}}\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2 d\xi=0.$$

This means that $\lim_{L\to\infty}\left|\widehat{\Phi}^{(j-L)}\left((\mathfrak{p}^{-1}N)^{-L}\xi\right)\right|^2 d\xi=0$. Hence, we have

$$\left|\widehat{\Phi}^{(j)}(\xi)\right|^2=\sum_{n=1}^{\infty}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\xi\right)\right|^2$$

This completes the proof.

Since $\left\{(qN)^{j/2}\Phi^{(j)}\left((\mathfrak{p}^{-1}N)^{j}x-\lambda\right):\lambda\in\Lambda\right\}$ is an orthonormal basis of $L^2\left(K,\mathbb{C}^M\right)$ for all $j\in\mathbb{Z}$, we have

$$1=\sum_{\lambda\in\Lambda}\left|\widehat{\Phi}^{(j)}\left(\xi+\lambda\right)\right|^2=\sum_{\lambda\in\Lambda}\sum_{n=1}^{\infty}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\left(\xi+\lambda\right)\right)\right|^2=D_{\Psi^{(j)}}\left(\xi\right).$$

Since $D_{\Psi^{(j)}}\left(\xi\right)=1$, we can choose the smallest $n\in\mathbb{N}$ such that for all $j\in\mathbb{Z}$ and for almost all $\xi\in K$,

$$\sum_{\lambda\in\Lambda}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\left(\xi+\lambda\right)\right)\right|^2\neq 0$$

and then for almost all $\xi\in K$, we define $\Phi^{(j)}(\xi)$ by

$$\widehat{\Phi}^{(j)}(\xi)=\frac{\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\xi\right)}{\sqrt{\sum_{\lambda\in\Lambda}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\left(\xi+\lambda\right)\right)\right|^2}}.$$

Moreover for a fixed $j\in\mathbb{Z}$ and $n\in\mathbb{N}$, we define an infinite vector of $l^2(\Lambda)$ as

$$\Psi_{j,n}(\xi)=\left\{\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\left(\xi+\lambda\right)\right)\right\}_{\lambda\in\lambda}\quad\text{for a.e }\xi\in K\tag{3.28}$$

THEOREM 3.8. Assume that $\Psi^{(j)}\in L^2\left(K,\mathbb{C}^M\right)$ for every $j\in\mathbb{Z}$, such that the system $\left\{(qN)^{j/2}\Psi^{(j)}\left((\mathfrak{p}^{-1}N)^{j}x-\lambda\right):\lambda\in\Lambda\right\}$ is an orthonormal basis of $L^2\left(K,\mathbb{C}^M\right)$. Then the mother wavelets $\Psi^{(j)}, j\in\mathbb{Z}$ come from a VVNUNMRA, if and only if

$$D_{\Psi^{(j)}}(\xi)=\sum_{n=1}^{\infty}\sum_{k\in\mathbb{N}_0}\left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}\left(\xi+\lambda\right)\right)\right|^2=1\quad\text{a.e}\,\xi\in K$$

PROOF. Necessary part of the Theorem follows from the Proposition 3.7. For the proof of the sufficient part, we need the following lemmas:

LEMMA 3.9. *For all $j\in\mathbb{Z}$, and for almost all $\xi\in K$, we have*

$$\Psi_{j,n}(\xi)=\sum_{h=1}^{\infty}\left\langle\Psi_{j,n}(\xi),\Psi_{j,h}(\xi)\right\rangle\Psi_{j,h}(\xi),\tag{3.29}$$

PROOF. The series in the Lemma converges absolutely for a.e. $\xi \in K$. Let us first show that

$$\widehat{\Psi}^{(j-n)}((\mathfrak{p}^{-1}N)^{-n}\xi) = \sum_{h=1}^{\infty}\sum_{\lambda\in\Lambda}\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right) \times \overline{\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}(\xi+\lambda)\right)}\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}\xi\right). \tag{3.30}$$

Let us denote by $\Gamma_{j,n}(\xi)$, the second member of the series (3.30). Then using Equation (3.7) and (3.30), we have

$$\begin{aligned}
\Gamma_{j,n}(\xi) &= \sum_{h=1}^{\infty}\sum_{\lambda\in\Lambda}\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right)\overline{\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}(\xi+\lambda)\right)}\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}\xi\right)\\
&= \sum_{\lambda\in\Lambda}\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right)\left\{\sum_{h=0}^{\infty}\overline{\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}(\xi+\lambda)\right)}\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}\xi\right)\right.\\
&\qquad\left. -\overline{\widehat{\Psi}^{(j)}(\xi+\lambda)}\widehat{\Psi}^{(j)}(\xi)\right\}\\
&= \sum_{h=0}^{\infty}\sum_{\lambda q\in\Lambda}\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+u(q\lambda))\right)\\
&\qquad\times\overline{\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}(\xi+u(qk))\right)}\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}\xi\right)\\
&= \sum_{h=1}^{\infty}\sum_{\lambda\in\Lambda}\widehat{\Psi}^{\left((j+1)-(n+1)\right)}\left((\mathfrak{p}^{-1}N)^{n+1}(\mathfrak{p}\xi+\lambda)\right)\\
&\qquad\times\overline{\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}(\mathfrak{p}\xi+\lambda)\right)}\widehat{\Psi}^{(j-h)}\left((\mathfrak{p}^{-1}N)^{-h}\mathfrak{p}\xi\right)\\
&= \Gamma_{j+1,n+1}(\mathfrak{p}\xi)
\end{aligned}$$

The above system is equivalent to

$$\Gamma_{j,n}(\xi) = \Gamma_{j-1,n-1}(\mathfrak{p}^{-1}N\xi).$$

In consequence, for $j \in \mathbb{Z}$, $n \in \mathbb{N}$ and almost all $\xi \in \mathbb{K}$, we have by recursion

$$\Gamma_{j,n}(\xi) = \Gamma_{j-(n-1),1}((\mathfrak{p}^{-1}N)^{n+1}\xi),$$

from which equation (3.30) follows as $\Gamma_{j-n+1,1}(\xi) = \widehat{\Psi}^{(j-n)}((\mathfrak{p}^{-1}N)^{-n}\xi)$.

Moreover, since $\left\langle \Psi_{j,n}(\xi), \Psi_{j,h}(\xi)\right\rangle$ is integral periodic, equation (3.29) holds. This completes the proof of the lemma.

From the above lemma, it can be seen that

$$\sum_{n=1}^{\infty} \left\|\Psi_{j,n}(\xi)\right\|^2_{l^2(\Lambda)} = \sum_{n=1}^{\infty}\sum_{\lambda\in\Lambda} \left|\widehat{\Psi}^{(j-n)}\left((\mathfrak{p}^{-1}N)^{-n}(\xi+\lambda)\right)\right|^2 = D_{\Psi^{(j)}}(\xi) = 1. \quad (3.31)$$

For all $j \in \mathbb{Z}$, and for almost all $\xi \in K$, we define

$$\mathscr{F}_j(\xi) = \overline{\text{span}}\{\Psi_{j,n}(\xi) : n \geq 1\}. \quad (3.32)$$

It is a subspace of $l^2(\Lambda)$ of dimension 1.

LEMMA 3.10. Let $\{\alpha_n : n \geq 1\}$ be a family of vectors in a Hilbert space $\mathbb{H}$ such that

$$\sum_{n=1}^{\infty} \|\alpha_n\|^2 = C < \infty \quad \text{and} \quad v_n = \sum_{m=1}^{\infty} \langle \alpha_n, \alpha_m\rangle \alpha_m \quad \text{for all } n \geq 1.$$

Then dimension of the subspace $\overline{\text{span}}\{\alpha_n : n \geq 1\}$ *of* $\mathbb{H}$ is equal to C.

Sufficient part of the Theorem: Using Lemma 3.10, it follows that the family $\mathscr{F}_j(\xi)$, defined by (3.32) is generated by only one unit vector $X_j(\xi)$. To construct it, we first make a partition of $\mathfrak{D}$ as follows

$$\mathscr{P}_{j,n} = \left\{\xi \in \mathfrak{D} : \Psi_{j,n}(\xi) \neq 0 \quad \text{and} \quad \Psi_{j,m}(\xi) = 0 \quad \text{for } m < n\right\}, n \geq 1,$$

and the null set

$$\mathscr{P}_{j,0} = \left\{\xi \in \mathfrak{D} : D_{\Psi^{(j)}}(\xi) = 0\right\}.$$

Let us now define the unit vector $X_j(\xi)$ on $\mathfrak{D}$ by

$$X_j(\xi) = \frac{\Psi_{j,n}(\xi)}{\left\|\Psi_{j,n}(\xi)\right\|_{l^2(\Lambda)}} \quad \text{if } \xi \in E_{j,n}.$$

We write $X_j(\xi) = \left\{u_\lambda^{(j)}(\xi)\right\}_{\lambda\in\Lambda}$ and define $\Phi^{(j)}$ almost everywhere on $\mathbb{K}$ by

$$\widehat{\Phi}^{(j)}(\xi) = u_\lambda^{(j)}\left(\xi-\lambda\right) \quad \text{if } \xi \in \mathfrak{D}+\lambda.$$

These $\Phi^{(j)}, j \in \mathbb{Z}$ are the required scaling functions.

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A stochastic equivalence approach for an Ornstein-Uhlenbeck process driven power system dynamics

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Abstract

This paper develops a stochastic equivalence approach for an Ornstein-Uhlenbeck process-driven power system. The concept of stochastic equivalence coupled with stochastic differential rule plays the important role to develop the stochastic equivalence approach of this paper. This paper also develops the prediction theory of power system dynamics with the OU process as well.

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Additional Key Words and Phrases: stochastic equivalence approach, stochastic differential equation, the OU process, Kuramoto oscillator, power system dynamics.

1. INTRODUCTION

The stochastic differential equation (SDE) is caused by random term. This random term can be represented in form of white noise process or coloured noise process. For deterministic representation of the swing equation of a single machine-infinite bus (SMIB) system, see Kundur (1994). The white noise driven-Itô SDE model for a power system dynamics, see (Wang and Crow 2013; Hirpara and Sharma 2015). The white noise driven-Stratonovich SDE model for a power system dynamics and circuits and systems, see (Hirpara 2019; Patil and Sharma 2015) and the coloured noise driven-SDE model for a power system dynamics, see(Hänggi and Jung 1995; Patel and Sharma 2012; Patil and Sharma 2014; Hirpara and Sharma 2015; Guo and Shi 2017) and recent publication Verdejo et al. (2019) as well. The well-known Kuramoto model (Dörfler and Bullo 2010; Tönjes 2010; Simpson-Porco 2012; Schäfer et al. 2017; Supplemental material 2017), explaining essential mechanisms underlying synchronization, is very similar to the power system dynamics when neglecting inertia.

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The problem becomes complex when the extended phase space approach increases the dimensionality of the augmented state vector with the Markovian property, see Patel and Sharma (2012). It becomes easier approaches that allow the Markovian property as well as reduce the dimensionality of the state vector and that is known as the stochastic equivalence approach, see Patil and Sharma (2014) for greater detail.

This paper is aimed to analyse power system dynamics with the stochastic equivalence approach. In contrast to Hirpara and Sharma (2015), we developed stochastic power system dynamics in the presence of an Ornstein-Uhlenbeck (OU) process with the stochastic equivalence approach and examine the effectiveness of the prediction equations with no observations or valueless observations. The Numerical simulations help to check stochastic equivalent SDE in MATLAB for a smaller correlation time.

The remainder of the paper is organized as follows. Section 2 describes mathematical preliminaries. Section 3 is about the stochastic equivalence approach for an OU driven power system dynamics and numerical simulations. Section 4 discusses concluding remarks.

2. MATHEMATICAL PRELIMINARIES

Consider the stochastic differential system described by

$$\dot{\mu}_t = A(\mu_t) + B(\mu_t)\eta_t, \tag{1}$$

where the input process η_t is a stationary process with zero mean, relatively smaller correlation time. The right-hand side $B(\mu_t)\eta_t$ of equation (1) has a multiplicative noise character. Two different SDEs can be regarded as the stochastically equivalent, if they are associated with the same Fokker–Planck equation. Importantly, the concept of stochastic equivalence is useful for the coloured noise-driven SDE, where the input noise process is a coloured noise process with a smaller correlation time. A simple calculation shows that the white noise-driven stochastic differential equation associated with the Fokker-Planck equation (Stratonovich 1963, p. 97) assumes the structure

$$\dot{\mu}_t = (S_1(\mu_t) - \frac{1}{2}\partial_{\mu_t} S_2(\mu_t)) + \sqrt{S_2(\mu_t)}w_0(t), \tag{2}$$

where the input process $w_0(t)$ is zero mean, stationary, Gaussian white noise process. Thanks to equation (4.180) of Stratonovich (1963), a special case of equation (4.180), i.e. the coefficients $S_1(\mu_t)$ and $S_2(\mu_t)$ for the stochastic differential system $\dot{\mu}_t = A(\mu_t) + B(\mu_t)\eta_t$, can be re-cast as

$$S_1(\mu_t) = A(\mu_t) + \frac{\gamma_1}{2}B(\mu_t)\partial_{\mu_t}B(\mu_t) + \gamma_2 B^2(\mu_t)\partial_{\mu_t}B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}), \tag{3}$$

$$S_2(\mu_t) = \gamma_1 B^2(\mu_t) + 2\gamma_2 B^3(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}), \tag{4}$$

Where, $\gamma_1 = 2\int_{-\infty}^{0} R_{\eta\eta}(\tau)d\tau$, $\gamma_2 = \int_{-\infty}^{0} |\tau| R_{\eta\eta}(\tau)d\tau$, and the autocorrelation $R_{\eta\eta}(\tau) = E\eta_{t+\tau}\eta_t$. Note that $\partial_{\mu_t} = \frac{\partial}{\partial\mu_t}$ and $\partial^2_{\mu_t} = \frac{\partial^2}{\partial\mu_t^2}$ are throughout the paper. Equations (3)-(4) are valid for the weakly coloured noise input process η_t. Equation (2) in conjunction with equations (3)-(4) assumes the structure

$$\dot{\mu}_t = A(\mu_t) - \frac{\gamma_2}{2}B^2(\mu_t)\partial_{\mu_t}B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}) - \frac{\gamma_2}{2}B^3(\mu_t)\partial^2_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)})$$
$$+\sqrt{\gamma_1}B(\mu_t)\sqrt{1 + \frac{2\gamma_2}{\gamma_1}B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)})}w_0(t), \tag{5}$$

Note that equation (5) is *stochastically* equivalent to the SDE, i.e. $\dot{\mu}_t = A(\mu_t) + B(\mu_t)\eta_t$, since both are associated with the same Fokker-Planck equation. Equation (5) can be‘re-formulated’ in the Itô sense, i.e

$$d\mu_t = (A(\mu_t) - \frac{\gamma_2}{2}B^2(\mu_t)\partial_{\mu_t}B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}) - \frac{\gamma_2}{2}B^3(\mu_t)\partial^2_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}))dt$$
$$+\sqrt{\gamma_1}B(\mu_t)\sqrt{1 + \frac{2\gamma_2}{\gamma_1}B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)})}dW_t, \tag{6}$$

where

$$a(\mu_t) = A(\mu_t) - \frac{\gamma_2}{2} B^2(\mu_t)\partial_{\mu_t} B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}) - \frac{\gamma_2}{2} B^3(x_t)\partial_{x_t}^2(\frac{A(x_t)}{B(x_t)}), \quad (7)$$

$$b(\mu_t) = \sqrt{\gamma_1} B(\mu_t)\sqrt{1 + \frac{2\gamma_2}{\gamma_1} B(\mu_t)\partial_{x_t}(\frac{A(\mu_t)}{B(\mu_t)})}. \quad (8)$$

Equation (6), the Itô stochastic differential equation, describes a rigorous interpretation of the SDE in contrast to equation (5).

The OU process is a Gauss-Markov process and satisfies the stochastic differential equation,

$$d\eta_t = -\frac{1}{\tau_{cor}}\eta_t dt + \frac{\sqrt{2N}}{\tau_{cor}} dW_t, \quad (9)$$

where the terms τ_{cor} and $\frac{\sqrt{2N}}{\tau_{cor}}$ has an interpretation as correlation time and the process noise coefficient. Furthermore, the autocorrelation $R_{\eta\eta}(\tau)$ of the OU process, a stationary process, is $\frac{N}{\tau_{cor}} e^{-\frac{|\tau|}{\tau_{cor}}}$. This expression suggests that the correlation time of the stationary coloured noise can be explained using the concept of the autocorrelation of the coloured noise process. For the OU process, a weakly coloured noise process, the terms γ_1, γ_2 of equations (3)-(4) become,

$$\gamma_1 = 2N,\ \gamma_2 = N\tau_{cor}. \quad (10)$$

From equation (10) and equations (7)-(8), we have the following system non-linearity $a(\mu_t)$ and process noise coefficient $b(\mu_t)$ for the OU process-driven stochastic differential system:

$$a(\mu_t) = A(\mu_t) - \frac{N\tau_{cor}}{2} B^2(\mu_t)\partial_{\mu_t} B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)}) - \frac{N\tau_{cor}}{2} B^3(\mu_t)\partial_{\mu_t}^2(\frac{A(\mu_t)}{B(\mu_t)}), \quad (11)$$

$$b(\mu_t) = \sqrt{2N} B(\mu_t)\sqrt{1 + \tau_{cor} B(\mu_t)\partial_{\mu_t}(\frac{A(\mu_t)}{B(\mu_t)})}. \quad (12)$$

For a coloured noise-driven SDE, the 'scalar' exact mean and variance evolutions, $d\langle\mu_t\rangle$ and dP_t are,

$$d\langle\mu_t\rangle = (A(\langle\mu_t\rangle) - \frac{\gamma_2}{2}B^2(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$-\frac{\gamma_2}{2}B^3(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}) + \frac{1}{2}P_t(\partial^2_{\langle\mu_t\rangle}A(\langle\mu_t\rangle) - \frac{\gamma_2}{2}(B^3(\langle\mu_t\rangle)\partial^4_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+7B^2(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial^3_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+5B^2(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}) + 10B(\langle\mu_t\rangle)(\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle))^2\partial^2_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+2(\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle))^3\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}) + 6B(\langle\mu_t\rangle)$$

$$\times\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}))))dt, \quad (13)$$

$$dP_t = (2P_t(\partial_{\langle\mu_t\rangle}A(\langle\mu_t\rangle) - \frac{\gamma_2}{2}(4B^2(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+B^2(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+2B(\langle\mu_t\rangle)(\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle))^2\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}) + B^3(\langle\mu_t\rangle)\partial^3_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)}))) + \gamma_1 B^2(\langle\mu_t\rangle)$$

$$\times(1+\frac{2\gamma_2}{\gamma_1}B(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})) + P_t(\gamma_2 B^3(\langle\mu_t\rangle)\partial^3_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+6\gamma_2 B^2(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+3\gamma_2 B^2(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}B(\langle\mu_t\rangle)\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+6\gamma_2 B(\langle\mu_t\rangle)(\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle))^2\partial_{\langle\mu_t\rangle}(\frac{A(\langle\mu_t\rangle)}{B(\langle\mu_t\rangle)})$$

$$+\gamma_1 B(\langle\mu_t\rangle)\partial^2_{\langle\mu_t\rangle}B(\langle\mu_t\rangle) + \gamma_1(\partial_{\langle\mu_t\rangle}B(\langle\mu_t\rangle))^2))dt. \quad (14)$$

Notably, the numerical coefficients of equations (13)-(14), approximate evolutions are attributed to the successive differentiation with respect to the term $\langle \mu_t \rangle$. Note that $\partial_{\langle \mu_t \rangle} = \frac{\partial}{\partial \langle \mu_t \rangle}, \partial^2_{\langle \mu_t \rangle} = \frac{\partial^2}{\partial \langle \mu_t \rangle^2}, \partial^3_{\langle \mu_t \rangle} = \frac{\partial^3}{\partial \langle \mu_t \rangle^3}$ and $\partial^4_{\langle \mu_t \rangle} = \frac{\partial^4}{\partial \langle \mu_t \rangle^4}$ are throughout the paper. For proof of equations (13)-(14), see the paper Patil and Sharma (2014).

Here, we state two special cases of evolution equations (13)-(14): (i) for the terms $\gamma_1 = 1$ and $\gamma_2 = 0$, equations (13)-(14) reduce to the *classical exact evolution equations*, see Eq. (4.159) of Jazwinski (1970, p. 137) (ii) the terms $\gamma_1 = 2N$ and $\gamma_2 = N\tau_{cor}$ lead to the exact evolution equations of the OU process-driven SDE.

3. POWER SYSTEM DYNAMICS AND NUMERICAL EXPERIMENTATIONS

In deterministic setting, the swing equation of a single machine-infinite bus (SMIB) system is given by the following second-order non-linear differential equation Kundur (1994):

$$M\ddot{\delta} + D\dot{\delta} + \frac{VE'_a}{X}\sin\delta = P_m. \tag{15}$$

Note that the terms $V, M, D, E'_a, \delta, X, P_m$ denotes the voltage magnitude of the infinite bus, combined inertia constant and the damping coefficient of the generator and turbine, the transient emf, the rotor angle of the generator, the total reactance, the input mechanical power respectively. After accounting random power fluctuations in the swing equation of the single machine-infinite bus system. we can get,

$$M\ddot{\delta} + D\dot{\delta} + \frac{VE'_a}{X}\sin\delta = P_m + \eta_t, \tag{16}$$

where η_t is the OU process. After accomplishing the phase space formulation, we have $\mu_1 = \delta_t, \mu_2 = \dot{\delta}_t$,

$$\dot{\mu}_1 = \mu_2 \tag{17}$$

$$\dot{\mu}_2 = -\frac{D}{M}\mu_2 - \frac{VE'_a}{M\ X}\sin\mu_1 + \frac{P_m}{M} + \frac{\eta_t}{M}. \tag{18}$$

Consider the contribution to the evolution of the phase variable $\delta_t = \mu_1$ coming from the damping term is considerably greater than the inertial term (Hänggi and Jung, 1995, p. 241), then the term $-\frac{D}{M}\mu_2 - \frac{VE'_a}{M\ X}\sin\mu_1 + \frac{P_m}{M} + \frac{\eta_t}{M}$ vanishes, i.e. If we model a system without physical or virtual inertia, power system is best described as a Kuramoto oscillator (Dörfler and Bullo 2010; Tönjes 2010; Simpson-Porco 2012; Schäfer et al. 2017; Supplemental material 2017) with the equation of motion,

$$\dot{\delta}_t = -\frac{VE'_a}{D\ X}\sin\delta_t + \frac{P_m}{D} + \frac{1}{D}\eta_t, \tag{19}$$

Thus, the above equation can be recast by utilizing a more convenient notation by choosing the state variable notation μ_t for the rotor angle δ_t. Thus,

$$\dot{\mu}_t = A(\mu_t, t) + B(\mu_t, t)\eta_t, \tag{20}$$

where

$$A(\mu_t, t) = -\frac{VE'_a}{D\ X}\sin\mu_t + \frac{P_m}{D}, \quad B(\mu_t, t) = B(t) = \frac{1}{D}.$$

Note that for a non-linear time varying system. The input argument of the right-hand side of equation (20) involves the time variable t as well. From equation (20) and equation (6) of the paper, we get

$$d\mu_t = \left(-\frac{VE'_a}{D\ X}\sin\mu_t\left(1 + \frac{\gamma_2}{2D^2}\right) + \frac{P_m}{D}\right)dt + \frac{\sqrt{\gamma_1}}{D}\sqrt{1 - \frac{2\gamma_2}{\gamma_1}\frac{VE'_a}{D\ X}\cos\mu_t}\,dW_t, \tag{21}$$

After combining equations (13)-(14) with equation (20). As a result of these, we have the following system of prediction equations for the power system driven by the OU process,

$$d\langle\mu_t\rangle = \left(\frac{VE'_a}{D\ X}\sin\langle\mu_t\rangle\left(1 + \frac{N\tau_{cor}}{2D^2}\right)\left(\frac{P_t}{2} - 1\right) + \frac{P_m}{D}\right)dt, \tag{22}$$

$$dP_t = \left(P_t\frac{VE'_a}{D\ X}\cos\langle\mu_t\rangle\left(-2 + \frac{N\tau_{cor}}{D^2}\right) - \frac{5}{2}N\tau_{cor}\frac{VE'_a}{D^3\ X}\cos\langle\mu_t\rangle + \frac{2N}{D^2}\right)dt. \tag{23}$$

Note that equation (20) and equation (21) are stochastically equivalent and suggest qualitative characteristics of the rotor angle of the machine. For numerical experimentations of equations (20)-(21), the initial conditions and system parameters for power system dynamics are the following Hirpara and Sharma (2015):

$$V = 1.0\,pu, E'_a = 1.2\,pu, D = 5\,pu/rad/\text{sec},$$
$$X'_d = 0.15\,pu, X_l = 0.1\,pu, X = 0.25\,pu, \tau_{cor} = 0.1$$
$$P_m = 1.0\,pu,\ \langle \mu(0) \rangle = 1\,\text{rad},\ P(0) = 0\,\text{rad}^2, N = 0.05$$

In first case, we consider that the correlation time τ_{cor} is smaller than second case. Note that the solid line trajectory (-) of figure 1 demonstrates the numerical simulation of equation (20) and the dotted line (...) of figure 1 denotes the numerical simulation of equation (21) of the paper. The terms $\gamma_1 = 2N$ and $\gamma_2 = N\tau_{cor}$ corresponding to correlation time $\tau_{cor} = 0.1$ become $\gamma_1 = 0.1$ and $\gamma_2 = 0.005$. Figure 1 suggested that qualitative characteristics of both SDEs conforms each other for a smaller correlation time. The trajectories of both SDEs are coinciding with each other. Thus, both SDEs are stochastically equivalent for the OU process with smaller correlation time. Figures 2 and 3 demonstrate numerical simulation of equation (22), mean trajectory and equation (23), variance trajectory with smaller correlation time.

For second case, we simulated equations (20) and (21) by choosing larger correlation time $\tau_{cor} = 1$ other parameters are same as first case. The terms $\gamma_1 = 0.1$ and $\gamma_2 = 0.05$ for larger correlation time.

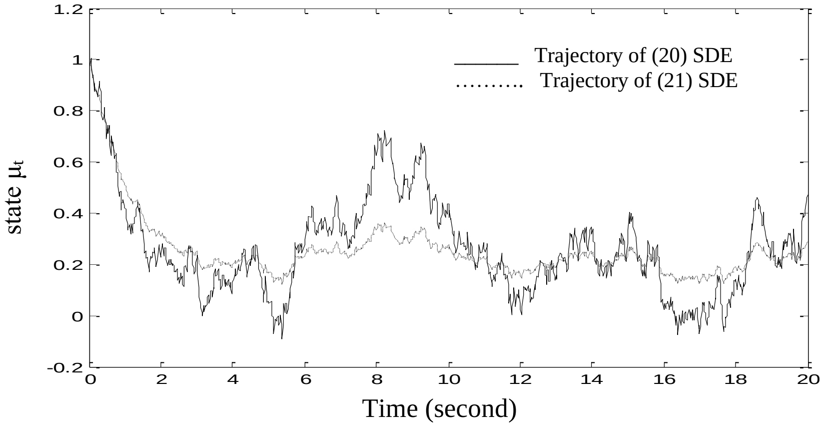

Figure 1: a comparison between trajectories

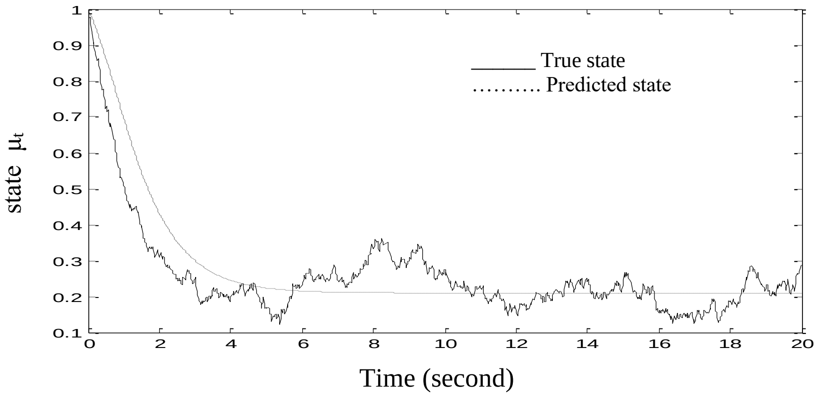

Figure 2: a comparison between trajectories

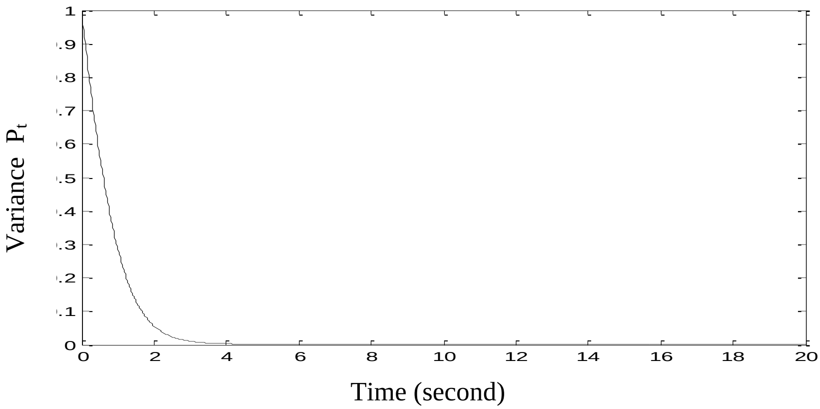

Figure 3: a variance trajectory

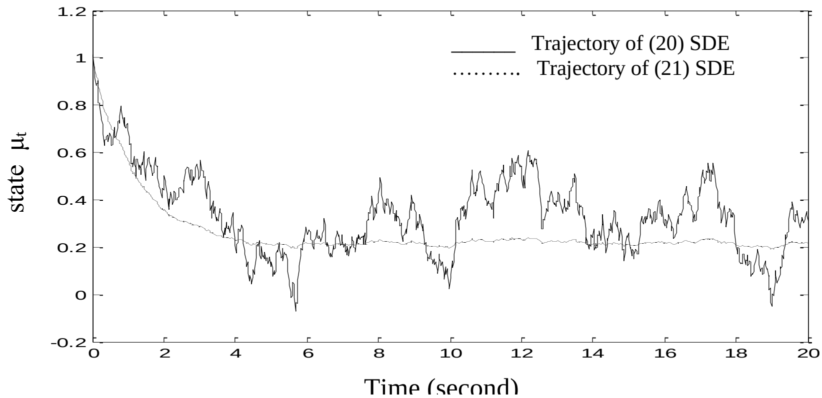

Figure 4: a comparison between trajectories

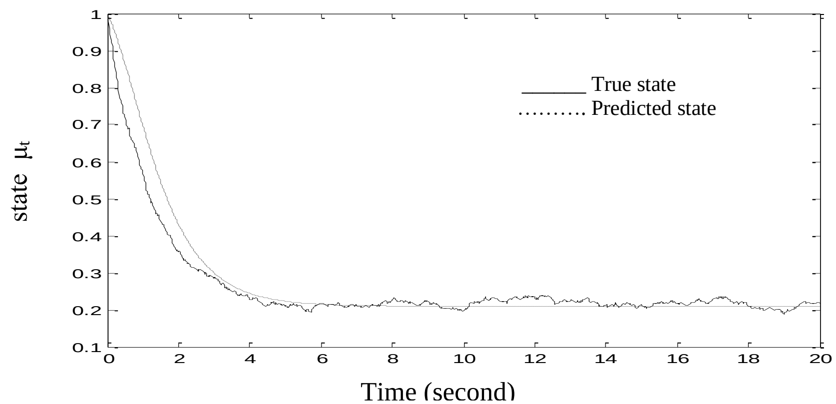

Figure 5: a comparison between trajectories

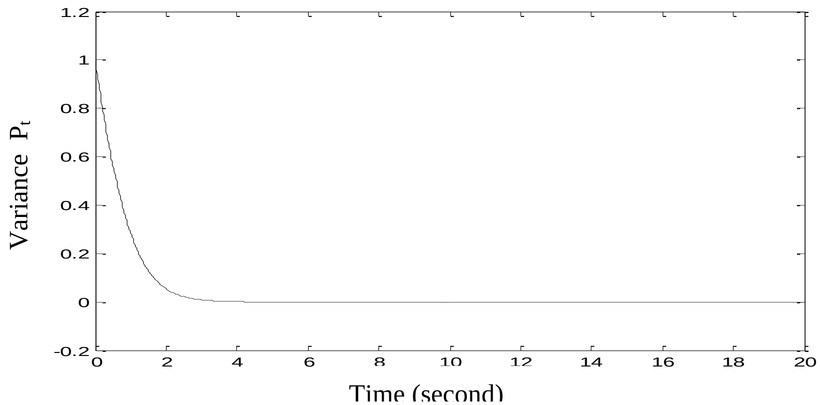

Figure 6: a variance trajectory

The graphical notations are same as first case. Figure 4 reveals that the difference between the state trajectories resulting from the SDEs stated in equation (20) and equation (21) are larger. Figures 5 and 6 demonstrate numerical simulation of equation (22), mean trajectory and equation (23), variance trajectory with larger correlation time. Thus, the numerical simulation suggests that both SDEs, equation (20) and equation (21), are stochastically equivalent for the input coloured noise process with ‘smaller correlation time’.

4. CONCLUSION

The main achievement of this paper is to develope prediction algorithm for the OU process-driven SDE. It is shown that for a smaller correlation time, the trajectories of SDEs stated in equation (20) and equation (21) are coinciding to each other. Thus, both SDEs are stochastically equivalent to each other and for larger correlation time, difference between both trajectories are larger. In this paper we have also developed conditional mean and variance equation for power system dynamics with stochastic equivalence approach. For future direction, in this paper we have considered no observation or value less observation. For available observation, we were required to develop filtering algorithm for the OU process driven power system dynamics.

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An extensive extension of exponentiated exponential distribution using alpha power transformation – statistical properties and applications in engineering science

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Abstract

In this article, an extension of exponentiated exponential distribution is familiarized by adding an extra parameter to the parent distribution using alpha power technique. The new distribution obtained is referred to as Alpha Power Exponentiated Exponential Distribution. Various statistical properties of the proposed distribution like mean, variance, central and non-central moments, reliability functions and entropies have been derived. Two real life data sets have been applied to check the flexibility of the proposed model. The new density model introduced provides the better fit when compared with other related statistical models.

Mathematics Subject Classification 2000: 2010: 60E05, 62E10, 62N05.
Keywords: Exponentiated Exponential Distribution, Alpha-Power Exponentiated Exponential Distribution, Survival Function, Hazard Function, Moment Generating Function, Characteristic Function.

1. INTRODUCTION

The advancement of conventional distributions has become a usual exercise in statistical theory, since last few decades. Designing a new distribution from a classical one by adding an additional parameter using different methods has got wide scope in recent years. The main purpose of such developments is to make the classical distributions more flexible for complex data sets. Recently, Mahdavi A, Kundu D, introduced a new technique called Alpha Power Transformation (APT), where an additional parameter is introduced in continuous probability distributions. The Alpha Power Transformation is defined as:

Let $f(x)$ be the probability density function (pdf) of any continuous random variable 'X', then the pdf of Alpha Power Transformation is given by

$$f_{APT}(x) = \begin{cases} \frac{\log\alpha}{\alpha-1}\alpha^{F(x)}f(x), & if\ \alpha > 0,\ \alpha \neq 1 \\ f(x) & ,if\ \alpha = 1 \end{cases} \tag{1.1}$$

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And the corresponding CDF of APT is given as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} & ,if\ \alpha > 0, \alpha \neq 1 \\ F(x) & ,if\ \alpha = 1 \end{cases} \tag{1.2}$$

The APT of survival function $S_{APT}(x)$ is given by

$$S_{APT}(x) = \begin{cases} \frac{\alpha}{\alpha-1}\left(1-\alpha^{F(x)-1}\right) & ,if\ \alpha \neq 1 \\ 1-F(x) & ,if\alpha = 1 \end{cases} \tag{1.3}$$

The APT of Hazard rate function $H_{APT}(x)$ is given by

$$H_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)-1}}{1-\alpha^{F(x)-1}} log\alpha & , \quad if\ \alpha \neq 1 \\ \frac{f(x)}{s(x)}, & if\ \alpha = 1 \end{cases} \tag{1.4}$$

The exponential distribution is the probability distribution of the time between events in a Poisson point process, which is described in probability theory and statistics as a process in which events occur continuously and independently at a constant average rate. It's a good example of how the gamma distribution works. It is the geometric distribution's continuous equivalent, and it has the essential virtue of being memory less. It is employed in a variety of additional applications in addition to the analysis of Poisson point processes. The exponential distribution is not the same as the exponential families of distributions, which is a vast class of probability distributions that contains the exponential distribution as one of its members, as well as the normal, binomial, gamma, Poisson, and many more.

The exponentiated exponential distribution formulated by Debasis Kunduis the two parameter right skewed unimodal distribution. It is the particular case of Gompertz-Verhulst distribution function. Therefore, a random variable ‘X’ is said to follow exponentiated exponential distribution if it follows the pdf

$$f(x;\beta,\lambda) = \beta\lambda\left(1-e^{-\lambda x}\right)^{\beta-1}e^{-\lambda x} \tag{1.5}$$

And the corresponding distribution function is given by

$$F(x;\beta,\lambda) = \left(1-e^{-\lambda x}\right)^{\beta}; x > 0 \tag{1.6}$$

2. ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

In this section, we introduce an extension of exponentiated exponential distribution by adding an extra parameter to the original distribution using the technique of alpha power transformation. Then the resultant distribution is known as Alpha Power Exponentiated Exponential Distribution (APEED).

Suppose 'X' be a random variable following Alpha Power Exponentiated Exponential distribution, then the PDF of 'X' is given as

$$f_{APEED}(x;\alpha,\lambda,\beta) = \frac{\beta\lambda}{\alpha-1} log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}}, \quad \alpha,\lambda,\beta > 0 \qquad (2.1)$$

And the corresponding CDF is given by

$$F_{APEED}(x) = \frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}, \quad \alpha,\lambda,\beta > 0 \qquad (2.2)$$

ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

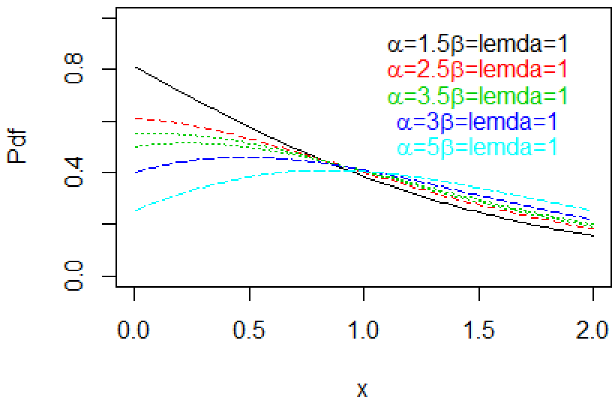


Fig 1. *Shows the graph of pdf of APEE distribution for various values of α*

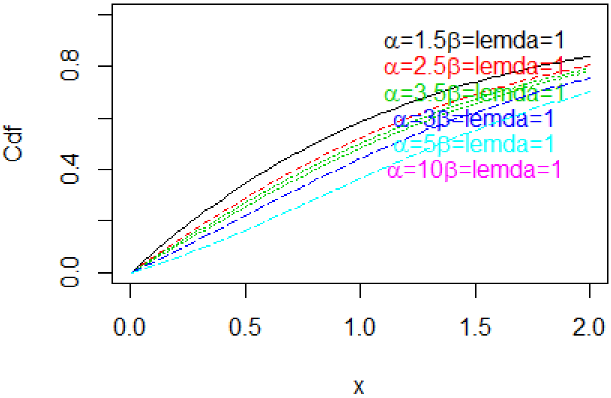


Fig.2 *Shows the graph of cdf of APEE Distribution*

3. RELIABILITY ANALYSIS

In this section, we will obtain the survival function, hazard rate, reverse hazard rate, cumulative hazard rate and mills ratio of APEE distribution.

3.1. Survival function of APEE distribution

The survival function of APEED is defined as under

$$S_{APEED}(x) = 1 - F_{APEED}(x)$$

Substituting the value of $F_{APPED}(x)$ from equation (1.6) we get

$$S_{APEED}(x) = 1 - \frac{\alpha^{(1-e^{-\lambda x})^{\beta}} - 1}{\alpha - 1}$$

$$S_{APEED}(x) = \frac{\alpha - \alpha^{(1-e^{-\lambda x})^{\beta}}}{\alpha - 1} \tag{3.1}$$

3.2. Hazard Rate Function of APEE distribution

The hazard rate function of APEED is defined as under

$$\mathrm{H}_{APEED}\ (x) = \frac{f(x)}{S(x)} = \frac{\lambda\beta log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}}{\alpha-\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}} \tag{3.2}$$

3.3. Reverse Hazard Rate Function of APEE distribution

The reverse hazard rate function of APEED is defined as under

$$\mathrm{RHR}_{APEED}(x) = \frac{f(x)}{F(X)} = \frac{\beta\lambda log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}}{\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}-1} \tag{3.3}$$

3.4. Cumulative Hazard Rate of APEE distribution

The cumulative hazard rate of APEED is defined as under

$$\mathrm{CHR}_{APEED}(x) = -log[S(x)]$$

$$= -log\left[\frac{\alpha-\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}}{\alpha-1}\right]$$

$$=\left(1-e^{-\lambda x}\right)^{\beta} log\alpha \tag{3.4}$$

3.5. Mills Ratio of APEE distribution

The mills ratio of APEED is defined as under

$$\mathrm{MR}_{APEED}(x) = \frac{F(x)}{S(x)} = \frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}} \tag{3.5}$$

Plots below show the graphs of the survival function, hazard rate, reverse hazard rate, cumulative hazard rate of APEED

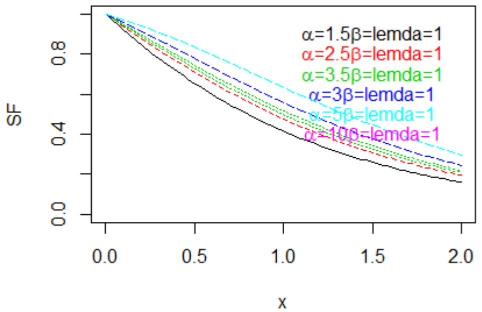


Fig 3. *Shows SF of APEED*

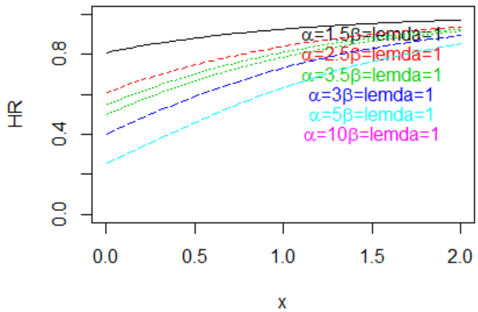
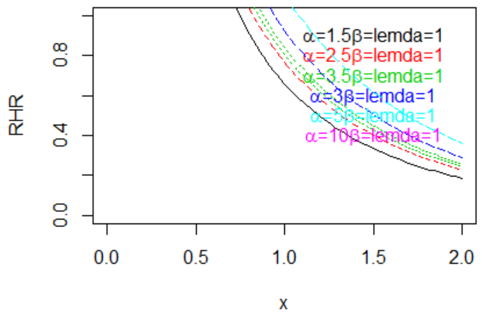


Fig 4. *Shows HR of APEED*



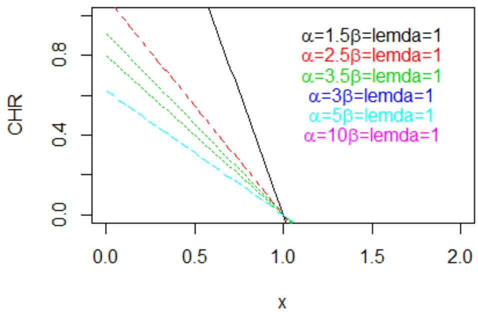


Fig 5. *Shows CHR of APEED*

Fig 6. *Shows RHR of APEED*

4. STRUCTURAL PROPERTIES OF APEEED

In this section, structural properties of APEE distribution are derived. These structural properties include moments about mean, moments about origin, variance, standard deviation, coefficient of variation and index of dispersion.

THEOREM: Let 'x' be a random variable following APEED, then r^{th} moment denoted by $\mu_r{}'$ is given by

$$\mu_r{}' = \int_0^\infty x^r f(x)dx$$

$$= \int_0^\infty x^r f_{APEED}(x)dx$$

$$= \int_0^\infty x^r \frac{\beta\lambda}{\alpha-1} log\alpha \, (1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta} dx$$

PROOF:

Take $(1-e^{-\lambda x})^\beta = t$, then solving the above integral we get

$$\mu_r{}' = \int_0^1 \left[-\frac{1}{\lambda} log(1-t^{\frac{1}{\beta}})\right]^r \frac{log\,\alpha}{\alpha-1} \alpha^t dt$$

$$= \left(\frac{log\,\alpha}{\alpha-1}\right)\frac{(-1)^r}{\lambda^r}\int_0^1 \alpha^t log(1-t^{\frac{1}{\beta}})^r \, dt \mu_r{}' = \frac{(-1)^r}{\lambda^r}\left[r\,log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] \quad (4.1)$$

Putting $r = 1,2,3 \, and \, 4$, we get the first four moments about origin as under

$$\mu_1{}' = \frac{-1}{\lambda}\left[log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] = \frac{1}{\lambda}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right] \quad (4.2)$$

$$\mu_2{}' = \frac{1}{\lambda^2}\left[2log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] \quad (4.3)$$

$$\mu_3{}' = \frac{(-1)^3}{\lambda^3}\left[3\,log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] = \frac{1}{\lambda^3}\left[1 - 3log\frac{1}{(1-e^{-\lambda x})^\beta}\right] \quad (4.4)$$

$$\mu_4{}' = \frac{1}{\lambda^4}\left[4\,log\,\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] \quad (4.5)$$

Therefore, Mean $\mu_1{}' = \frac{1}{\lambda}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]$

The moments about mean are derived by using the relationship between moments about origin and moments about mean

$$\mu_2 = \mu_2{}' - (\mu_1{}')^2$$

$$= \frac{1}{\lambda^2}\left[2log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right]\left\{\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]\right\}^{\mathbf{2}}$$

$$= \frac{1}{\lambda^2}\left[4\,log\;\frac{1}{(1-e^{-\lambda x})^\beta} - \left[log\frac{1}{(1-e^{-\lambda x})^\beta}\right]^2 - 2\right] \quad (4.6)$$

$$\mu_3 = \mu_3{}' - 3\mu_2{}'\mu_1{}' + 2\mu_1{}'^2$$

$$= \frac{(-1)^3}{\lambda^3}\left[3\,log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] - 3\left\{\frac{1}{\lambda^2}\left[2log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right]\right\}\left\{\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]\right\} + 2\left\{\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]\right\}^2$$

$$= \frac{(-1)}{\lambda^3}\left[\left(3\,log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right) + 6\left(log\frac{1}{(1-e^{-\lambda x})^\beta}\right)^2 - 9log\frac{1}{(1-e^{-\lambda x})^\beta} - 3\right] + \frac{2}{\lambda^2}\left[log\frac{1}{(1-e^{-\lambda x})^\beta}\right]^2 \quad (4.7)$$

$$\mu_4 = \mu_4{}' - 4\mu_3{}'\mu_1{}' + 6\mu_2{}'\mu_1{}'^2 - 3\mu_1{}'^4$$

$$= \frac{1}{\lambda^4}\left[4\,log\;\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right] - 4\left\{\frac{(-1)^3}{\lambda^3}\left[3\,log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right]\right\}\left\{\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]\right\} + 6\left\{\frac{1}{\lambda^2}\left[2log\frac{1}{(1-e^{-\lambda x})^\beta} - 1\right]\right\}\left\{\frac{\mathbf{1}}{\boldsymbol{\lambda}}\left[1 - log\frac{1}{(1-e^{-\lambda x})^\beta}\right]\right\}$$

$$= \frac{1}{\lambda^4}\left[4log\;\frac{1}{(1-e^{-\lambda x})^\beta} - 12\left(log\;\frac{1}{(1-e^{-\lambda x})^\beta}\right)^2 + 16log\;\frac{1}{(1-e^{-\lambda x})^\beta} - 5\right] - \frac{6}{\lambda^3}\left[2\left(log\;\frac{1}{(1-e^{-\lambda x})^\beta}\right)^2 - 3log\;\frac{1}{(1-e^{-\lambda x})^\beta} + 1\right] \quad (4.8)$$

The standard deviation, coefficient of variation and index of dispersion are obtained as under

Standard deviation

$$\sigma = \sqrt{variance} = \frac{1}{\lambda}\sqrt{\left[4\,log\,\frac{1}{(1-e^{-\lambda x})^{\beta}} - \left[log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}\right]^2 - 2\right]} \qquad (4.9)$$

$$\mathbf{C.V} = \frac{\sigma}{\mu_1{'}} = \frac{\sqrt{\left[4\,log\,\frac{1}{(1-e^{-\lambda x})^{\beta}} - \left[log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}\right]^2 - 2\right]}}{\left[1-log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}\right]} \qquad (4.10)$$

$$\textbf{Index of dispersion } \boldsymbol{\gamma} = \frac{\boldsymbol{\sigma^2}}{\boldsymbol{\mu_1{'}}} = \frac{\frac{1}{\lambda}\left[4\,log\,\frac{1}{(1-e^{-\lambda x})^{\beta}} - \left[log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}\right]^2 - 2\right]}{\left[1-log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}\right]} \qquad (4.11)$$

5. MOMENT GENERATING FUNCTION AND CHARACTERISTIC FUNCTION OF ALPHA POWER EXPONENTIATED EXPONENTIAL DISTRIBUTION

In this section, moment generating function and characteristic function of APEED are derived. Moment Generating Function is generally defined as

$$M_x(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx}\, \frac{\beta\lambda}{\alpha-1} log\alpha(1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} dx$$

Using Taylors Expansion, we get

$$M_x(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r x^r}{r!} \frac{\beta\lambda}{\alpha-1} log\alpha(1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} x^r \frac{\beta\lambda}{\alpha-1} log\alpha(1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}} dx$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{(-1)^r}{\lambda^r}\left[r\, log \frac{1}{(1-e^{-\lambda x})^{\beta}} - 1\right]\right] \quad (5.1)$$

The characteristics function of APEED is generally defined as

$$\emptyset_x(t) = \int_0^{\infty} e^{itx} f(x)\, dx$$

Using Taylor expansion, we get

$$\emptyset_x(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{(it)^r x^r}{r!} f(x) dx$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x)\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r \frac{\beta\lambda}{\alpha-1} log\alpha\left(1-e^{-\lambda x}\right)^{\beta-1} e^{-\lambda x} \alpha^{\left(1-e^{-\lambda x}\right)^{\beta}} dx =$$

$$\sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[\left[\frac{(-1)^r}{\lambda^r}\left[rlog \frac{1}{(1-e^{-\lambda x})^{\beta}} - 1\right]\right]\right] \quad (5.2)$$

6. ENTROPIES

Shanon Entropy:

THEOREM: Let X be a random variable following APEED then Shanon entropy is defined as

$$H(x;\alpha,\lambda,\beta) = E\left(-logf(x;\alpha,\lambda,\beta)\right)$$

PROOF:

$$H(x;\alpha,\lambda,\beta) = E\left[-log\left(\frac{\beta\lambda}{\alpha-1} log\alpha(1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^{\beta}}\right)\right]$$

$$= -E\left(log\left(\frac{\beta\lambda}{\alpha-1}\right)\right) + (\beta-1)E\left(log\left(1-e^{-\lambda x}\right)\right) + log\alpha E\left(1-e^{-\lambda x}\right)^{\beta} -$$

$$E[log(log\alpha)] - \lambda E(x) \quad (6.1)$$

Now

$$E\left(1-e^{-\lambda x}\right)^{\beta}=\int_0^{\infty}\left(1-e^{-\lambda x}\right)^{\beta}f(x)\,dx$$

$$=\int_{\mathbf{0}}^{\infty}\left(1-e^{-\lambda x}\right)^{\beta}\frac{\beta\lambda}{\alpha-1}log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}}dx$$

Taking $\left(1-e^{-\lambda x}\right)^{\beta}=t$, then solving the above integral we get

$$E\left(1-e^{-\lambda x}\right)^{\beta}=\left[\frac{\alpha}{\alpha-1}-\frac{\alpha}{(\alpha-1)log\alpha}+\frac{1}{(\alpha-1)log\alpha}\right] \tag{6.2}$$

and

$$E\left(1-e^{-\lambda x}\right)=\int_0^{\infty}\left(1-e^{-\lambda x}\right)\,f(x)dx$$

$$=\int_{\mathbf{0}}^{\infty}\left(1-e^{-\lambda x}\right)\frac{\beta\lambda}{\alpha-1}log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}}dx$$

Making the substitution of $\left(1-e^{-\lambda x}\right)^{\beta}=t$, then solving the above integral we get

$$E\left(1-e^{-\lambda x}\right)=-\frac{1}{\beta} \tag{6.3}$$

Substituting the values of equation (6.2) and (6.3) in equation (6.1) we have

$$H(x;\alpha,\lambda,\beta)=-\left(log\left(\frac{\beta\lambda}{\alpha-1}\right)\right)+(\beta-1)\left(-\frac{1}{\beta}\right)+log\alpha\left[\frac{\alpha}{\alpha-1}-\frac{\alpha}{(\alpha-1)log\alpha}+\frac{1}{(\alpha-1)log\alpha}\right]-log(log\alpha)-\lambda\left\{\frac{\mathbf{-1}}{\boldsymbol{\lambda}}\left[log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}-1\right]\right\}$$

$$=log\frac{1}{\left(1-e^{-\lambda x}\right)^{\beta}}+\frac{1}{\alpha-1}\{\alpha(log\alpha-1)+1\}-log\left(\frac{\beta\lambda}{\alpha-1}\right)-log(log\alpha)+\frac{1}{\beta}-2 \tag{6.4}$$

RENYI ENTROPY

$$I_\delta(x) = (1-\delta)^{-1} log \int_{-\infty}^{\infty} \{f_{APEED}(x;\alpha,\lambda,\beta)\}^\delta dx \tag{6.5}$$

Substituting the value of $f_{APEED}(x;\alpha,\lambda,\beta)$ from equation (2.1) in equation (6.5) we have

$$I_\delta(x) = (1-\delta)^{-1} log \int_0^\infty \left\{\frac{\beta\lambda}{\alpha-1} log\alpha (1-e^{-\lambda x})^{\beta-1} e^{-\lambda x} \alpha^{(1-e^{-\lambda x})^\beta}\right\}^\delta dx$$

Putting $(1-e^{-\lambda x})^\beta = t$ in the above equation, we get

$$I_\delta(x) = (1-\delta)^{-1} log \left[\frac{log\alpha}{\alpha-1}\right]^\delta \int_0^1 \alpha^{\delta t} dt$$

Solving the above integral we get

$$I_\delta(x) = (1-\delta)^{-1} \left\{log\left[\frac{log\alpha}{\alpha-1}\right]^\delta - log\left[\frac{\alpha^\delta-1}{\delta log\alpha}\right]\right\} \tag{6.6}$$

7. ORDER STATISTICS

Suppose $X_1, X_2, \ldots \ldots .. X_n$ be n random samples of size n from APEE distribution with pdf (2.1) and cdf (2.2), then pdf of k^{th} order statistics is given by

$$f_{x(k)}x = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x) \tag{7.1}$$

Substituting the values from equation (2.1) and (2.2) we get the pdf of k^{th} order statistics as

$$f_{x(k)}x = \left[\frac{n!}{(k-1)!(n-k)!}\left[\frac{\alpha^{(1-e^{-\lambda x})^\beta}-1}{\alpha-1}\right]^{k-1}\left[1-\frac{\alpha^{(1-e^{-\lambda x})^\beta}-1}{\alpha-1}\right]^{n-k}\frac{\beta\lambda}{\alpha-1}log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^\beta}\right] \tag{7.2}$$

The pdf of Ist order statistics is derived by putting k=1in the above equation

$$f_{x(1)}x = \frac{n!}{(1-1)!\,(n-1)!}\left[\frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}\right]^{1-1}\left[1-\frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}\right]^{n-1}\frac{\beta\lambda}{\alpha-1}log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}}$$

Solving the above equation we get

$$f_{x(1)}x = \frac{n\lambda\beta log\alpha}{(\alpha-1)^{n}}\left[\alpha-\alpha^{(1-e^{-\lambda x})^{\beta}}\right]\left[\alpha^{(1-e^{-\lambda x})^{\beta}}e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}\right] \qquad (7.3)$$

And the pdf of nth order statistics is derived by putting k = n in equation (7.2), we get

$$f_{x(n)}x = \left[\frac{n!}{(n-1)!(n-n)!}\left[\frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}\right]^{n-1}\left[1-\frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}\right]^{n-n}\frac{\beta\lambda}{\alpha-1}\log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}}\right]$$

Solving above equation we get

$$f_{x(n)}x = n\left[\frac{\alpha^{(1-e^{-\lambda x})^{\beta}}-1}{\alpha-1}\right]^{n-1}\frac{\beta\lambda}{\alpha-1}\log\alpha(1-e^{-\lambda x})^{\beta-1}e^{-\lambda x}\alpha^{(1-e^{-\lambda x})^{\beta}} \qquad (7.4)$$

Application in Real Data Analysis:

In this section, we'll look at how to apply the proposed distribution to a real-world situation. As a result, two genuine data sets are employed, and analyses are carried out using R software (1.4.1717).

The first set of data concerns the breaking stress of carbon fibres with a length of 50 mm (GPa). Nichols and Padgett (2006), Cordeiro and Lemonte (2011), and Al-Aqtashet al. had already used the data (2014). The following is the information:

0.39 0.85 1.08 1.25 1.47 1.57 1.61 1.61 1.69 1.80 1.84 1.87 1.89 2.03 2.03 2.05 2.12 2.87 2.88 2.93 2.95 2.96 2.97 3.09 3.11 3.11 3.15 3.15 3.19 3.22 3.22 3.27 3.28 3.31 2.35 2.41 2.43 2.48 2.50 2.53 2.55 2.55 2.56 2.59 2.67 2.73 2.74 2.79 2.81 2.82 2.85 3.31 3.33 3.39 3.39 3.56 3.60 3.65 3.68 3.70 3.75 4.20 4.38 4.42 4.70 4.90

Table 1 presents a summary of the data.

DATA 1	Min	Mean	Median	variance	1st Qu.	3rd Qu.	max
	0.390	2.760	2.835	0.7947	2.178	3.277	4.900

Table 2. Performance of the distributions:

Distribution	AIC	BIC	CAIC	HQIC
IE	274.057	276.247	274.247	277.787
GIE	203.240	207.619	203.431	204.971
EE	194.745	199.124	194.935	196.475
L	246.768	248.958	246.959	250.499
$APEE_{pro}$	194.320	198.699	194.510	196.050

Table 3. The estimation of parameters for the first data set:

Distribution	Estimated Parameters
IE	2.2992439
GIE	13.2850451 7.6013295
EE	9.199203 1.007550
L	0.59025384
$APEE_{pro}$	11.3002074 1.3687862

The strength of 1.5 cm glass fibres is the subject of the second data set. Smith and Naylor (1987) and Bourguignon et al. (1987) utilised data originally gathered by employees at the UK National Physical Laboratory (2014). The following is the information:

1.42 1.48 1.48 1.49 1.49 1.50 1.50 1.51 1.52 1.53 1.54 1.55 1.55 1.58 1.59 1.60 1.61

0.55 0.74 0.77 0.81 0.84 0.93 1.04 1.11 1.13 1.24 1.25 1.27 1.28 1.29 1.30 1.36 1.39

1.76 1.76 1.77 1.78 1.81 1.82 1.84 1.84 1.89 2.00 2.01 2.24 1.61 1.61 1.61 1.62 1.62 1.63 1.64 1.66 1.66 1.66 1.67 1.68 1.68 1.69 1.70 1.70 1.73

Table 4 presents a summary of the data.

DATA 2	Min	Mean	Median	variance	1st Qu.	3rd Qu.	max
	0.550	1.507	1.590	0.1050	1.375	1.685	2.240

Table 5. Performance of the distributions:

Distribution	**AIC**	**BIC**	**CAIC**	**HQIC**
IE	180.878	183.021	181.078	184.564
GIE	64.243	68.529	64.443	65.929
EE	66.767	71.053	66.967	68.453
L	164.557	166.700	164.757	168.243
$APEE_{pro}$	55.323	59.609	55.523	57.009

Table 6. The estimation of parameters for the second data set:

05	**Estimated Parameters**
IE	1.4083964
GIE	163.206558 8.1508240
EE	31.348914 2.611571
L	0.99611636
$APEE_{pro}$	39.0846906 3.5083968

CONCLUSION

In this article, a modified distribution namely Alpha Power Exponentiated Exponential Distribution is derived. Various structural properties including mgf, cf, moments about mean, moments about origin, variance, standard deviation, and index of dispersion are obtained. Apart from the above properties some reliability measures of the said distribution are drawn. These reliability measures contain survival function, hazard rate function, reverse hazard rate, cumulative hazard rate and mills ratio. Moreover Shanon's entropy and Renyi entropy of the new distribution are derived. Finally the newly developed distribution was applied to the two real life data sets which show APEE distribution resulted better than the models compared to it.

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Restructured class of estimators for population mean using an auxiliary variable under simple random sampling scheme

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Abstract

With this article in mind, we have found some results using eigenvalues of graph with sign. It is intriguing to note that these results help us to find the determinant of *Normalized Laplacian* matrix of signed graph and their coefficients of characteristic polynomial using the number of vertices. Also we found bounds for the lowest value of eigenvalue.

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1. INTRODUCTION

The readers should refer to [6] for expression and notations of graph theory and only simple and finite graphs are considered.
A Signed graph $\Gamma = (G(V, E), \nabla)$ is a graph with positive and negative signs in every edge, where G is the underlined graph without signs and ∇ is the function from the collection of edges E to the set having positive and negative signs.
One of the main applications of signed graphs is to represent the relationship among people where we assign a positive sign if the relationship between two individuals is pleasant, otherwise we assign a negative sign. [10] & [5].

The *balanced* signed graph was introduced by F.Harary[7] and he defines that every cycle of a *balanced* signed graph has negative edges in even number if not Γ is said to be *unbalanced*. In [8], Harary and Kebel showed a simple algorithm for balancing of a signed graph.

A graph that has been marked Γ_ν is a signed graph with positive or negative signs assigned to its vertices. The process of assigning signs to the vertices is called marking ν. For $v \in V(\Gamma)$, marked graph Γ_ν is defined as

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$$\nu(v) = \prod_{uv \in E(\Gamma)} \nabla(uv).$$

Switched signed graph $\Gamma_\nu(\Gamma)$ was defined by R.P Abelson and Rosenburg [14] which paved the way for the study of social behavior and mathematical analysis in graph theory.

A signed graph Γ_2 is obtain from a signed graph Γ_1 by reversing the sign of edges of Γ_1 whose end vertices are having opposite sign, and their underlined graphs G_1 and G_2 are isomorphic. The signed graph Γ_1 *switching equivalent* to Γ_2, is represented as $\Gamma_1 \sim \Gamma_2$.

Following is the characterization of *switched signed graphs.*

Proposition 1. [15] Any two signed graphs whose underlying graphs are same are cycle isomorphic if, and only if they are switching equivalent.

In a signed graph, degree of each vertex can be calculated by $d = d^+ + d^-$ so that degree of vertices in a signed graph Γ and their underlined graph is the same.

In *adjacent* matrix $A(\Gamma)$, if two vertices are adjacent then the entry a_{ij} is 1 along with the sign of the edge, otherwise the entry is zero.

In a *Laplacian* matrix $L(\Gamma)$, if vertices v_i and v_j are adjacent then the entry a_{ij} is 1 with the opposite sign of corresponding adjacent edge v_iv_j, otherwise a_{ij} is zero and the diagonal entries a_{ii} being the degree of the vertex . Also $L(\Gamma)$=$S(\Gamma) - A(\Gamma)$, where S is the diagonal matrix.

Here $(\Gamma, -)$ is a signed graph in which each edge is assigned by minus sign and $L(\Gamma, -)$ is the *Laplacian* matrix of $(\Gamma, -)$. Eigenvalues of *Laplacian* matrix of a signed graph are $\lambda_1 \ \geq \ \lambda_2 \ \geq \ \lambda_3 \ .... \geq \lambda_n$.

2. *NORMALIZED LAPLACIAN* MATRIX OF SIGNED GRAPH

F.R.K. Chung [17] introduced the *Normalized Laplacian* matrix. Lower bounds of *Normalized Laplacian* were investigated by Grossman [13] and its properties were stated by Chen et. al. in [11]. Also in [4] Cvetkovic et.al. mentioned deeply about normalized Laplacian and their bounds of eigen values.

The *Normalized Laplacian* matrix $\mathsf{L}(\Gamma)$ of a signed graph Γ with vertices u and v is given by

$$\mathsf{L}_{uv} = \begin{cases} 1, \text{ if } u = v \text{ and } d_u \neq 0 \\ -\nabla(uv)\frac{1}{\sqrt{d_u d_v}}, \text{ if } u \text{ and } v \text{ are adjacent} \\ 0 \text{ otherwise.} \end{cases}$$

Let $0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \ .... \leq \mu_n$ be the eigenvalues of *Normalized Laplacian* matrix of Γ, with n vertices. Also $\mathsf{L}(\Gamma) = S^{-1/2} L(\Gamma) S^{-1/2}$.

In 2003 Yaoping Hou. et. al. [16] established new bounds in the following theorem.

THEOREM 2. [16] Let Γ be a signed graph with n vertices. Then

$$\lambda_1 \leq 2(n-1),$$

equality applies if and only if Γ is switching equivalent to a complete graph with all edges being negative.

Some of the novel results prompted by the above theorem are presented in this article.

THEOREM 3. Let $\Gamma = (G, \nabla)$ be a signed graph. The greatest eigenvalue of *Normalized Laplacian* matrix $\mathsf{L}(\Gamma)$ is 2 if and only if Γ is switching equivalent to a complete graph with all edges being negative.

PROOF. If $\Gamma \sim (K_n, -)$ then $\mu_n(\mathsf{L}(\Gamma)) = \mu_n(S^{-1/2}\ L(\Gamma)\ S^{-1/2})$

$$= \mu_n(S^{-1/2}\ (S(\Gamma) - A(\Gamma))\ S^{-1/2})$$

$$= \mu_n(S^{-1/2}\ (S(\Gamma)) S^{-1/2} - S^{-1/2}(A(\Gamma))\ S^{-1/2})$$

$$=\mu_n(S^{-1/2}\ (S(\Gamma))S^{-1/2}) - \mu_n(S^{-1/2}(A(\Gamma))\ S^{-1/2})$$

$$=\mu_n(S^{-1/2}\ (S(\Gamma))S^{-1/2}) + \mu_n(S^{-1/2}(-A(\Gamma))\ S^{-1/2})$$

$$= 1 + 1$$

$$= 2.$$

If $\mu_n = 2$ *then* $\mu_n(S^{-1/2}(S(\Gamma)S^{-1/2}) = \mu_n(S^{-1/2}(-A(\Gamma))S^{-1/2})$.

Thus, $\mu_n(A(\Gamma)) = \mu_n(-(J-I))$, *where* J *is the all one matrix.*

Hence $\Gamma \sim (K_n, -)$.

THEOREM 4. Let Γ be a signed graph and K_n be the complete graph with n vertices, $\Gamma \sim (K_n, -)$ if and only if $\mu_k = \frac{n-2}{n-1}$, for $k < n$.

PROOF. *If* $\Gamma \sim (K_n, -)$ then $\mu_k(\mathsf{L}(\Gamma)) = \mu_k(S^{-1/2}(L(\Gamma))\ S^{-1/2})$

$$=\mu_k(S^{-1/2}\ (S(\Gamma) - A(\Gamma))\ S^{-1/2})$$

$$=\mu_k(S^{-1/2}\ (S(\Gamma))S^{-1/2} - S^{-1/2}(A(\Gamma))\ S^{-1/2})$$

$$=\mu_k(S^{-1/2}\ (S(\Gamma))S^{-1/2}) - \mu_k(S^{-1/2}(A(\Gamma))\ S^{-1/2})$$

$$=\mu_k(S^{-1/2}\ (S(\Gamma))S^{-1/2}) + \mu_k(S^{-1/2}(-A(\Gamma))\ S^{-1/2})$$

$$= 1 + \frac{-1}{n-1}$$

$$= \frac{n-2}{n-1}.$$

If $\mu_k = \frac{n-2}{n-1}$ *then* $\mu_k(S^{-1/2}\ (S(\Gamma))S^{-1/2}) = \mu_k(S^{-1/2}(-A(\Gamma))S^{-1/2})$.

Thus, $\mu_k(S^{-1/2}\ A(\Gamma)\ S^{-1/2}) = \mu_k\ (S^{-1/2}(-(J-I))\ S^{-1/2})$, *where* J *is the* *all one matrix. Hence* $\Gamma \sim (K_n, -)$.

COROLLARY 5. Let Γ be a signed graph. The greatest eigenvalue of *Normalized Laplacian* matrix $\mathsf{L}(\Gamma)$ is 2 if and only if Γ is switching equivalent to a complete bipartite graph with all edges being negative.

PROPOSITION 6. Let Γ be a graph with sign. If $\Gamma \sim (K_n, -)$ then $\sum_{i=1}^{n} \mu_i = n$.

PROOF.
$$\sum_{i=1}^{n} \mu_i = \mu_1 + \mu_2 + \mu_3 + ... + \mu_n$$

$$= 2 + (n-1)\ \frac{n-2}{n-1}$$

$$= 2 + \text{n - 2}$$

$$= \text{n}.$$

3. DETERMINANT OF *NORMALIZED LAPLACIAN* MATRIX OF SIGNED GRAPH

3.1. Matrix Tree Theorem for a *Laplacian* matrix

If b_c be the number of essential spanning subgraphs which contain c negative cycles, then

$$Det(L(\Gamma)) = \sum_{c=0}^{n} 4^c\ b_c.$$

From the above matrix tree theorem, we determine the determinant of *Normalized Laplacian* matrix of a graph with sign and n number of vertices.

PROPOSITION 7. Let Γ be a signed graph. If $\Gamma \sim (K_n, -)$ then

$$Det(L(\Gamma)) = 2\{\frac{(n-2)}{(n-1)}\}^{(n-1)}\ .$$

PROOF.
$$Det(\mathsf{L}(\Gamma)) = \prod_{i=1}^{n} \mu_i$$

$$= 2 \cdot \{1 - \tfrac{1}{n-1}\} \cdot \{1 - \tfrac{1}{n-1}\} \cdots \{1 - \tfrac{1}{n-1}\}$$

$$= 2 \cdot \tfrac{n-2}{n-1} \cdot \tfrac{n-2}{n-1} \cdot \tfrac{n-2}{n-1} \cdots \tfrac{n-2}{n-1}$$

$$= 2 \cdot \{\tfrac{n-2}{n-1}\}^{(n-1)} .$$

4. CHARACTERISTIC POLYNOMIAL COEFFICIENTS OF A *NORMALIZED LAPLACIAN* MATRIX

In the study of chemical properties of molecules and their bond structures, coefficients of a characteristic polynomial play a vital role. Ivailo M. Mladenov et. al. [12] introduced an algorithm to find the coefficients of characteristic polynomial of adjacent matrix of a graph. Kel'man expanded the latter formula and it is known as Kel'man formula.

Kel'man formula is the method to find the coefficients of a characteristic polynomial of a matrix which is given as follows.

Theorem 8. [1]

Let G be a simple graph. Then the characteristic polynomial coefficients of a *Normalized Laplacian* matrix of the graph are provided by using

$$b_{n-k} = (-1)^{n-k} \sum_{F \in F_k} \gamma(F) \qquad \text{where } k \geq 1 \text{ (for } k = 0, b_n = 0.)$$

F_k denotes the set of forests in G having k components and

$$\gamma(G) = \prod_{i=1}^{k} |F_i|$$

is the product of the orders of the components of the forest F.

Also in [3] Carla Silva Oliveria et. al. have found second and third *Laplacian* coefficients of a characteristic polynomial in 2002. Francesco Belardo and Slobodan K. Simic [9] have found *Laplacian* coefficients of signed graph by the following theorem:

Theorem 9. [9] The *Laplacian* characteristic polynomial of Γ is given by $\psi(\Gamma, x) = x^n + b_1 x^{n-1} + ... + b_{n-1} x + b_n$ for any signed graph Γ, then

$$b_i = (-1)^i \sum_{H \in H_i} w(H)$$

where H_i denotes the set of signed TU- subgraphs of Γ containing i edges.

We now present a simplified way of finding the coefficient of *Normalized Laplacian matrix* using the number of vertices.

PROPOSITION 10. Let $\Gamma = (G, \nabla)$ be a signed graph. If $\Gamma \sim (K_n, -)$, then for a positive integer t,

$$tr(\mathsf{L}^t) = 2^t + \{\frac{(n-2)^t}{(n-1)^{(t-1)}}\}$$

.

PROOF.
$tr(\mathsf{L}) = \sum_{i=1}^{n} \mu_i$

$$= 2 + (n-1)\tfrac{(n-2)}{(n-1)}$$

$$tr(\mathsf{L}^2) = 2^2 + (n-1)\{\tfrac{n-2}{n-1}\}^2.$$

$$tr(\mathsf{L}^3) = 2^3 + (n-1)\{\tfrac{n-2}{n-1}\}^3.$$

Similarly for an integer k,

$$tr(\mathsf{L}^k) = 2^k + (n-1)\{\tfrac{n-2}{n-1}\}^k$$

$$tr(\mathsf{L}^{k+1}) = \textstyle\sum_{i=1}^{n} \mu_i^{k+1}$$

$$= 2^{k+1} + \{\tfrac{n-2}{n-1}\}^{k+1} + ... + \{\tfrac{n-2}{n-1}\}^{k+1}$$

$$= 2^{k+1} + (n-1)\{\tfrac{n-2}{n-1}\}^{k+1}.$$

Hence by induction,

$$tr(\mathsf{L}^t) = 2^t + (n-1)\{\tfrac{n-2}{n-1}\}^t$$

$$tr(\mathsf{L}^t) = 2^t + \{\tfrac{(n-2)^t}{(n-1)^{(t-1)}}\}.$$

Examples:

$$tr(\mathsf{L}) = 2 + (n-1)\{\tfrac{n-2}{n-1}\} = n.$$

$$tr(\mathsf{L}^2) = 2^2 + (n-1)\{\tfrac{n-2}{n-1}\}^2 = \tfrac{n^2}{n-1}.$$

$$tr(\mathsf{L}^3) = 2^3 + (n-1)\{\tfrac{(n-2)^3}{(n-1)^3}\} = \tfrac{n^3+2n^2-4n}{(n-1)^2}.$$

$$tr(\mathsf{L}^4) = 2^4 + (n-1)\{\tfrac{(n-2)^4}{(n-1)^4}\} = \tfrac{n^4+8n^3-24n^2+16n}{(n-1)^3}.$$

Coefficients of characteristic polynomial of a *Normalized Laplacian* matrix of signed graph Γ, a_1, a_2, a_3, a_4 are calculated as follows.

$$a_1 = -tr(\mathsf{L}) = -n.$$

$$a_2 = \tfrac{-1}{2} tr(B_1\mathsf{L}) \quad \text{where } B_1 = \mathsf{L} + a_1 I$$

$$= \tfrac{-1}{2}(tr(\mathsf{L}^2) - ntr(\mathsf{L}))$$

$$= \tfrac{1}{2}\{\tfrac{n^2(n-2)}{(n-1)}\}.$$

$$a_3 = \tfrac{-1}{3} tr(B_2\mathsf{L}) \quad \text{where } B_2 = B_1\mathsf{L} + a_2 I$$

$$= \tfrac{-1}{3} tr(B_1\mathsf{L}^2 + a_2\mathsf{L})$$

$$= \tfrac{-1}{3}(tr(\mathsf{L}^3) + a_1 tr(\mathsf{L}^2) + a_2 tr(\mathsf{L}))$$

$$= \tfrac{-1}{3}\{8 + \tfrac{(n-2)^3}{(n-1)^2} - n(4 + \tfrac{(n-2)^2}{(n-1)}) + \tfrac{n}{2}(\tfrac{n^2(n-2)}{(n-1)})\}$$

$$= \tfrac{-1}{6}(\tfrac{n^5-5n^4+6n^3+4n^2-8n}{(n-1)^2}).$$

$$a_4 = \tfrac{-1}{4} tr(B_3\mathsf{L}) \quad \text{where } B_3 = B_2\mathsf{L} + a_3 I$$

$= \frac{-1}{4} tr(B_2 \mathsf{L}^2 + a_3 \mathsf{L})$

$= \frac{-1}{4} tr((B_1 \mathsf{L} + a_2)\mathsf{L}^2 + a_3 \mathsf{L})$

$= \frac{-1}{4} tr((\mathsf{L} + a_1)\mathsf{L}^3 + a_2 \mathsf{L}^2 + a_3 \mathsf{L})$

$= \frac{-1}{4} (tr(\mathsf{L}^4) + a_1 tr(\mathsf{L}^3) + a_2 tr(\mathsf{L}^2) + a_3 tr(\mathsf{L}))$

$= \frac{-1}{4} (\frac{n^4+8n^3-24n^2+16n}{(n-1)^3} - \frac{n^4+2n^3-4n^2}{(n-1)^2} + \frac{n^5-2n^4}{2(n-1)^2} - \frac{n^6-5n^5+6n^4+4n^3-8n^2}{6(n-1)^2})$

$= \frac{1}{24} (\frac{n^7-9n^6+26n^5-8n^4-96n^3+176n^2-96n}{(n-1)^3}).$

THEOREM 11. Let Γ be any signed graph which is switching equivalent to a complete graph in which each edge is negative and $\psi(\Gamma, x) = x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$ be the *Normalized Laplacian* characteristic polynomial of Γ with $a_0 = 1$. Then,

$$a_\varsigma = \frac{-1}{\varsigma} \sum_{m=0}^{\varsigma-1} a_m tr(\mathsf{L}^{\varsigma-m})$$

where, a_ς is the coefficient of characteristic polynomial and $\varsigma \neq 0$.

PROOF.

Since $a_1 = -tr(\mathsf{L}) = -n$,

$a_2 = \frac{-1}{2} tr(B_1 \mathsf{L})$ *where* $B_1 = \mathsf{L} + a_1 I$

$= \frac{-1}{2} (a_0 tr(\mathsf{L}^2) + a_1 tr(L)).$

Similarly for an integer k,

$$a_k = \frac{-1}{k}(a_0 tr(\mathsf{L}^k) + a_1 tr(\mathsf{L}^{k-1}) + a_2 tr(\mathsf{L}^{k-2}) + ... + a_{k-1} tr(\mathsf{L}))$$

i.e., $a_k = \frac{-1}{k}\sum_{m=0}^{k-1} a_m tr(\mathsf{L}^{k-m})$.

$$a_{k+1} = \frac{-1}{k+1} tr(B_k \mathsf{L}) \ \textit{where} \ B_k = B_{k-1}\mathsf{L} + a_k I$$

$$= \frac{-1}{k+1} tr(B_{k-1}\mathsf{L}^2 + a_k \mathsf{L})$$

$$= \frac{-1}{k+1} tr((\mathsf{L}B_{k-2} + a_{k-1})\mathsf{L}^2 + a_k \mathsf{L})$$

$$= \frac{-1}{k+1} tr(\mathsf{L}^3 B_{k-2} + a_{k-1}\mathsf{L}^2 + a_k \mathsf{L})$$

$$= \frac{-1}{k+1}(tr(\mathsf{L}^3 B_{k-2}) + a_{k-1} tr(\mathsf{L}^2) + a_k tr(\mathsf{L}))$$

$$a_{k+1} = \frac{-1}{k+1}(tr(\mathsf{L}^4 B_{k-3}) + tr(\mathsf{L}^3) a_{k-1} + tr(\mathsf{L}^2) a_k + tr(\mathsf{L}))$$

. . .

$$a_{k+1} = \frac{-1}{k+1}(a_0 tr(\mathsf{L}^{k+1}) + a_1 tr(\mathsf{L}^k) + a_2 tr(\mathsf{L}^{k-1}) + ... + a_k tr(\mathsf{L}))$$

i.e., $a_{k+1} = \frac{-1}{k+1}\sum_{m=0}^{k} a_m tr(\mathsf{L}^{k+1-m})$.

Hence by induction,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m tr(\mathsf{L}^{\varsigma-m}) \qquad \textit{where } \varsigma \neq 0.$$

Corollary 12. For any signed graph Γ and $\Gamma \sim (K_n, -)$, the *Normalized Laplacian* characteristic polynomial of Γ is $\psi(\Gamma, x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$ with $a_0 = 1$ then,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m \{2^{\varsigma-m} + \frac{(n-2)^{\varsigma-m}}{(n-1)^{(\varsigma-(m+1))}}\} \qquad \textit{where}, \varsigma \neq 0.$$

PROOF. *By* Proposition 10,

$$tr(\mathsf{L}^t) = 2^t + \{\frac{(n-2)^t}{(n-1)^{(t-1)}}\}.$$

By Theorem 11,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m tr(\mathsf{L}^{\varsigma-m}) \qquad where, \varsigma \neq 0.$$

i.e.,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m \{2^{\varsigma-m} + \frac{(n-2)^{\varsigma-m}}{(n-1)^{(\varsigma-(m+1))}}\}.$$

COROLLARY 13. any signed graph Γ and $\Gamma \sim (K_n, -)$, the *Normalized Laplacian* characteristic polynomial of Γ is $\psi(\Gamma, x) = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ with $a_0 = 1$ and μ_k be the *Normalized Laplacian* eigenvalue of Γ then,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m (2^{\varsigma-m} + (n-1)\mu_k^{\varsigma-m})$$

where, $\varsigma \neq 0$ and $k < n$.

PROOF. *From* theorem 4, $\mu_k = \frac{(n-2)}{(n-1)}$ hence by corollary 12,

$$a_\varsigma = \frac{-1}{\varsigma}\sum_{m=0}^{\varsigma-1} a_m (2^{\varsigma-m} + (n-1)\mu_k^{\varsigma-m})$$

where, $\varsigma \neq 0$ and $k < n$.

5. BOUNDS OF EIGENVALUES OF *NORMALIZED LAPLACIAN* MATRIX OF SIGNED GRAPH.

As we know, $\mathsf{L} = S^{-1/2}LS^{-1/2}$, where $S^{-1/2}$ is invertible.
The vectors g and g_j are defined as:
If f is the eigen function of L corresponding to eigenvalue μ_k, then $g = U^{1/2}f$,

$g_j = U^{1/2}\ f_j$

$$\mu_k = inf\,\frac{\sum(f(u) - \nabla(u,v)f(v))^2}{\sum_u f^2(u)d_u}$$

where degree of the vertex u is d_u.

If $\Gamma \sim (K_n, -)$ and for a vertex v

$$(1-\mu_k)f(v) = \frac{1}{d_v}\sum_{u\sim v}\nabla(u,v)f(u)$$

where, $u \sim v$ means u and v vertices are adjacent. Let $v_1, v_2, ..., v_{m+1}$ be adjacent vertices sequence, $f(v_1)$ be maximal and $f(v_{m+1}) \leq 0$. Let $y_i = f(v_i)$ and $\beta = 1 - \mu_k$. We get

$$\beta y_1 = \frac{1}{d_{v_1}}\sum_{u\sim v_1}\nabla(u,v_1)f(u) \leq \frac{y_2}{d_{v_1}} + \frac{(deg(v_1)-1)y_1}{d_{v_1}} \leq \frac{y_2}{d} + \frac{(d-1)y_1}{d}$$

.

Assume $y_1 = 1$, so that $y_2 \geq 1 - \mu_k d$.
In view of the fact, v_i is adjacent to v_{i+1} and v_{i-1} for $2 \leq i \leq m$, we get

$$\beta y_i \leq \frac{y_{i-1} + y_{i+1}}{d} + \frac{(d-2)y_1}{d}.$$

This implies

$$y_{i+1} \geq \beta d y_i - y_{i-1} - (d-2).$$

If $\Gamma \sim (K_n, -)$ we observe the following:

(1) $(1-\mu_k) = \frac{1}{(n-1)}$

(2) $f(v_1) = \sum_{u\sim v_1} \nabla(u, v_1) f(u)$

(3) $y_2 \geq 3 - n$

(4) $y_{m+1} \geq 3 - 2n$.

PROPOSITION 14. For $3 \leq r \leq m+1$, $y_r \geq 1 - \mu_k \beta^{r-3} d^{r-2} - \mu_k \beta^{r-2} d^{r-1}$.

PROOF. *We have,*

$$y_2 \geq 1 - \mu_k d \tag{1}$$

$$y_{i+1} \geq \beta d y_i - y_{i-1} - (d-2). \tag{2}$$

Proof is by induction on r.
From (2),

$$y_3 \geq 1 - \mu_k d - \mu k \beta d^2$$

.

Suppose result holds for $r \leq i$, *where* $i \geq 3$.

From (2)

$$y_3 \geq \beta d y_2 - 1 - (d-2)$$

$$y_4 \geq \beta d y_3 - y_2 - (d-2)$$

$$y_5 \geq \beta d y_4 - y_3 - (d-2)$$

. . .

$$y_i \geq \beta d y_{i-1} - y_{i-2} - (d-2)$$

$$y_{i+1} \geq \beta d y_i - y_{i-1} - (d-2).$$

$\therefore\ (y_2+y_3+y_4+...+y_i+y_{i+1}) \geq \beta d(y_2+y_3+y_4+...+y_{i-1}+y_i)-(y_1+y_2+y_3+y_4+...+y_{i-2}+y_{i-1})-(i-1)(d-2)+1-\mu_k d.$

$y_{i+1} \geq \beta d(y_2+y_3+y_4+...+y_{i-1}+y_i)-(2y_2+2y_3+2y_4+...+2y_{i-1}+2y_i)-(i-1)(d-2)+y_i-\mu_k d.$

$$i.e.,\quad y_{i+1} \geq (\beta d-2)(y_2+y_3+y_4+...+y_{i-1}+y_i)+y_i-(i-1)(d-2)-\mu_k d \quad (3)$$

From (2) we have,

$$y_3 \geq 1-\mu_k d-\mu_k \beta d^2$$

$$y_4 \geq 1-\mu_k \beta d^2-\mu_k \beta^2 d^3.$$

In general

$$y_i \geq 1-\mu_k \beta^{i-3} d^{i-2}-\mu_k \beta^{i-2} d^{i-1}$$

so,

$(y_2+y_3+y_4+...+y_{i-1}+y_i) \geq (i-1)-2\mu_k d-2\mu_k\beta d^2-2\mu_k\beta^2 d^3 ...2\mu_k\beta^{i-3}d^{i-2}-\mu_k\beta^{i-2}d^{i-1}.$

Also we have

$$y_i \geq 1-\mu_k \beta^{i-3} d^{i-2}-\mu_k \beta^{i-2} d^{i-1}.$$

From (3) we get $y_{i+1} \geq (\beta d-2)((i-1)-2\mu_k d-2\mu_k\beta d^2-2\mu_k\beta^2 d^3-...-2\mu_k\beta^{i-3}d^{i-2}-\mu_k\beta^{i-2}d^{i-1})+(1-\mu_k\beta^{i-3}d^{i-2}-\mu_k\beta^{i-2}d^{i-1})-\mu_k d-(i-1)(d-2)$

$y_{i+1} \geq 1-(i-4)\mu_k d+2\mu_k\beta d^2+2\mu_k^2 d^3 ...2\mu_k\beta^{i-4}d^{i-3}+\mu_k\beta^{i-3}d^{i-2}-\mu_k\beta^{i-1}d^i-\mu_k\beta^{i-2}d^{i-1}$

.

$$y_{i+1} \geq 1+(i-3)\mu_k d+2\mu_k d+2\mu_k d+...+2\mu_k d+\mu_k d-\mu_k\beta^{i-2}d^{i-1}-\mu_k\beta^{i-1}d^i$$

$$y_{i+1} \geq 1+(i-3)\mu_k d-\mu_k\beta^{i-2}d^i-\mu_k\beta^{i-2}d^{i-1}$$

$$y_{i+1} \geq 1-\mu_k\beta^{i-1}d^i-\mu_k\beta^{i-2}d^{i-1}.$$

As a result, $y_r \geq 1-\mu_k\beta^{r-3}d^{r-2}-\mu_k\beta^{r-2}d^{r-1}.$

Theorem 15. Let Γ be a graph with sign having n vertices and if $\Gamma \sim (K_n,-)$. Then $\mu_k \geq \frac{1}{n}$.

Proof. *From* Proposition 14,

$$0 \geq y_{m+1} \geq 1-\mu_k\beta^{m-2}d^{m-1}-\mu_k\beta^{m-1}d^m$$

$$0 \geq 1 - \mu_k d^{m-1} - \mu_k d^m$$

$$\mu_k \geq \frac{1}{(d+1)d^{m-1}}.$$

The distance between a vertex that maximises f and one that minimises f is at most the graph's diameter 'D'., therefore $m \leq \lceil D/2 \rceil$. Since the diameter is 1,

$$\mu_k \geq \frac{1}{(d+1)}.$$

Hence the result follows.

6. CONCLUSION

Usually the coefficients of the characteristic polynomial of a graph or a signed graph are found using the concept of trees and TU subgraphs. But in this paper, we have given a simple and an elegant proof of finding the *Laplacian* coefficients of the characteristic polynomial of a signed graph using the number of vertices of the graph. We believe that this new approach will pave way for further research in this area.

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Improved strategy for computation of population mean under double stratified sampling framework

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Abstract

The article contains a new technique to estimate the mean of the variate of the interest of the finite population with the help of two auxiliary variates. The technique complies well with the stratified population in which each strata proportion is predicted by taking an initial sample called the first phase sample. When the first phase sample is taken, a second sample is taken from the first sample which is called the second phase sample which is used to estimate the mean of the variate of the interest. In our study, we have considered the population which has two correlated auxiliary variates that pass almost through the origin. In such a situation ratio estimation technique and product estimation technique that provides improved estimates of the mean of the variate of the interest. Our technique considers a ratio-product type exponential estimator of which we have established efficiency theoretically as well as empirically.

Mathematics Subject Classification 2000: 62D05
Keywords: Auxiliary variate, Study variate, Double stratified sampling scheme, Bias, Mean square error.

1. INTRODUCTION

In sample survey, surveyors use the auxiliary variates in order to increase the accuracy in the estimation of population parameters when the associated variates are readily and inexpensively available to the entire population. Many times the associated variates are not readily available, in which case, a larger sample is taken provided the collection of information on auxiliary variate is cheap, easy and economical. In many surveys, gathering information on variate of the interest is quite costly and gaining information on auxiliary variates are cheap, easy and economical or auxiliary variate are easily available. If the variate of the interest and auxiliary variates are highly correlated then the efficiency of the estimators may be increased by using auxiliary variate. But this will benefit only if the attainment in precision that is achieved by using the sample auxiliary variate weigh up the loss in precision that has been incurred because of the reduction in the size of the sample for computing mean of the variate of the interest. Surveyors also use auxiliary variates to estimate the population heterogeneity when there is absence of information about

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heterogeneity. Thus, in order to estimate the population mean of different levels of study, our population study is divided into different levels and sample observations have been prepared from each level using the double sampling method.

This article has drawn inspiration from many of the articles like Chand, 1975 and its chain type estimator for the double sampling, Bahl and Tuteja, 1991 product and ratio type exponential estimators, Upadhyay and Singh, 1999 noteworthily contributed in this area and motivated Kadilar and Cingi, 2003, Kadilar and Cingi, 2005, Shabbir and Gupta, 2005. Singh and Vishwakarma, 2007 evolved the method further. Samiuddin and Hanif, 2007, Singh et. al., 2011, Sanaulla et. al., 2014 and Vishwakarma and Singh, 2015 have used two auxiliary variates for double stratified sampling method. All in all, inspired by their work, a need was felt to propose a new methodology in this area, and the results are discussed in a subsequent section.

2. DOUBLE STRATIFIED RANDOM SAMPLING

Let there be a finite population, U_j (j = 1, 2, 3, …, N), of size N that can be divided into strata. Let the variates that can be measured with each unit be a target variate y and x and z be the associated variates that information is either fully available or if not then it can be measured easily and economically. Let L be the number of strata that can be formed from the population. Let N_h (h = 1, 2, 3, …, L) be size of h^{th}- stratum such that $\sum_a^b N_h = N$ and $W_h = N_h / N$. Let $\left(\bar{y}_h = \sum_{i=1}^{n_h} y_{hi}/n_h,\ \bar{x}_h = \sum_{i=1}^{n_h} x_{hi}/n_h,\ \bar{z}_h = \sum_{i=1}^{n_h} z_{hi}/n_h\right)$ be the unbiased estimators of the variate of the interest and the associated auxiliary variates of the population mean $\left(\bar{Y}_h = \sum_{i=1}^{N_h} y_{hi}/N_h,\ \bar{X}_h = \sum_{i=1}^{N_h} x_{hi}/N_h,\ \bar{Z}_h = \sum_{i=1}^{N_h} z_{hi}/N_h\right)$, based on n_h observations. When information on $\bar{X}_h$ is unknown Double Stratified Sampling method comes to rescue by using first phase sample to note down value on auxiliary variates. The procedures for double stratified sampling method is to select a sample of size n'_h from the h^{th} -stratum using simple random sampling without replacement (*SRSWOR*) where $\sum_{h=1}^{L} n'_h = n'$ and note down the value on the observed auxiliary variate(s) for these units. This will be called a stratified first phase sample. Now, for noting down the information on variate of the interest, another sample of size $n'_h (n_h < n'_h)$ is drawn from the each stratum using SRSWOR such that $\sum_{h=1}^{L} n_h = n$. This will be called second phase sample.

Now, for the given conditions below:

$$\left.\begin{aligned}
&\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h, \bar{y}_h = \bar{Y}_h(1+e_{0h}), e_0 = \sum_{h=1}^{L} W_h \bar{Y}_h e_{0h}/\bar{Y}, \\
&\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h, \bar{x}_h^{'} = \bar{X}_h(1+e_{1h}^{'}), e_1 = \sum_{h=1}^{L} W_h \bar{X}_h e_{1h}^{'}/\bar{X}, \\
&\bar{x}_h = \bar{X}_h(1+e_{1h}), e_1 = \sum_{h=1}^{L} W_h \bar{X}_h e_{1h}/\bar{X}, \\
&\bar{Z} = \sum_{h=1}^{L} W_h \bar{Z}_h, \bar{z}_h^{'} = \bar{Z}_h(1+e_{2h}^{'}), e_2 = \sum_{h=1}^{L} W_h \bar{Z}_h e_{2h}^{'}/\bar{Z}, \\
&\bar{z}_h = \bar{Z}_h(1+e_{2h}), e_2 = \sum_{h=1}^{L} W_h \bar{Z}_h e_{2h}/\bar{Z},
\end{aligned}\right\} \tag{1}$$

The expectations are defined as,

$$\left.\begin{aligned}
&E(e_0) = E(e_1) = E(e_1^{'}) = E(e_2) = E(e_2^{'}) = 0 \\
&E(e_0^2) = \frac{1}{\bar{Y}^2} \sum_{h=1}^{L} W_h^2 f_{1h} S_{yh}^2 = V_{200} \\
&E(e_1^2) = \frac{1}{\bar{X}^2} \sum_{h=1}^{L} W_h^2 f_{1h} S_{xh}^2 = V_{020} \\
&E(e_1^{'2}) = \frac{1}{\bar{X}^2} \sum_{h=1}^{L} W_h^2 f_{2h} S_{xh}^2 = V_{020}^{'} \\
&E(e_2^2) = \frac{1}{\bar{Z}^2} \sum_{h=1}^{L} W_h^2 f_{1h} S_{zh}^2 = V_{002} \\
&E(e_2^{'2}) = \frac{1}{\bar{Z}^2} \sum_{h=1}^{L} W_h^2 f_{2h} S_{zh}^2 = V_{002}^{'}
\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
&E(e_0e_1^{'}) = \frac{1}{\bar{Y}\bar{X}}\sum_{h=1}^{L}W_h^2 f_{2h}S_{yxh} = V_{110}^{'},\ E(e_0e_1) = \frac{1}{\bar{Y}\bar{X}}\sum_{h=1}^{L}W_h^2 f_{1h}S_{yxh} = V_{110},\\
&E(e_0e_2^{'}) = \frac{1}{\bar{X}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{2h}S_{yzh} = V_{101}^{'},\ E(e_0e_2) = \frac{1}{\bar{Y}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{1h}S_{yzh} = V_{101},\\
&E(e_1^{'}e_1) = \frac{1}{\bar{X}^2}\sum_{h=1}^{L}W_h^2 f_{2h}S_{xh}^2 = V_{020}^{'},\ E(e_1^{'}e_2^{'}) = \frac{1}{\bar{X}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{2h}S_{zxh} = V_{011}^{'},\\
&E(e_1^{'}e_2) = \frac{1}{\bar{X}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{2h}S_{zxh} = V_{011}^{'},\ E(e_1e_2^{'}) = \frac{1}{\bar{X}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{2h}S_{zxh} = V_{011}^{'},\\
&E(e_1e_2) = \frac{1}{\bar{X}\bar{Z}}\sum_{h=1}^{L}W_h^2 f_{1h}S_{zxh} = V_{011},\ E(e_2^{'}e_2) = \frac{1}{\bar{Z}^2}\sum_{h=1}^{L}W_h^2 f_{2h}S_{zh}^2 = V_{002}^{'}
\end{aligned}\right\} \tag{3}$$

where $f_{1h} = \left(\frac{1}{n_h} - \frac{1}{n_h^{'}}\right)$, $f_{2h} = \left(\frac{1}{n_h^{'}} - \frac{1}{N_h}\right)$ $f_{3h} = (f_{1h} - f_{2h})$

$$S_{hy}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)^2,\ S_{hx}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N}(x_{hi} - \bar{X}_h)^2,$$

$$S_{hz}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(z_{hi} - \bar{Z}_h)^2,\ S_{hyx} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)(x_{hi} - \bar{X}_h),$$

$$S_{hyz} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(y_{hi} - \bar{Y}_h)(z_{hi} - \bar{Z}_h),\ S_{hxz} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}(x_{hi} - \bar{X}_h)(z_{hi} - \bar{Z}_h).$$

3. SOME AVAILABLE ESTIMATOR

Some of the popular estimators that are available in the literature are reproduced here. Usual unbiased estimator for population mean $\bar{Y}$ in case of double stratified random sampling, is given by

$$\bar{y}_{(st)} = \sum_{h=1}^{L}W_h\bar{y}_h, \tag{4}$$

where

$$W_h = \frac{N_h}{N}$$

The variance of the unbiased estimator is,

$$Var(\bar{y}_{(st)}) = \bar{Y}^2\left(\frac{\sum_{h=1}^{L}W_h^2 f_{1h}S_{yh}^2}{\bar{Y}^2}\right) = \bar{Y}^2 V_{200} \tag{5}$$

The ratio and product estimators for double stratified random sampling is,

$$\hat{\bar{Y}}_{Rd} = \bar{y}_{st}\frac{\bar{x}'_{st}}{\bar{x}_{st}} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\sum_{h=1}^{L} W_h \bar{x}'_h}{\sum_{h=1}^{L} W_h \bar{x}_h} \right) \tag{6}$$

and

$$\hat{\bar{Y}}_{Pd} = \bar{y}_{st}\frac{\bar{x}_{st}}{\bar{x}'_{st}} = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\sum_{h=1}^{L} W_h \bar{x}_h}{\sum_{h=1}^{L} W_h \bar{x}'_h} \right) \tag{7}$$

The mean square error of above estimators, for the first order approximation are,

$$MSE(\hat{\bar{Y}}_{Rd}) = \bar{Y}^2\left(V_{200} + V_{020} - V'_{020} - 2(V_{110} - V'_{110})\right) \tag{8}$$

and

$$MSE(\hat{\bar{Y}}_{Pd}) = \bar{Y}^2\left(V_{200} + V_{020} - V'_{020} + 2(V_{110} - V'_{110})\right) \tag{9}$$

Mohanty (1967) regression-cum-ratio and regression-cum-product type estimator for Y in two-phase stratified sampling, is given by

$$y^R_{reg} = \left[\bar{y}_{st} + \beta_1\left(\bar{x}'_{st} - \bar{x}_{st}\right)\right]\frac{\bar{Z}}{\bar{z}'_{st}} \tag{10}$$

and

$$y^P_{reg} = \left[\bar{y}_{st} + \beta_2\left(\bar{x}'_{st} - \bar{x}_{st}\right)\right]\frac{\bar{z}'_{st}}{\bar{Z}} \tag{11}$$

where β_1 and β_2 are constants.

The MSE of y^R_{reg} and y^P_{reg} with β_1 and β_2 i.e. $\beta_1 = \beta_2 = \dfrac{\bar{Y}\left(\bar{V}_{110} - \bar{V}'_{110}\right)}{\bar{X}\left(V_{020} - V'_{020}\right)}$ is given by

$$MSE(y^R_{reg}) = \bar{Y}^2\left[V_{200} + V'_{020} - 2V'_{101} - \frac{(V'_{110} - V_{110})^2}{V_{020} - V'_{020}}\right] \tag{12}$$

and

$$MSE(y^R_{reg}) = \bar{Y}^2\left[V_{200} + V'_{020} + 2V'_{101} - \frac{(V'_{110} - V_{110})^2}{V_{020} - V'_{020}}\right] \tag{13}$$

Sanauallah et. al. (2014) generalized exponential chain ratio type estimators for the stratified two-phase sampling method is,

$$\hat{\bar{Y}}_S = \lambda\sum_{h=1}^{L} W_h\bar{y}_h\left[\alpha_h exp\left\{\alpha\sum_{h=1}^{L}W_h\left(\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}-\bar{x}_h\right)\middle/\sum_{h=1}^{L}W_h\left(\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}+\left(a-1\right)\bar{x}_h\right)\right\}\right] \tag{14}$$

where λ, α and a are contants and minimum MSE of the above estimator is

$$MSE\left(\hat{\bar{Y}}_S\right)_{min} = \bar{Y}^2\left[1+\left\{1+V_{200}-\frac{(V_{110}^{'}-V_{110}-V_{101}^{'})^2}{V_{020}-V_{020}^{'}+V_{002}^{'}}\right\}^{-1}\right] \tag{15}$$

4. PROPOSED ESTIMATOR

Motivated by Sanaulla et. al. (2014), the following class of exponential estimator for estimation of population mean under double stratified random sampling has been proposed,

$$\hat{\bar{Y}}_{RP}^{s} = \left[\sum_{h=1}^{L}W_h\bar{y}_h + k\left\{W_h\left(\bar{x}_h^{'}-\bar{x}_h\right)\right\}\right]\left[\alpha H_1+\left(1-\alpha\right)H_2\right] \tag{16}$$

where

$$H_1 = exp\left[\left\{\sum_{h=1}^{L}W_h\left(\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}-\bar{x}_h\right)\right\}\middle/\left\{\sum_{h=1}^{L}W_h\left(\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}+\bar{x}_h\right)\right\}\right]$$

and

$$H_2 = exp\left[\left\{\sum_{h=1}^{L}W_h\left(\bar{x}_h-\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}\right)\right\}\middle/\left\{\sum_{h=1}^{L}W_h\left(\bar{x}_h+\bar{x}_h^{'}\frac{\bar{Z}}{\sum_{h=1}^{L}W_h\bar{z}_h^{'}}\right)\right\}\right]$$

Further simplifying using (1), (2) and (3) , we have

$$\left(\hat{\hat{Y}}_{RP}^{s}-Y\right)^{2}=\bar{Y}^{2}\Big[\alpha^{2}\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)^{2}+\frac{1}{4}\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)^{2}+e_{0}^{2}-\alpha\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)^{2}-e_{0}\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)$$
$$+2\alpha e_{0}\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)+k^{2}\bar{X}^{2}\left(e_{1}^{'}-e_{1}\right)^{2}+(2\alpha-1)k\bar{X}\bar{Y}\left(e_{1}^{'}-e_{1}\right)\left(e_{1}^{'}-e_{1}-e_{2}^{'}\right)+2k\bar{X}\bar{Y}e_{0}\left(e_{1}^{'}-e_{1}\right)\Big] \quad (17)$$

Taking expectation, we have

$$MSE\left(\hat{\hat{Y}}_{RP}^{s}\right)=\bar{Y}^{2}\Big[\alpha^{2}\left(V_{020}-V_{020}^{'}+V_{002}^{'}\right)+\frac{1}{4}\left(V_{020}-V_{020}^{'}+V_{002}^{'}\right)+V_{200}-\alpha\left(V_{020}-V_{020}^{'}+V_{002}^{'}\right)-\left(V_{110}^{'}-V_{110}+V_{101}^{'}\right)$$
$$+2\alpha\left(V_{110}^{'}-V_{110}+V_{101}^{'}\right)\Big]+k^{2}\bar{X}^{2}\left(V_{020}-V_{020}^{'}\right)+(2\alpha-1)k\bar{X}\bar{Y}\left(V_{020}-V_{020}^{'}\right)+2k\bar{X}\bar{Y}e_{0}\left(V_{110}^{'}-V_{110}\right)\Big] \quad (18)$$

Finding optimum value of k and α from (11) we get

$$k_{(opt)}=\frac{\bar{Y}}{\bar{X}}\left[\frac{V_{101}^{'}}{V_{002}^{'}}-\frac{\left(V_{110}^{'}-V_{110}\right)}{\left(V_{020}-V_{020}^{'}\right)}\right] \quad (19)$$

$$\alpha_{(opt)}=\frac{1}{2}+\frac{V_{101}^{'}}{V_{200}^{'}} \quad (20)$$

By substituting the optimum value of k_{opt} and α_{opt} the minimum MSE is,

$$MSE(\hat{\hat{Y}}_{RP}^{s})_{min}=\bar{Y}^{2}\left[V_{200}-\frac{V_{101}^{'2}}{V_{002}^{'}}-\frac{(V_{110}^{'}-V_{110})^{2}}{V_{020}-V_{020}^{'}}\right] \quad (21)$$

Putting $k=0$ and finding optimum value of α, we get

$$\alpha_{opt}=\frac{1}{2}-\left[\frac{V_{110}^{'}-V_{110}-V_{101}^{'}}{V_{020}-V_{020}^{'}+V_{002}^{'}}\right] \quad (22)$$

and using (19) in (11), we have

$$MSE(\hat{\hat{Y}}_{RP}^{s})_{min}=\bar{Y}^{2}\left[V_{200}-\frac{(V_{110}^{'}-V_{110}-V_{101}^{'})^{2}}{V_{020}-V_{020}^{'}+V_{002}^{'}}\right]$$

5. EFFICIENCY COMPARISONS

For carrying out efficiency comparison, the proposed estimator is compared with the above discussed estimators and modified form of Hansen-Hurwitz unbiased estimator.

$MSE(\hat{\hat{Y}}_{RP}^{s})<Var(\bar{y}_{st})$ if

$$\frac{V_{101}^{'2}}{V_{002}^{'}}+\frac{(V_{110}^{'}-V_{110})^2}{V_{020}-V_{020}^{'}}\geq 0 \tag{23}$$

$MSE(\hat{\hat{Y}}_{RP}^{s}) < MSE(\bar{y}_{Rd})$ if

$$\frac{V_{101}^{'2}}{V_{002}^{'}}+\frac{(V_{020}-V_{020}^{'}+V_{110}^{'}-V_{110})^2}{V_{020}-V_{020}^{'}} \tag{24}$$

$MSE(\hat{\hat{Y}}_{RP}^{s}) < MSE(\bar{y}_{Pd})$ if

$$\frac{V_{101}^{'2}}{V_{002}^{'}}+\frac{(V_{020}-V_{020}^{'}-V_{110}^{'}+V_{110})^2}{V_{020}-V_{020}^{'}} \tag{25}$$

$MSE(\hat{\hat{Y}}_{RP}^{s}) < MSE(y_{reg}^{R})$ if

$$\frac{V_{101}^{'2}}{V_{002}^{'}}-2V_{101}^{'}+V_{002}^{'}\geq 0 \tag{26}$$

$MSE(\hat{\hat{Y}}_{RP}^{s}) < MSE(y_{reg}^{P})$ if

$$\frac{V_{101}^{'2}}{V_{002}^{'}}+2V_{101}^{'}+V_{002}^{'}\geq 0 \tag{27}$$

$MSE(\hat{\hat{Y}}_{RP}^{s}) < MSE\left(\hat{\hat{Y}}_{S}\right)_{min}$ if

$$\frac{\left[V_{110}^{'}\left(V_{020}-V_{020}^{'}\right)+V_{020}^{'}\left(V_{110}^{'}-V_{110}\right)\right]^2}{V_{002}^{'}\left(V_{020}-V_{020}^{'}\right)\left(V_{110}^{'}-V_{110}\right)}\geq 0 \tag{28}$$

6. EMPIRICAL STUDY

To carry out the numerical illustrations some industrial datasets are used to check the usefulness of the proposed estimators of $\bar{Y}$,

Dataset-I: "Source: (Murthy, 1967) **Y**: Production (Output), **X**: No. of workers, **Z**: Fixed Capital"

Dataset-II: "Source: (Sardanal et al., 1992) **Y**: 1983 Workers (in millions), **X**: 1980 Workers (in millions), **Z**: 1982 Gross National Product (in tens of millions of U.S. dollars)"

Dataset-III: "Source: (Gujarati, 2003) **Y**: Average miles per gallon, **X**: Engine horsepower, **Z**: Top speed, miles per hour"

Table 1: Dataset-I

Strata	N_h	n_h	n'_h	S_{yh}	S_{xh}	S_{zh}	$\bar{Y}_h$	$\bar{X}_h$	$\bar{Z}_h$	ρ_{xyh}	ρ_{xzh}	ρ_{yzh}
1	19	5	11	757.08	11.18	109.45	65.16	2669.25	349.68	0.81	0.90	0.93
2	32	8	17	669.11	44.35	109.22	140.00	4657.62	706.60	0.88	0.84	0.92
3	14	3	8	417.00	81.11	277.18	403.21	6537.21	1539.60	0.92	0.93	0.98
4	15	4	9	645.69	171.44	370.96	763.20	7843.67	2620.53	0.97	0.94	0.96

Table 2: Dataset-II

Strata	N_h	n_h	n'_h	S_{yh}	S_{xh}	S_{zh}	$\bar{Y}_h$	$\bar{X}_h$	$\bar{Z}_h$	ρ_{xyh}	ρ_{xzh}	ρ_{yzh}
1	38	16	27	16.46	14.91	1915.17	11.88	13.03	1029.16	0.99	0.74	0.74
2	14	6	10	60.23	58.48	78097.92	26.18	27.35	25671.57	0.99	0.97	0.96
3	11	4	8	34.89	32.66	7588.80	21.88	23.13	5028.82	0.99	0.97	0.97
4	33	14	23	209.08	200.07	20672.75	75.24	79.65	7533.94	0.99	0.30	0.29
5	24	10	17	18.80	18.69	19782.83	20.09	20.28	16314.42	0.99	0.90	0.90

Table 3: Dataset-III

Strata	N_h	n_h	n'_h	S_{yh}	S_{xh}	S_{zh}	$\bar{Y}_h$	$\bar{X}_h$	$\bar{Z}_h$	ρ_{xyh}	ρ_{xzh}	ρ_{yzh}
1	21	06	15	12.14	76.71	19.48	37.55	116.57	114.14	-0.7914	0.9894	-0.7781
2	34	04	17	8.34	31.94	07.10	37.25	93.00	106.50	-0.8339	0.8820	-0.6651
3	26	02	08	5.47	49.55	13.21	26.39	26.39	118.88	-0.7696	0.9669	-0.5935

In Table 5 the Percentage Relative Efficiencies (PREs) of different suggested estimators of $\bar{Y}$ with respect to usual estimator $\bar{y}_{st}$ using the formula given below are tabulated.

$$PRE\left(*, \bar{y}_{st}\right) = \frac{V\left(\bar{y}_{st}\right)}{V\left(*\right)} \tag{30}$$

where $*$ stands for $MSE(\bar{y}_{Rd})$, $MSE(\bar{y}_{Pd})$, $MSE(y^{R}_{reg})$, $MSE(y^{P}_{reg})$, $MSE\left(\hat{\bar{Y}}_S\right)$ and $MSE\left(\hat{\bar{Y}}^{s}_{RP}\right)$.

Table 5: PREs of the estimators of $\bar{Y}$

Estimators	PREs		
	Dataset-I	***Dataset-II***	***Dataset-III***
$\bar{y}_{st}$	100.00	100.00	100.00
$\bar{y}_{Rd}$	13.00	101.00	*
$\bar{y}_{Pd}$	*	*	29.00
y_{reg}^{R}	18.73	151.97	*
y_{reg}^{P}	*	*	52.60
$\hat{\bar{Y}}_S$	136.05	152.32	193.58
$\hat{\bar{Y}}_{RP}^{s}$	143.00	203.90	208.30

* Data not applicable for this estimator

7. CONCLUSION

In Table-4, it can be noticed with the help of percent relative efficiencies (PREs) that the proposed class of estimators – clearly - outdo the usual unbiased estimator in double stratified sampling $\bar{y}_{st}$, ratio and product estimators $\bar{y}_{Rd}$ and $\bar{y}_{Pd}$ respectively. Also it outperform Mohanty (1967) regression-cum-ratio and regression-cum-product estimators y_{reg}^{R} and y_{reg}^{P} respectively. It is also superior to Sanaullah et al. (2014) estimators $\hat{\bar{Y}}_S$ for the population mean $\bar{Y}$. So, in practice the use of the suggested class of estimators could be preferred instead of existing methods to computing statistical inference for industrial data and others for higher accuracy in the estimates.

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