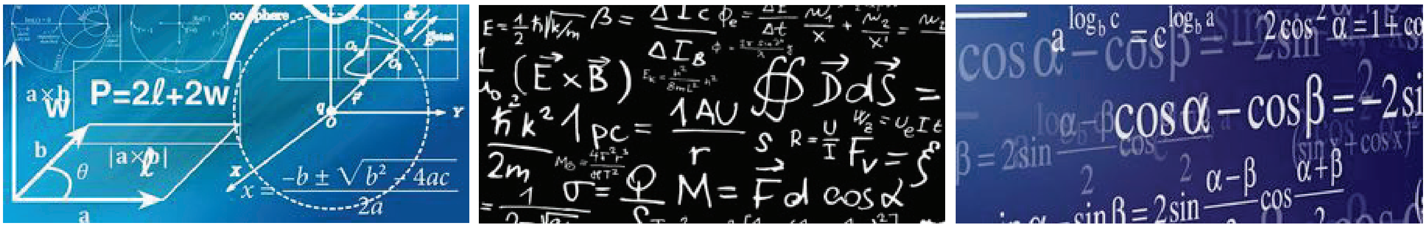

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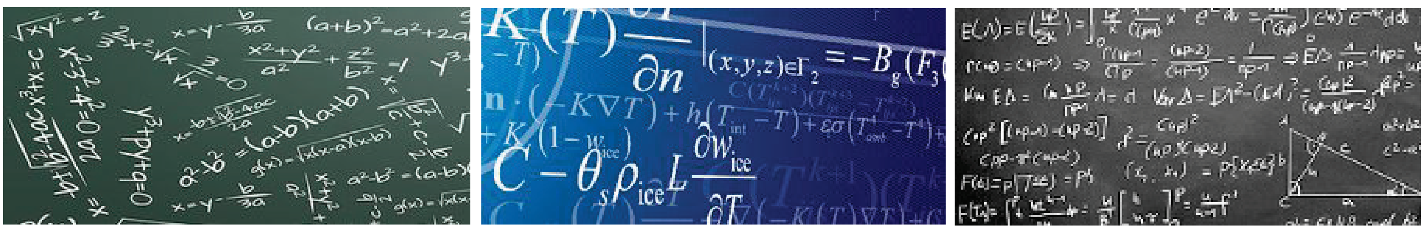

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Original paper

Derivations of the Bézier Curve

Dušan Páleš*, Jozef Rédl

Slovak University of Agriculture, Faculty of Engineering, Department of Machine Design, Nitra, Slovak Republic

ABSTRACT

On selected polygon of control points we create Bézier curve. This we differentiate using a direct derivation of the polygon and receive hodograph of a new Bézier curve. The process is repeated for the second derivation, and these results are compared with numerical derivation of the parametric specified curve.

KEYWORDS: tangent curve, hodograph, numerical derivation

JEL CLASSIFICATION: M55, N55

INTRODUCTION

Bézier curve needs for its definition a set of control points that determine it completely [3]. Its application, as well as the use of B-spline curve we addressed our previous works [2]. In finding tangents and normals of Bézier curve is necessary to calculate its derivation [1], [3]. Similarly, the first and the second derivation is used in determining of the length and curvature of any curve. The article briefly describes method for obtaining the derivations of a Bézier curve and on a simple example this method is compared with the numerical derivation for the parametric specified curve.

MATERIAL AND METHODS

Derivations of a Bézier curve

Bézier curve is defined by the polygon of (n+1) control points P_0, P_1, ... , P_n and follows the equation

* Corresponding author: Dušan Páleš, Slovak University of Agriculture, Faculty of Engineering, Department of Machine Design, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic. E-mail: dusan.pales@uniag.sk

$$P(t)=\sum_{i=0}^{n} B_i^n(t).P_i \tag{1}$$

where Bernstein polynomial is defined

$$B_i^n(t)=\binom{n}{i}.t^i.(1-t)^{n-i} \tag{2}$$

The control points P_i are independent on parameter t and therefore is sufficient to differentiate $B_i^n(t)$

$$\frac{d}{dt}B_i^n(t)=B_i^{n\prime}(t)=n.(B_{i-1}^{n-1}(t)-B_i^{n-1}(t)) \tag{3}$$

Then the derivation throughout the Bézier curve has the form

$$\frac{d}{dt}P(t)=P'(t)=\sum_{i=0}^{n-1} B_i^{n-1}(t).\left[n.(P_{i+1}-P_i)\right] \tag{4}$$

If we introduce a set of control points $Q_0 = n(P_1 - P_0)$, $Q_1 = n(P_2 - P_1)$, $Q_2 = n(P_3 - P_2)$, ..., $Q_n = n(P_n - P_{n-1})$ writing of derivation (4) simplify on

$$P'(t)=\sum_{i=0}^{n-1} B_i^{n-1}(t).Q_i \tag{5}$$

When (5) compared to (1) we see that by differentiation of the Bézier curve arises again Bézier curve of degree (n-1), with control points $n(P_1 - P_0)$, $n(P_2 - P_1)$, $n(P_3 - P_2)$, ... , $n(P_n - P_{n-1})$. This new differentiated curve is usually called the hodograph of the original Bézier curve. The form $n(P_{i+1} - P_i)$ represents n-times direction vector from the point P_i to the point P_{i+1}.

For t = 0, $P'(0) = n(P_1 - P_0)$, which means that the first tangent of the curve is in the direction of $(P_1 - P_0)$ n-times multiplied. Similarly for t = 1, $P'(1) = n(P_n - P_{n-1})$, the last tangent of the curve is in the direction of $(P_n - P_{n-1})$ n-times enlarged.

In calculating the second derivation we start from equation (5) and again analogously we differentiate as for the first derivation

$$\begin{aligned} P''(t)&=\sum_{i=0}^{n-2} B_i^{n-2}(t).\left[(n-1).(Q_{i+1}-Q_i)\right] \\ &=\sum_{i=0}^{n-2} B_i^{n-2}(t).\left[n.(n-1).(P_{i+2}-2P_{i+1}+P_i)\right] \end{aligned} \tag{6}$$

The higher derivations can be obtained likewise by applying the same process.

RESULTS AND DISCUSSION

We chose ten points in the plane, n = 9, where x and y-coordinates are given in the Table 1. These control points have been plotted and we constructed the Bézier curve according to [2]. The result is shown in the Figure 1.

By derivation of the Bézier curve (4) we obtained its hodograph drawn in the Figure 2. The control points of the new curve are calculated in the middle two columns of the Table 1. Degree of the new Bézier curve was reduced.

The process was repeated and it was found the second derivation of the original curve according to (6). The graph is shown in the Figure 3 and the coordinates of the control points are given in the Table 1 in the last two columns. Degree of the Bézier curve of second derivation again decreased.

The values of both obtained derivations are presented in the Table 2 for selected parameter step t = 0.01. Each curve is parameterized through t∈<0, 1>. From all 101 values we point out every tenth value.

Tab. 1 Coordinates of control points of a Bézier curve and its first and second derivations

Bézier curve, n = 9			1st derivation, n = 8		2nd derivation, n = 7	
i	$P_i[x]$	$P_i[y]$	$P_i'[x]$	$P_i'[y]$	$P_i''[x]$	$P_i''[y]$
0	14	0	-72	36	-144	576
1	6	4	-90	108	1296	-648
2	-4	16	72	27	144	576
3	4	19	90	99	-576	-792
4	14	30	18	0	576	-792
5	16	30	90	-99	-2232	576
6	26	19	-189	-27	2880	-648
7	5	16	171	-108	-1944	576
8	24	4	-72	-36	-	-
9	16	0	-	-	-	-

Tab. 2 Points of Bézier curve and curves forming the first and second derivations

	Bézier curve		1st derivation		2nd derivation	
	P[x]	P[y]	P´[x]	P´[y]	P´´[x]	P´´[y]
0	14	0	-72	36	-144	576
10	7.357225	5.402068	-51.62585	64.071936	418.985914	85.74912
20	4.450247	11.921203	-6.478664	63.790848	426.933965	-69.76512
30	5.652148	17.775412	27.243613	51.558912	233.447731	-174.19392
40	9.176722	21.894451	39.682437	29.263104	20.73047	-266.84928
50	12.960938	23.390625	33.46875	0	-130.5	-306
60	15.531450	21.894451	17.232261	-29.263104	-172.40279	-266.84928
70	16.497654	17.775412	3.741085	-51.558912	-74.676211	-174.19392
80	16.783494	11.921203	4.746424	-63.790848	85.544755	-69.76512
90	17.649271	5.402068	9.410566	-64.071936	-123.408634	85.74912
100	16	0	-72	-36	-1944	576

For the parameter t = 0.01, in every tenth value, obtained by direct calculation

Tab. 3 Points of Bézier curve and curves forming the first and second derivations

Bézier curve			1st derivation		2nd derivation	
	P[x]	P[y]	P´[x]	P´[y]	P´´[x]	P´´[y]
0	14	0	-72.5565	38.7414	-49.39	496.06
10	7.357225	5.402068	-49.4982	64.4611	436.96	62.87
20	4.450247	11.921203	-4.3674	63.4239	412.36	-80.57
30	5.652148	17.775412	28.3737	50.6708	211.19	-184.46
40	9.176722	21.894451	39.7546	27.9168	2.01	-273.87
50	12.960938	23.390625	32.7993	-1.5297	-140.23	-305.51
60	15.531450	21.894451	16.3753	-30.5848	-168.88	-259.18
70	16.497654	17.775412	3.3947	-52.4127	-58.19	-163.85
80	16.783494	11.921203	5.1888	-64.1212	93.19	-58.60
90	17.649271	5.402068	8.6690	-63.6011	-203.46	112.09
Last	16	0	-62.8257	-38.7414	-1629.74	496.06

For the parameter t = 0.01, in every tenth value, obtained by numerical calculation

Since the Bézier curves for a set of control points are defined parametrically, can be differentiated also numerically. The results of these numerical derivations are given in the Table 3. The last line in this table identified Last means for the original curve i = 100, for the first derivation i = 99 and for the second derivation i = 98. The number of points is reduced, which directly follows from numerical differentiation algorithm.

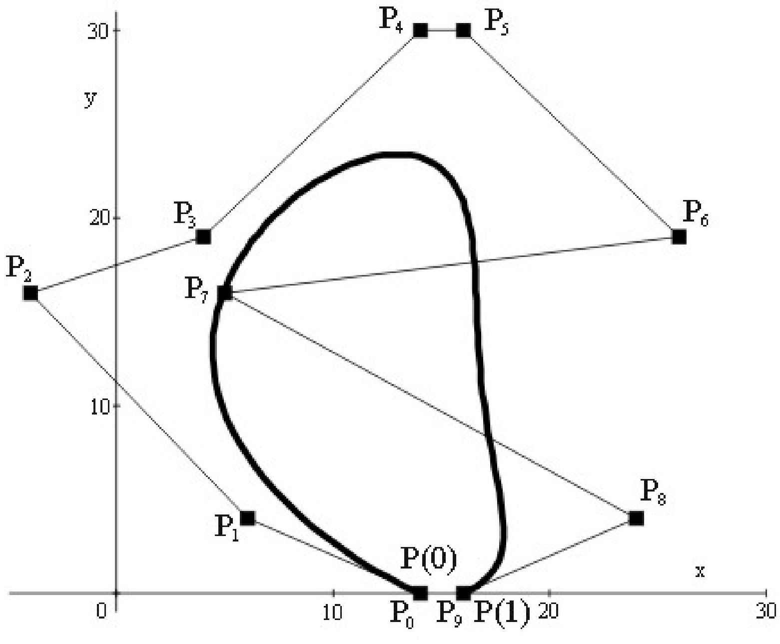

Fig. 1 Bézier curve and its control points drawn for the parameter t = 0.01

Obtained curve of the first derivation is shown in the Figure 4 and the second derivation curve obtained is in the Figure 5.

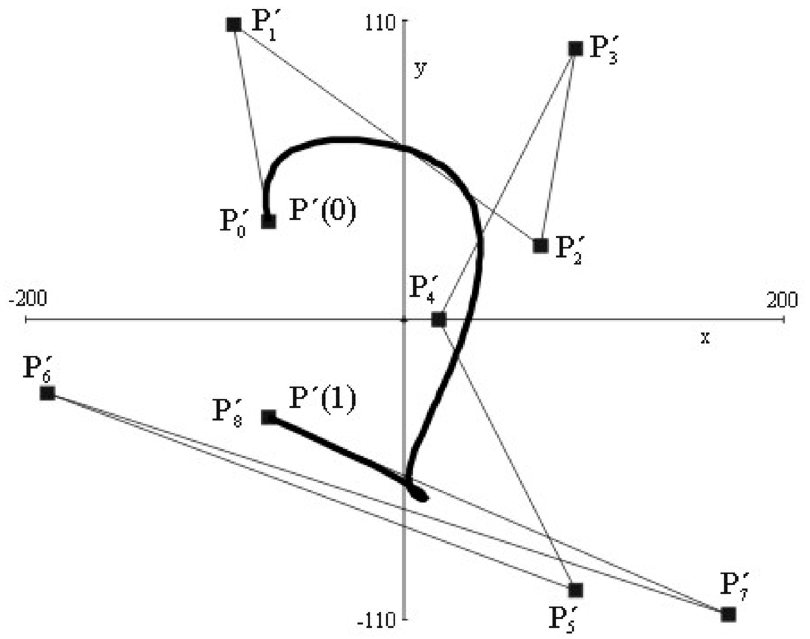

Fig. 2 The first derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by direct calculation

Comparing the pairs of the Figures 2 and 4 we see that result of the first derivation found by direct calculation (5) corresponds to the result of numerical derivation. Similar finding belongs to the comparison of the second derivations in the Figures 3 and 5.

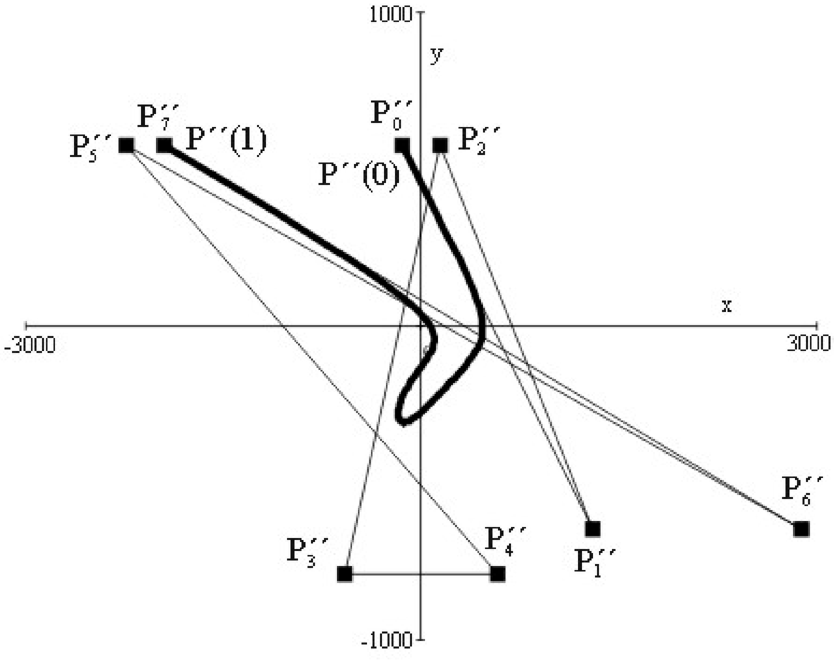

Fig. 3 The second derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by direct calculation

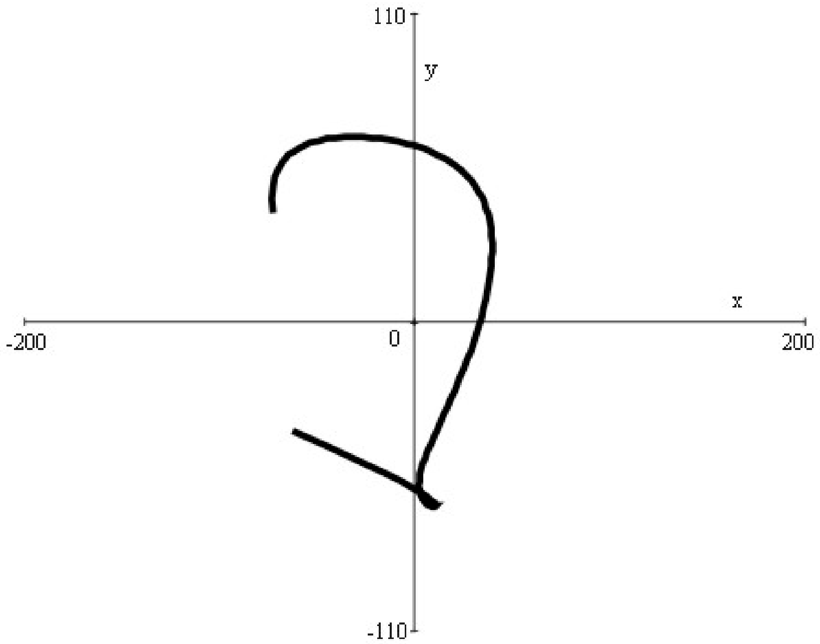

Fig. 4 The first derivation of the Bézier curve with its control points
For the parameter t = 0.01, obtained by numerical calculation

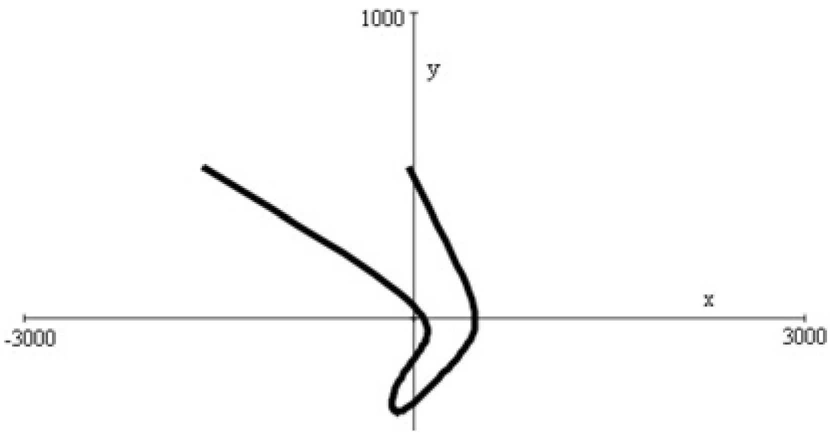

Fig. 5 The second derivation of the Bézier curve with its control points.
For the parameter t = 0.01, obtained by numerical calculation

When comparing the values of the first and second derivation in the Tables 2 and 3 there is not already excessive identity and mainly the second derivations are quite different. However, it should be noted scales of the Figures 1 - 5. They result that with increasing derivation also scale is increasing, which is a direct consequence of the relationships (4) and (5). Therefore also relatively large differences between direct and numerical derivation significantly do not affect the outcome and shape of the curve remains the same by both methods.

CONCLUSIONS

We created Bézier curve and its first two derivations, which again formed a new Bézier curves. This direct calculation of the derivations was compared with the numerical derivations of the original curve. The results in the Figures 2 - 5 are coincided for both derivatives. By comparing the respective values in the Tables 2 and 3 resulting differences were caused by inaccuracy of the numerical derivation. The inaccuracy increases with degree of derivation. Nevertheless numerical derivation very well approximated a direct derivation of the Bézier curve.

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Original paper

Numerical differentiation of stochastic function

Jozef Rédl*, Dušan Páleš

Slovak University of Agriculture, Faculty of Engineering, Department of Machine Design
Nitra, Slovak Republic

ABSTRACT

In this paper we are presenting the derivation of the finite difference approximation of the first derivative of the discrete stochastic function. The set of data was represented with time depending stochastic function of center of gravity dislocation and velocity with constant step size. The stochastic function was obtained from real experimental measurement and was processed with *Dynstab (Dynamic Stability)* software. Applied forward, backward and central differences algorithms were written in Microsoft Visual C#® 2010 programming language as a part of software named *Dynstat (Dynamic Statistics)*. These algorithms were used to calculate the first derivative of the function and error of differentiation. We compared the results of known precise solutions with calculated results. We also confront the results calculated of known generated trigonometric function. With this manner we are evaluated the inaccuracy of the stochastic function numerical differentiation.

KEYWORDS: numerical differentiation, algorithm, programming, error evaluating

JEL CLASSIFICATION: N40

INTRODUCTION

Solutions of many technical problems are based on obtaining the differentials of technical functions. The searched differentials of function are often evaluated algebraically where the accuracy of results is guaranteed. If the values technical functions are measured continuously its differentials must be evaluated numerically. In this case the accuracy is not guaranteed. The high instability of differentiating process is the reason for the algebraic evaluation if it's possible. In the specific case the algebraic evaluation is not possible and we have to design an algorithm of numerical solution. Methods of numerical differentiation algorithms are known and published by many authors.(e.g. [1], [2], [4]). In this contribution we deal with the derivation and the application of the finite difference method. This method has three variants

* Corresponding author: Jozer Rédl, Slovak University of Agriculture, Faculty of Engineering, Department of Machine Design, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic. E-mail: jozef.redl@uniag.sk

namely forward, backward and central differences and they has defined in [1], [12]. Practical applications were published with [1], [3], [11]. Mathematical solution accuracy depends first of all on the chosen step of differentiation and from the order of approximation polynomial. The next factors are the applied method as well as the chosen numerical data type. When the exact solution is available and is obtained from numerical integration how is defined in [8], feasible solution of the inaccuracy we can get from results comparison.

MATERIAL AND METHODS

Finite difference method

The basic method of numerical differentiation (forward, backward, central) are defined in [4]. Its derivation we can realize from the initial value problem of the ordinary differentials equations in accordance with [2]. This method is the start point for derivation of numerical integration how is described in [6]. Let us assume the differential equation in the next form:

$$\dot{y} = f\left(x, y(x)\right). \tag{1}$$

The solution of this equation is the function y, and its satisfied at certain point $x_0 \in \langle a, b \rangle$, values η, hence $y(x_0) = \eta$. This initial value problem we call as a *Cauchy initial value problem*. To guarantee the existence of solution is necessary for the *Cauchy uniqueness* of solution in the area of function continuous substitute with *Lipschitz continuous condition*. The equation (1) rewriting to the form:

$$\dot{y} = f(x, y), \tag{2}$$

where $f(x, y)$ is the real vector function and is defined and continuous on the set $S = \langle a, b \rangle \times \mathcal{R}^m$. If the *Lipschitz condition* must be satisfied, there have to exist a constant L, which is independent on x, y such that: $\| f(x, y) - f(x, z) \| \leq L \| y - z \|$ for all $x_0 \in \langle a, b \rangle$ and all vectors y, z. Foremost we have to define the origin of the solving of initial value problem with numerical integration. The goal of the solution is the points calculation, $x_0 = a, x_1, x_2, \dots, x_n$, that are the approximations of exact solution points $(x_0), y(x_1), y(x_2) \dots y(x_n)$, in points $x_0, x_1, x_2, \dots x_n$, where step of integration method is $h_n = x_{n+1} - x_n$ or $h_{n-1} = x_n - x_{n-1}$ and must have to be satisfied that $h_n > 0, \forall n$. If $h_n = h$ we talking about method with constant step. The numerical integration method is based on the application of the *Taylor's series*, where the exact solution is approximated with the *Taylor's polynomial* of the highest order in the form:

$$\begin{aligned} y(x_{n+1}) &= y(x_n + h_n) = \\ &= y(x_n) + h_n \dot{y}(x_n) + \frac{h_n^2}{2!}\ddot{y}(x_n) + \dots + \frac{h_n^p}{p!} y(x_n) + \frac{h_n^{p+1}}{(p+1)!} y^{p+1}(\xi_n), \end{aligned} \tag{3}$$

as are published in [6], [8], [9]. Modifying equation (3) and neglecting the highest degree members we get:

$$y(x_{n+1}) = y(x_n + h_n) = y(x_n) + h_n \dot{y}(x_n) + \frac{h_n^{p+1}}{(p+1)!} y^{p+1}(\xi_n). \tag{4}$$

The application of the basic equation for differentiating we get:

$$\dot{y} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - \frac{1}{2!} h \ddot{f}(\xi), \tag{5}$$

where $E(f) = -\frac{1}{2!} h \ddot{f}(\xi)$ is the error of numerical differentiation. Substituting in the equation (5) with $x = x_i, h = x_{i+1} - x_i$, we get forward-difference method in the form:

$$f(\dot{x}_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{1}{2!} h \ddot{f}(\xi). \tag{6}$$

Changing indexing like that $x = x_i, h = x_i - x_{i-1}$, we get backward-difference method:

$$f(\dot{x}_{i-1}) = \frac{f(x_i) - f(x_{i-1})}{h} - \frac{1}{2!} h \ddot{f}(\xi_{i-1}). \tag{7}$$

If we change $x = \frac{1}{2}(x_{i-1} - x_i)$, where x_{i-1} and x_i are symmetrical about x, we get $x_{i-1} = x - h,\ h = \frac{1}{2}(x_i - x_{i-1})$.

Inserting to (5) we got:

$$f(\dot{x}_{i-1}) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{h^2}{6} f(\overline{\overline{\xi}}_{i-1}), \tag{8}$$

what is the central-difference formula. If we apply the same manner as defined above and at the deriving process we change the values of $x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}$ for solution as defined in [6] and combining the four equations of *Taylor series*, we get the formulations of highest degrees. There is the fourth degree form:

$$f(\dot{x}_{i-2}) = \frac{f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2})}{12h}. \tag{9}$$

Stochastic data set

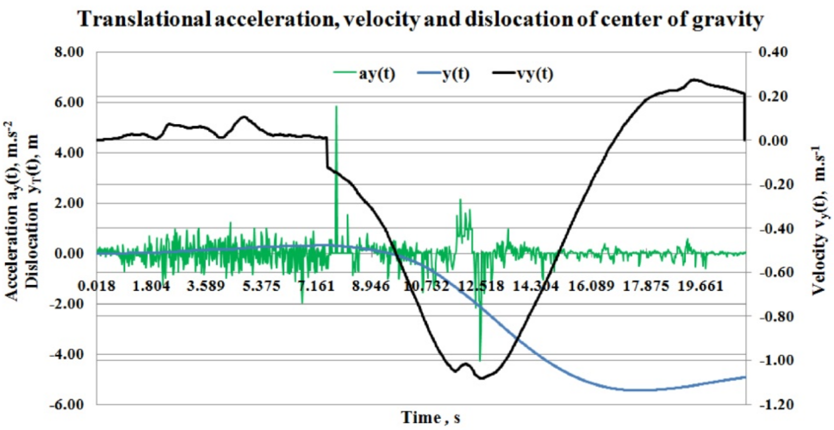

Fig. 1 Used technical function

We obtained the technical functions from the real experimental measurement of which we chose translational acceleration, translational velocity and dislocations of center of gravity moving with respect to the *y* axis. The methodology of discrete data processing in *Dynstab* software is described in [8]. Statistical processing of the stochastic technical functions is defined in [9]. The applied random data set is stationary and ergodic with zero mean value defined in the next form: $X = \{X(t); t \in (-\infty, +\infty)\}$. The graphs of stochastic functions $a_y(t), v_y(t), y(t)$ are depicted on the figure1.

RESULTS AND DISCUSSION

Visual C# algorithm

The relevant parts of algorithm were written in Microsoft Visual C# 2010 and we presented them in the abbreviated form. Non-mathematical parts of the algorithms (controls, forms, etc.) are not included in the list. Declarations of applied variables in the source code are:

```
private int dim; // dimension for arrays
private double[] h,fnc,dfnc; //Arrays of step, function, diff. function
```

Solving algorithms:

```
double stp; // step of differentiation
switch (diffmethComboBox.SelectedIndex)// select chosen method
       { case 0: //forward diff
         for (int i = 0; i < dim - 1; i++)
         { stp = h[0]; dfnc[i] = (1 / stp) * (fnc[i + 1] - fnc[i]);} break;
         case 1: //backward
         for (int i = 1; i < dim; i++)
         { stp = h[0]; dfnc[i - 1] = (1 / stp) * (fnc[i] - fnc[i - 1]);} break;
         case 2://central
         for (int i = 1; i < dim - 1; i++)
         { stp = 2 * h[0]; dfnc[i - 1] = (1 / stp) * (fnc[i + 1] –
                                         -fnc[i - 1]);} break;
         case 3://central-4th. order
         for (int i = 2; i < dim - 2; i++)
         { stp = 12 * h[0]; dfnc[i - 2] = (1 / stp) *(fnc[i - 2] -
                           8 * fnc[i - 1] + 8 * fnc[i+1] - fnc[i+2]);}}
```

To define the numeric variables we chose the type *Double* as defined in [5].

Evaluated results

Results of the numerical differentiation are depicted in figure 2. There is shown the exact solution as well as solution realized with variants of finite-difference method. We generated a sample function $y = \sin(x)$ and its exact solution $\dot{y} = \cos(x)$ in interval $x \in \langle 0,2\pi \rangle$ for comparison. The step of numerical differentiating of goniometric function was adapted to stochastic data count $n = 1178$. Solved step was $h = 0{,}00533377360541561$.

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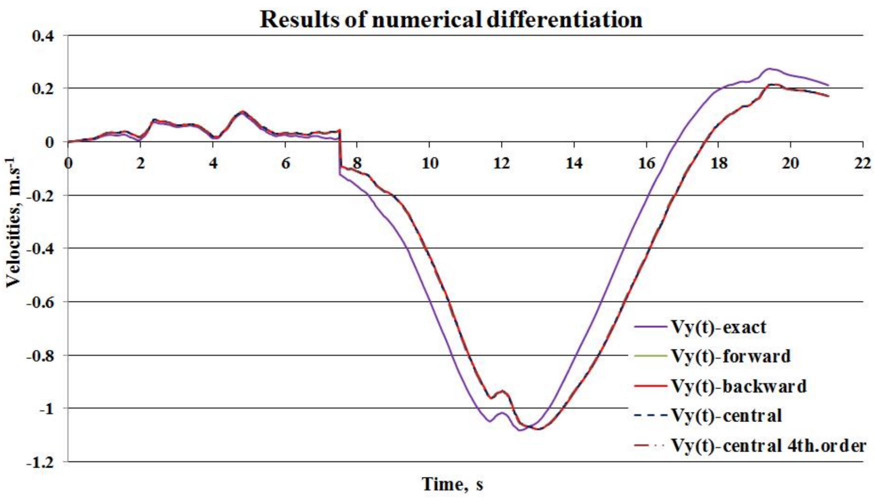

Fig. 2 Results of differentiating

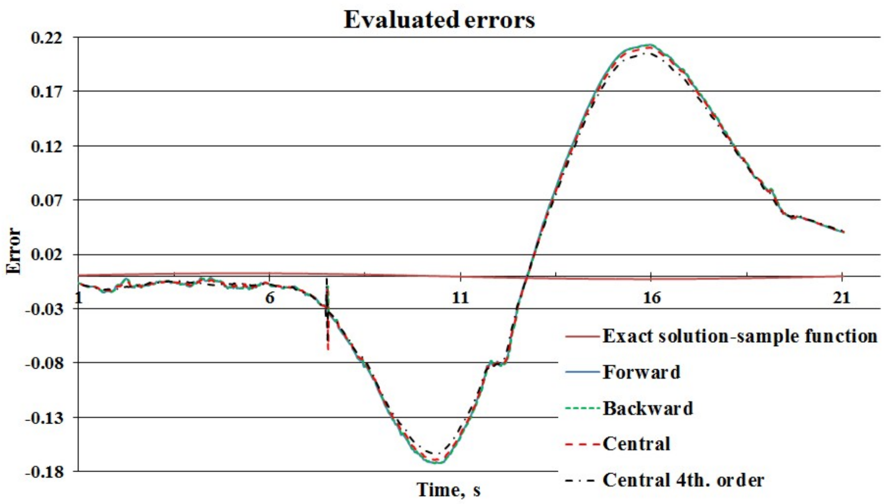

Fig. 3 Evaluated errors

The error of numerical differentiating for individual variants we evaluated with the next formula:

$$E(f)_{(i)} = {}_{exact}f'(x)_{(i)} - {}_{num}f'(x)_{(i)} \ . \tag{10}$$

Results are depicted in figure 3. The percentage expression of the average error is evaluated by:

$$E(f)_{(i)av\%} = \left[\frac{1}{n}\sum_{i=0}^{n} E(f)_{(i)}\right].100 \ . \tag{11}$$

CONCLUSIONS

In this paper we are dealing with the algorithm and the application of numerical differentiation. Applied methods were forward, backward a central a central 4th degree. Algorithm was written in the Microsoft Visual C# 2010 Professional programing language. The input data set was obtained from experimental measurement and processed with the *Dynstab* simulation software. Technical function had the stochastic character and was processed with the *Dynstab* software. For numerical differentiating purpose, we chose the functions of trajectory of center of gravity moving with respect to the y axis. The exact solution of differentiation we obtained from simulation program. According to the realized analysis we got the conclusion that the numerical differentiating of stochastic function is very unstable process. For the individual methods we evaluated the percentage average errors as follows. For forward we evaluated the error 19.19 %, for backward we evaluated the error 19.15 %, for central we evaluated the error 19.0 % and for central 4th degree we evaluated the error 18.7 %. Listed errors include also the error of the method as well as rounding errors. For comparison with generated sample periodical function $\sin(\boldsymbol{x})$ and its numerical differentiation, where we know an exact solution, is the average error $0.675 \cdot 10^{-6}$ %, what is classified as very high accuracy. In accordance with terms mentioned above, we can conclude that the stochastic characters of technical function have high degree of the participation on the instability and inaccuracy of the numerical differentiation.

Acknowledgement

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Original Paper

The sum of the series of reciprocals of the quadratic polynomials with double positive integer root

Radovan Potůček*

University of Defence, Faculty of Military Technology, Department of Mathematics and Physics, Brno, Czech Republic

ABSTRACT

This contribution, which is a follow-up to author's papers [3] and [4], deals with the series of reciprocals of the quadratic polynomials with double positive integer root. The formula for the sum of this kind of series expressed by means of harmonic numbers are derived and verified by several examples evaluated using the basic programming language of the computer algebra system Maple 16. There is stated another formula using generalized harmonic numbers, too. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: telescoping series, harmonic numbers, CAS Maple, Riemann zeta function

JEL CLASSIFICATION: I30

INTRODUCTION AND BASIC NOTIONS

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n \ .$$

* Corresponding author: Radovan Potůček, University of Defence, Faculty of Military, Technology Department of Mathematics and Physics, Kounicova 65, 662 10 Brno, Czech Republic. E-mail: Radovan.Potucek@unob.cz

The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a *sum* s and write $\sum_{k=1}^{\infty} a_k = s$.

The n-th *harmonic number* is the sum of the reciprocals of the first n natural numbers: $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n = \sum_{k=1}^{n} 1/k$. The *generalized harmonic number* of order n in power r is the sum

$$H_{n,r} = \sum_{k=1}^{n} \frac{1}{k^r}, \tag{1}$$

where $H_{n,1} = H_n$ are harmonic numbers. Every generalized harmonic number of order n in power m can be written as a function of generalized harmonic number of order n in power $m-1$ using formula (see [6])

$$H_{n,m} = \sum_{k=1}^{n-1} \frac{H_{k,m-1}}{k(k+1)} + \frac{H_{n,m-1}}{n}, \tag{2}$$

whence

$$H_{n,2} = \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n}. \tag{3}$$

From formula (1), where $r = 1,2$ and $n = 1,2,\ldots,10$, we get the following table 1.

Tab.1: Some harmonic and generalized harmonic numbers

n	1	2	3	4	5	6	7	8	9	10
H_n	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$
$H_{n,2}$	1	$\frac{5}{4}$	$\frac{49}{36}$	$\frac{205}{144}$	$\frac{5269}{3600}$	$\frac{5369}{3600}$	$\frac{266681}{176400}$	$\frac{1077749}{705600}$	$\frac{771817}{352800}$	$\frac{1968329}{1270080}$

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH DOUBLE POSITIVE INTEGER ROOT

We deal with the problem to determine the sum $s(a,a)$ of the series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty} \frac{1}{(k-a)^2}$$

for positive integers a, i.e. to determine the sum $s(1,1)$ of the series

$$\sum_{k=2}^{\infty} \frac{1}{(k-1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots, \tag{4}$$

the sum $s(2,2)$ of the series

$$\sum_{\substack{k=1\\k\neq 2}}^{\infty}\frac{1}{(k-2)^2}=\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=s(1,1)+1\,,$$

the sum $s(3,3)$ of the series

$$\sum_{\substack{k=1\\k\neq 3}}^{\infty}\frac{1}{(k-3)^2}=\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=s(1,1)+\left(1+\frac{1}{2^2}\right)=s(1,1)+\frac{5}{4},$$

etc. Clearly, we get the formula

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}=s(1,1)+s_{a-1}(1,1)\,, \tag{5}$$

where $s_{a-1}(1,1)$ is the $(a-1)$th partial sum of the series (4), and also the formula

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}=2s_{a-1}(1,1)+\sum_{k=a}^{\infty}\frac{1}{k^2}\,. \tag{6}$$

A problem to determine the sum $s(1,1)=1/1^2+1/2^2+1/3^2+1/4^2+\cdots$ is so called Basel problem. This problem was posed by *Pietro Mengoli* (1625-1686) in 1644. In 1689 Jacob Bernoulli (1654-1705) proved that the series $\sum_{k=1}^{\infty}1/k^2$ converges and its sum is less than 2. In 1737 *Leonhard Euler* (1707-1783) showed his famous result $\sum_{k=1}^{\infty}1/k^2=\pi^2/6$. This sum presents the value $\zeta(2)$ of the *Riemann zeta function*

$$\zeta(s)=\sum_{k=1}^{\infty}k^{-s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots\,.$$

The values of the n-th partial sum $s_n(1,1)=1/1^2+1/2^2+1/3^2+\cdots+1/n^2$ correspond to the values $H_{n,2}$, so their first ten values are presented in the third row of the table 1. Some another values of the n-th partial sums $s_n(1,1)$, computed by CAS Maple 16, are $s_{100}\doteq 1.634984$, $s_{1000}\doteq 1.643935$, $s_{10000}\doteq 1.644834$, $s_{100000}\doteq 1.644924$, $s_{1000000}\doteq 1.644933$, whereas the series $s(1,1)$ converges to the number $1.644934066\ldots$.

The partial sums $s_n(1,1)$, i.e. generalized harmonic numbers $H_{n,2}$ are also determined by the formula (see [5])

$$s_n(1,1)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(i-1)!\,(j-1)!}{(i+j)!}+\frac{3}{2}\sum_{l=1}^{n}\frac{1}{l^2\binom{2l}{l}}\,. \tag{7}$$

This surprising identity was derived by the contemporary brilliant amateur French mathematician Benoit Cloitre (see [1]).

According to formula (5) is

$$s(a,a)=\zeta(2)+H_{a-1,2}\,. \tag{8}$$

Using formulas (5) and (7) we get

Theorem 1 The series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2},$$

where $a>0$ is integer, has the sum

$$s(a,a)=\frac{\pi^2}{6}+\frac{1}{2}\left[\sum_{i=1}^{a-1}\sum_{j=1}^{a-1}\frac{(i-1)!\,(j-1)!}{(i+j)!}+3\sum_{l=1}^{a-1}\frac{1}{l^2\binom{2l}{l}}\right]. \tag{9}$$

Remark 1 In [2] it is stated the equality

$$\int_0^{\infty}\frac{te^{-Nt}}{e^t-1}\mathrm{d}t=\sum_{k=N+1}^{\infty}\frac{1}{k^2}, \tag{10}$$

which can be proved using a geometric sum-type expansion of the denominator and evaluation of the subsequent integrals by means of the integration by parts and L'Hôpital's Rule.

Using formula (6) we get

Theorem 2 The series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2},$$

where $a>0$ is integer, has the sum

$$s(a,a)=2H_{a-1,2}+\int_0^{\infty}\frac{te^{-(a-1)t}}{e^t-1}\mathrm{d}t\,. \tag{11}$$

Example 1 Evaluate the sum of the series

$$\sum_{\substack{k=1\\k\neq 5}}^{\infty}\frac{1}{(k-5)^2}=\frac{1}{4^2}+\frac{1}{3^2}+\frac{1}{2^2}+\frac{1}{1^2}+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

by formula **i)** (9), **ii)** (11), and **iii)** (8) and compare the obtained results.

Solution

i) The series has by formula (9) from Theorem 1, where $a=5$, the sum

$$s(5,5)=\frac{\pi^2}{6}+\frac{1}{2}\left[\sum_{i=1}^{4}(i-1)!\sum_{j=1}^{4}\frac{(j-1)!}{(i+j)!}+3\sum_{l=1}^{4}\frac{1}{l^2\binom{2l}{l}}\right]=\frac{\pi^2}{6}+\frac{S}{2}.$$

At first, we successive evaluate the sum S. We get

$$S=0!\left(\frac{0!}{2!}+\frac{1!}{3!}+\frac{2!}{4!}+\frac{3!}{5!}\right)+1!\left(\frac{0!}{3!}+\frac{1!}{4!}+\frac{2!}{5!}+\frac{3!}{6!}\right)+2!\left(\frac{0!}{4!}+\frac{1!}{5!}+\frac{2!}{6!}+\frac{3!}{7!}\right)$$
$$+3!\left(\frac{0!}{5!}+\frac{1!}{6!}+\frac{2!}{7!}+\frac{3!}{8!}\right)+3\left(\frac{1}{1\cdot\binom{2}{1}}+\frac{1}{4\cdot\binom{4}{2}}+\frac{1}{9\cdot\binom{6}{3}}+\frac{1}{16\cdot\binom{8}{4}}\right)=\frac{205}{72}.$$

Now, we have

$$s(5,5)=\frac{\pi^2}{6}+\frac{1}{2}\cdot\frac{205}{72}\doteq 3.0685451780.$$

ii) By formula (11) from Theorem 2, where $a=5$, by means of integration using Maple 16, we also get the required sum:

$$s(5,5)=2H_{4,2}+\int_0^\infty \frac{te^{-4t}}{e^t-1}\mathrm{d}t\doteq 2\cdot\frac{205}{144}-0.2211322959\doteq 3.0685451809.$$

iii) The third and much more easily way, how to determine the sum $s(5,5)$, is to use formula (8) and the value of $H_{a-1,2}=H_{4,2}=205/144$ from the table 1. So we immediately obtain the required result:

$$s(5,5)=\zeta(2)+H_{4,2}=\frac{\pi^2}{6}+\frac{205}{144}\doteq 3.0685451780.$$

Formulas (9) and (8) give the identical result 3.06854517807 and the result obtained by formula (11) differs from them only about $3\cdot 10^{-9}$.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(a,a)$ of the series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty}\frac{1}{(k-a)^2}$$

for $a=1,2,\ldots,10,50,99,100,500,999,1000$. We use on the one hand an approximative direct evaluation of the sum

$$s(a,a,t)=\sum_{\substack{k=1\\k\neq a}}^{t}\frac{1}{(k-a)^2},$$

where $t=10^6$, using the basic programming language of the CAS Maple 16, and on the other hand the formula (9) for evaluation the sum $s(a,a)$. We compare 16 pairs of these two ways obtained sums $s(a,a,10^6)$ and $s(a,a)$ to verify formula (9). We use following simple procedure rp2raapos and one for statement:

```
rp2raapos:=proc(a,t)
    local i,j,A,s1,s2,saa,sumaa;
    A:=a-1; s1:=0; s2:=0; saa:=0; sumaa:=0;
    for i from 1 to A do
        for j from 1 to A do
```

```
            s1:=s1+((i-1)!*(j-1)!)/((i+j)!);
        end do;
    end do;
    for i from 1 to A do
        s2:=s2+1/(i*i*binomial(2*i,i));
    end do;
    saa:=Pi*Pi/6+s1/2+3*s2/2;
    print("a=",a,": saa=",evalf[20](saa));
    for i from 1 to t do
        if i <> a then
        sumaa:=sumaa+1/((i-a)*(i-a));
        end if;
    end do;
    print("sumaa(",t,")=",evalf[20](sumaa));
    print("diff=",evalf[20](abs(sumaa-saa)));
end proc:
for a in [1,2,3,4,5,6,7,8,9,10,50,99,100,500,999,1000] do
    rp2raapos(a,1000000);
end do;
```

The approximative values of the sums $s(a,a)$ obtained by the procedure rp2raapos and rounded to 6 decimals, are written into the following table 2.

Tab. 2 Some approximative values of the sums $s = s(a,a)$

a	1	2	3	4	5	6	7	8
s	1.644934	2.644934	2.894934	3.006045	3.068545	3.108545	3.136323	3.156731
a	9	10	50	99	100	500	999	1000
s	3.172356	3.184701	3.269667	3.279716	3.279818	3.287866	3.288867	3.288868

Computation of 16 couples of the sums $s(a,a,10^6)$ and $s(a,a)$ took over 16 hours. The relative errors, i.e. the ratios $|[s(a,a) - s(a,a,10^6)]/s(a,a)|$, range between 10^{-7} for $a = 1$ and 10^{-9} for $a = 1000$.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the quadratic polynomials with double positive integer root a, i.e. with the series

$$\sum_{\substack{k=1\\k\neq a}}^{\infty} \frac{1}{(k-a)^2} \,.$$

We derived that the sum $s(a,a)$ of this series is given by the formula

$$s(a,a) = \frac{\pi^2}{6} + \frac{1}{2}\left[\sum_{i=1}^{a-1}\sum_{j=1}^{a-1} \frac{(i-1)!\,(j-1)!}{(i+j)!} + 3\sum_{l=1}^{a-1} \frac{1}{l^2\binom{2l}{l}}\right].$$

We verified this main result by computing 16 sums by using the CAS Maple 16.

Two another ways how to calculate the sum $s(a,a)$ is using the value of generalized harmonic number $H_{a-1,2}$ of order $a-1$ in power 2 and the improper integral

$$s(a,a) = 2H_{a-1,2} + \int_0^{\infty} \frac{te^{-(a-1)t}}{e^t - 1} \mathrm{d}t$$

or the short formula with the value of the generalized harmonic number $H_{a-1,2}$

$$s(a,a) = \frac{\pi}{2} + H_{a-1,2} \ .$$

The series of reciprocals of the quadratic polynomials with double positive integer root so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

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Original Paper

Comparison of study results in selected subjects depending on study program in Faculty of Economics and Management, Slovak University of Agriculture in Nitra

Radomíra Hornyák Gregáňová[1*], Miriam Pietriková[2]

[1] Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
[2] Slovak University of Agriculture, Faculty of Economics and Management, Department of Statistics and Operations Research, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic

ABSTRACT

Feedback is a crucial part of the educational process and the means by which by exam results and the quality of the educational process are being assessed. This article presents the analysis of the examination assessment results in Mathematics and Statistics in the Faculty of Economics and Management in the Slovak University of Agriculture in Nitra that depend upon the Bachelor Study Program. The main aim of this paper is to analyze the exam results in the compulsory subjects of Mathematics I, A, Mathematics II, B and Statistics. We have verified the following hypothesis: does the data dependence exist among the exam outcomes of the mentioned subjects that depend upon the bachelor study program. The following basic methods of the descriptive statistics and hypotheses testing were utilized in the assessment of the survey results. The existence of the statistically significant relations among the acquired assessments was verified by means of the χ^2–test.

KEYWORDS: study program, graduated subject, exam result, average mark, impact of student enlistment on acquired assessment

JEL CLASSIFICATION: I 21, C12

INTRODUCTION

The teaching of Mathematics and Statistics has a long tradition in the Faculty of Economics and Management in the Slovak University of Agriculture in Nitra (hereinafter referred to as FEM SPU). Mandatory subjects of Mathematics and Statistics provide the apparatus and

[*] Corresponding author: Radomíra Hornyák Gregáňová, Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
E-mail: radomira.greganova@uniag.sk

methods applied in the scientific activities in various areas. Content, methods and forms applied in the mathematical education in universities are changing in accordance with the actual requirements of scientific departments, new education trends and practice. Appropriate attitude of students towards Mathematics is conditioned by several factors. Therefore it is crucial to create it since the primary school [4], [5].

Mandatory subjects are taught within the bachelor study as follows: Mathematics I, A in the winter term of the 1st study year, Mathematics II, B in the summer term of the 1st study year and Statistics in the winter term of the 2nd study year. Department of Mathematics of FEM SPU in Nitra provides the subjects of Mathematics I,A, Mathematics II, B. Department of Statistics and Operations Research of the FEM SPU in Nitra secures the subject of Statistics.

The main goal of our research, as part of internal evaluation process at FEM SPU in Nitra, was to find out whether there exist statistically significant differences between student exam results in the mentioned subjects depending on the study program (SP).

MATERIAL AND METHODS

The statistical sample included all students from the Faculty of Economics and Management (of the Slovak University of Agriculture in Nitra), in particular, a group of students of selected the accredited study program in the bachelor degree. Students of the bachelor study in FEM SPU are offered to study in 8 study programs (SP), out of which 7 study programs have been selected (Table 1). Subjects of Mathematics I, A, Mathematics II, B and Statistics form the knowledge core in all selected study programs studied in the bachelor study in FEM SPU in Nitra. Basic knowledge and skills obtained from the compared subjects are further developed in the subsequent taught specialized subjects in FEM SPU in Nitra within the bachelor and as well as engineering study. The exam results of before mentioned subjects reflect the fact whether respectively how students were successful in these subjects. Those data have been drawn from UIS in academic years 2012/13, respectively. 2013/14 and processed through MS Excel and SAS.

Tab. 1 List of offered SPs in FEM SPU and their determination

EKP	Company economics	**MAP**	Company management
EMA	Economics and management of agro sector	**OBP**	Commercial entrepreneurship
MPA	International business with agrarian commodities	**UCT**	Accounting
IBA	International business with agrarian commodities – SP in the English language		

Source: authors

The main task of our research was to find out whether there exist statistically significant differences between student exam results in the mentioned subjects depending on the study program (SP). Study results of mandatory subjects of Mathematics I, A, Mathematics II, B and Statistics were assessed using the standard statistical methods.

The following basic methods of descriptive statistics and hypotheses testing were utilized in the assessment of survey results. The existence of statistically significant relations between acquired assessments was verified by mean of χ^2–test. The chi-square statistic is most appropriate for use with categorical variables, such as marital status [1].

Statistically demonstrated differences in the assessment were based on the significance of testing (*p*-value), presenting the error probability which is reached when the H_0 hypothesis is rejected even it is true. In case the p-value of testing characteristic is lower than 0.05, a null hypothesis about the equality of observed features is rejected and the difference in values of a statistical feature is considered as statistically significant [6].

In our case we dealt with the statistical samples of range *n* and analysed two statistical features – the first observed feature *X* presents student exam results classified according to study program and the second observed feature *Y* present the results of total assessment of student knowledge conducted in a regular term of before mentioned subjects.

We tested the following null hypothesis H_0: There in no dependence between the observed features *X* and *Y*. The alternate hypothesis H_1 as opposite: There is dependence between the observed features *X* and *Y*.

Pearson was looking for a simple statistic, a value that could be easily computed and that would indicate whether the results of an experiment deviated from expected results [2].

The statistics χ^2 is used as a testing criterion and is presented by the following ratio

$$\chi^2 = \sum_{i=1}^{m} \sum_{j=1}^{r} \frac{\left(a_i \cdot b_j - \left(a_i \cdot b_j\right)_0\right)^2}{\left(a_i \cdot b_j\right)_0}$$

The testing statistics χ^2 has the χ^2 - division with the number of variance levels (m-1)*(r-1) under the validity of testing hypothesis H_0. The testing hypothesis H_0 is rejected on the significance level α, if the value of testing criterion χ^2 exceeds the critical value $\chi^2(\alpha;(m-1)*(r-1))$. The critical value χ^2 can be found in the table of critical values [3] (we use abbreviation "CritV" in the table 4).

The applying of χ^2 goodness of fit test finds out that there exists the dependence between the compared features; therefore it is suitable to determine the intensity of such a dependence. Several measures were defined for the determination of dependence intensity between categorical features out of which the mostly used are Pearson's contingency coefficients. Pearson's coefficient of square contingency is defined as follows

$$C = \sqrt{\frac{\chi^2}{n + \chi^2}}$$

Disadvantage of such a constructed coefficient is that the maximum coefficient value is strongly influenced by the size of pivot table. This feature is removed in the following so called Adjusted Pearson's contingency coefficient

$$C_{adj} = \frac{C}{\min\left\{\sqrt{1-\frac{1}{r}};\ \sqrt{1-\frac{1}{m}};\right\}}.$$

This adjusted coefficient takes valued from the interval <0; 1> for a pivot table of optional size and values are mutually comparable [6].

The program Microsoft Excel 2010 was used for the realization of calculations and determination of critical values.

RESULTS AND DISCUSSION

The article compares the exam results from the subject Mathematics I, A (Mat I), Mathematics II, B (Mat II) a Statistics (Stat) aimed to find out whether there exist statistically significant differences in achieved students' assessment studied in individual study programs in FEM SPU.

We used the exam results from the academic years 2013/14 (Mat I a Mat II) and subsequently 2014/15 (Stat). Exam results from individual subjects are assessed by a standard scale ECTS: A(1), B(1,5), C(2), D(2,5), E(3) and FX(4).

Table 2 presents the assessment of student's successfulness in individual subjects in a regular term. As regards the total students' successfulness assessment in individual study programs of comparable subjects we can state that the students' share who were unsuccessful in the exam of a regular term from the subject Statistics is almost twice as high as in Mathematics, even all students who passed the education in the English language succeeded this subject and similarly as well as Mat II in a regular term. It is worth noting the comparison of students' successfulness EMA, out of whom 55.56 % was unsuccessful in the exam from Mat I in a regular term but in the subsequent subject Mat II the successfulness of such students reached even 94.44 % in a regular term. Totally the highest successfulness was achieved by the students of SP UCT in the subject Mat I (95.12 %), which simultaneously occurred in the total lowest average mark as regards the assessment of SP or as well as the compared subjects (Fig. 1, Tab. 3).

Tab. 2 Percentage of students' successfulness in subjects when taking the exam in a regular term

Subject		**Mat I**		**Mat II**		**Stat**		**Students number**
Assessment		**A - E**	**FX**	**A - E**	**FX**	**A - E**	**FX**	
Study program	**EKP**	85.06 %	14.94 %	91.95 %	8.05 %	68.97 %	31.03 %	**87**
	EMA	44.44 %	55.56 %	94.44 %	5.56 %	72.22 %	27.78 %	**18**
	IBA	75.00 %	25.00 %	**100.00 %**	**0.00 %**	**100.00 %**	**0.00 %**	**16**
	MAP	79.73 %	20.27 %	77.03 %	22.97 %	63.06 %	36.94 %	**74**
	MPA	72.73 %	27.27 %	81.82 %	18.18 %	72.73 %	27,27 %	**11**
	OBP	88.64 %	11.36 %	61.36 %	38.64 %	56.82 %	43.18 %	**44**
	UCT	95.12 %	4.88 %	87.80 %	12.20 %	80.49 %	19.51 %	**41**
Total FEM		**82.13 %**	**17.87 %**	**83.16 %**	**16.84 %**	**69.42 %**	**30.58 %**	**291**

If we assess the students successfulness from the point of mean mark (Tab. 3, Fig. 1) we find out that the mean mark is getting worse in the compared subjects (2.309; 2.318; 2.758) and as well the median value of mark with the term number spent in the university is arising.

The big differences exist between compared subjects concerning the students' successfulness assessment of mostly occurred results as well as within the result comparison from one subject among SPs. Students of SP EKP in both subjects of Mathematics mostly acquired the mark A, but in the subject Statistics they worsened as well as in the majority of other SP, and they were most often assessed with the mark FX.

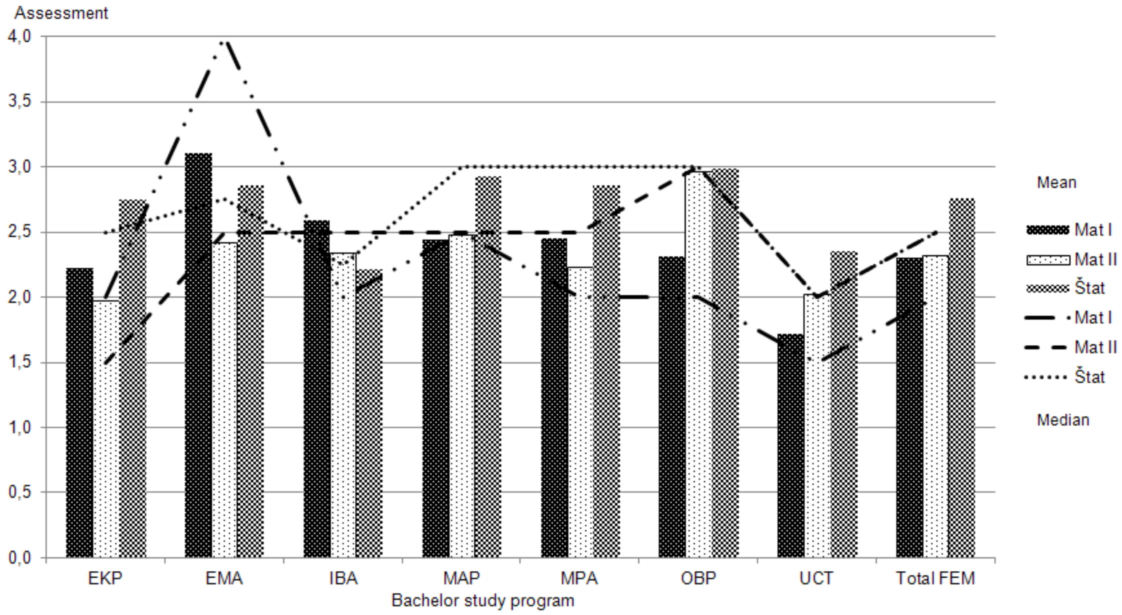

Fig. 1 Comparison of students' percentage in terms of mean and median assessment

As regards the students' successfulness of compared SP in individual subjects the best result were obtained by the students of Accounting in the subject Mat I and in the subject Mat II were only slightly worse. If we take into account that the majority of these students are graduate of secondary schools where the mathematics teaching does not belong to the main subject taught during the whole study period, so it is a surprising finding which is simultaneously confirmed by the results in the subject Statistics where the students of SP Accounting achieved the best results, as the median and modus of students' assessment reached the value 2 in comparison with other students.

Tab. 3 Comparison of students' successfulness in subjects

Subject	**Mat I**			**Mat II**			**Stat**		
SP	**Mean**	**Modus**	**Median**	**Mean**	**Modus**	**Median**	**Mean**	**Modus**	**Median**
EKP	2.230	1	2	1.977	1	1.5	2.753	4	2.5
EMA	3.111	4	4	2.417	2.5	2.5	2.861	4	2.75
IBA	2.594	-	2	2.344	3	2.5	2.219	3	2.25
MAP	2.446	2.5; 4	2.5	2.480	3; 4	2.5	2.926	4	3
MPA	2.455	2	2	2.227	1	2.5	2.864	3; 4	3
OBP	2.318	3	2	2.966	4	3	2.989	4	3
UCT	1.720	1	1.5	2.024	1	2	2.354	2	2
Total FEM	**2.309**	**1**	**2**	**2.318**	**1**	**2.5**	**2.758**	**4**	**2.5**

Based on the before mentioned we suppose the existence of differences in achieved assessment of compared subjects between students classified in several SPs. Examination was therefore focused on the difference determination in knowledge assessment in a regular exam term arising between students of individual SPs. As the subject Statistics A is characterised by the high percentage of unsuccessfulness therefore the final results were used as so called final students' assessments after the graduation of all three exam terms.

Tab. 4 Verification of difference and dependence existence

	Acquired assessment	**Value of testing statistics**	
	***p*-value**	χ^2	**CritV$_{0.05\ (0.01)}$**
SP vs. Mat I	0.002619	63.1266	43.7730 (50.8922)
SP vs. Mat II	0.000990	65.6792	
SP vs. Stat	0.037263	48.9548	

Table 4 presents the verification results of students' SP enlistment impact on the acquired assessment in three associate subjects. In case of subjects Mathematics I respectively II we can state the existence of statistically high influence of SP in the exam result as there exists only 0.26 % respectively 0.099 % probability of independence between the study program and final student's assessment. In case of the subject Statistics there exists 3.73 % risk that there is no mutual contingency between the observed features.

Based on the before mentioned findings we quantified the intensity of confirmed dependence by means of Pearson's contingency coefficient (Table 5).

Tab. 5 Quantification of analysed dependence tightness

Tightness level of assessed dependence	**Study program vs.**		
	Mat I	**Mat II**	**Stat**
Pearson's contingency coefficient	0.36218	0.42912	0.37948
Adjusted Pearson's contingency coefficient	0.39675	0.47007	0.41570

The coefficient values acquired the value higher than 0.33 for all analysed dependences which is generally considered as the limit between weak and medium strong dependence between compared categorical features. In all three cases we can state the existence of medium strong impact of studied SPs on the assessment result of compared subjects.

CONCLUSIONS

The comparison of students' result assessment in the exams from the subjects Mathematics I, II a Statistics graduated within the bachelor study in FEM SPU in Nitra found out the assessment differences. The proven impact is represented by the students' enlistment in study programs. Surprisingly the best results were achieved by the students of study program Accounting as regards the character of compared subjects where the majority of students were assessed by the mark „A“ in case of subjects Mat I, II. Students of this program achieve the best study results at high school and are better placed to be successful in the university study. Concerning the subject Statistics it was the mark „C“ in comparison with the mostly occurred mark of all students (FX). The analysis found out that there exists statistically proven impact of a study program on achieved students' assessment in the exam of selected subjects. In case of arranging from the weakest to the strongest impact the intensity size of such an impact is following: Mathematics I, Statistics and Mathematics II.

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Original Paper

Impact of education on the ability to use mathematical tools to solve financial tasks. A case study of Slovakia

Janka Drábeková[1*], Soňa Švecová[2]

[1] Slovak University of Agriculture in Nitra, Faculty of Economics and Management, Department of Mathematics, Nitra, Slovak Republic
[2] Borodáčova 5; 949 11 Nitra, Slovak Republic

ABSTRACT

Between financial literacy and mathematics has been found a high correlation according OECD study. In the Slovak Republic, relative performance in financial literacy was lower than expected at all levels of mathematics performance. The aim of the paper was to examine the impact of education focus on the ability to use mathematical tools to solve financial tasks on a sample of students from Slovakia. The examined sample consisted of 103 students from two faculties of the Slovak University of Agriculture in Nitra. We performed a questionnaire consisted of 7 multiple choice questions (financial tasks according to ISCED 2, ISCED 3A). We found statistically significant differences in the answers of respondents according to their university education focus (economic, non-economic). The most problematic area of the questionnaire was to calculate value added tax.

KEYWORDS: numerical skills, financial literacy, education focus, university students, Slovakia

JEL CLASSIFICATION: D10, D35, M34

INTRODUCTION

Finances are part of our daily life with great influence on it. Whether positive or negative way, it depends on the financial literacy of each of us. Nowadays financial literacy has become an universally necessary skill for life because dynamic and rapidly changing development on the financial markets coupled with impact of the global economic crisis makes the financial decisions and personal money management more challenging than ever before [16]. Financial literacy is knowledge and understanding of financial concept and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order

[*] Corresponding author: Janka Drábeková, Slovak University of Agriculture in Nitra, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic
E-mail: janka.drabekova@uniag.sk

to make effective decisions across a range of financial contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life [10]. The growth of financial literacy of population through financial education can be perceived as a tool to improve living conditions through better decision-making [3]. Higher level of financial literacy can increase level of living standard [17]. Increasing consumer financial literacy is a public policy objective to improve welfare through better decision making [4, 18]. Researches have been shown that levels of financial literacy worldwide are unacceptably low [1], particularly among young people [10]. In 2008, the OECD established the organization named the International Network on Financial Education, which is directly focused on support of financial education in the OECD countries [7]. Due to the recognizing an importance of financial literacy, a growing number of countries have developed and have implemented national strategies for financial education in order to improve the financial literacy of their populations in general, often with a particular focus on younger generations [5, 10].

In 2008, Ministry of Education, Science, Research and Sport of the Slovak Republic emphasized the importance of financial literacy development by formulating the National Standard of Financial Literacy in 2008 [17]. It is the initial document for incorporating financial literacy into the school education programs in the Slovak republic [8]. This document was updated to version 1.1 in september 2014 [9]. It that time methodology was also developed for incorporation and application of financial literacy topics into The National Education Programs – Mathematics ISCED 2, ISCED 3A [14, 15].

The ability to use mathematical tools to solve numerical tasks in financial decision making (numerical literacy) goes “hand to hand” with financial literacy [7]. On average across the 13 OECD countries and economies, the correlation between financial literacy and mathematics was 0.83 [11], which indicates that financial literacy was strongly correlated with mathematics. Some basic knowledge of mathematics is necessary to develop proficiency in financial literacy [11]. Conversely, interest in financial matters and financial literacy competencies can also support the development of mathematics and reading skills as well as provide a potentially engaging, real-life context to other school subjects [10].

The aim of the paper was to examine the impact of education focus on the ability to use mathematical tools to solve financial tasks on a sample of students from Slovakia.

MATERIAL AND METHODS

The examined sample consisted of 103 students from two faculties of the Slovak University of Agriculture in Nitra (SAU). We selected the Faculty of Economics and Management (FEM, 50 students) and the Faculty of Biotechnology and Food Sciences (FBFS, 53 students). We wanted to compare numerical ability of students focused on economic studies and students focused on non-economic studies. The examined sample consisted of first-year students of the university, 42 graduated from secondary grammar schools (SGS), 26 graduated from business colleges (BC) and 35 graduated from secondary vocational colleges (SVC, technical, chemical, agricultural). The sample structure is shown in table 1.

We performed a questionnaire consisted of 7 multiple choice questions. Questions were formulated as financial tasks. We focused on The National Educational Program Mathematics ISCED 2, ISCED 3A. The tasks were focused on the ability to calculate value added tax, to calculate gross salary, to use of exchange list, to compare the offers of insurance companies, to use simple and compound interest, to understanding the links between the interest rate and the length of the interest period. Students solved the tasks in questionnaire in 20 minutes and they were allowed to use a calculator.

Tab. 1 Numbers of students participating in research

	Faculty of Economics and Management		Faculty of Biotechnology and Food Sciences		total
	women	*men*	*women*	*men*	
secondary grammar schools	8	2	21	11	42
business colleges	23	3	0	0	26
secondary vocational colleges	8	6	18	3	35
total	39	11	39	14	103

We created index of successfulness of respondents for each question according to selected determinants. It is an average score of correct answers of respondents. The highest possible $I-SR$ value can be 1, the lowest 0. We calculated $I-SR$ by the formulas:

1) Index of selected determinants (FEM, FBFS, SGS, BC, SVC, women, men, questions):

$$I-SR=\frac{Number\ \ of\ \ correct\ \ answers\ \ of\ \ respondents}{Number\ \ of\ \ respondents\ according\ to\ determinant}$$

2) Total index:

$$I-SR=\frac{Number\ \ of\ \ correct\ \ answers\ \ of\ \ respondents}{\left(Number\ of\ \ questions\right)\cdot\left(Number\ \ of\ \ respondents\ according\ to\ determinant\right)}$$

We used SAS software to realized statistical analysis of obtained data. We created contingency tables, because they provide a basic view of the interrelation between two or more variables and can help find interactions between them. Analysis of contingency tables includes chi-square tests and measures of association.

RESULTS AND DISCUSSION

The index of successfulness of respondents in each question according to selected determinants (education focus, graduated school, gender, question) is shown in table 2.

Tab. 2 Index of successfulness of respondents according to selected determinants

Determinants	*Questions*							
	1	2	3	4	5	6	7	Total
FEM	0.400	0.760	0.400	0.820	0.720	0.720	0.740	0.651
FBFS	0.019	0.377	0.623	0.868	0.434	0.321	0.547	0.456
SGS	0.119	0.429	0.619	0.952	0.429	0.405	0.619	0.510
BC	0.500	0.846	0.346	0.654	0.731	0.615	0.692	0.626
SVC	0.086	0.514	0.514	0.857	0.629	0.571	0.629	0.543
women	0.218	0.564	0.487	0.821	0.551	0.474	0.654	0.538
men	0.160	0.560	0.600	0.920	0.640	0.640	0.600	0.589
question	0.204	0.563	0.515	0.845	0.573	0.515	0.641	0.551

The overall successfulness of our sample measured by total $I-SR$ was 55.1 %. It means that is average each respondent answered correctly more than 3 answers out of 7.

As can be seen in table 2 the main differences in $I-SR$ calculated for selected determinants are in education focus. The $I-SR$ of the students of the Faculty of Economics and Management (FEM) was 65.1 %. It was the best result. The $I-SR$ of the students of the Faculty of Biotechnology and Food Sciences (FBFS) was 45.6 %. Twenty percentage point difference was probably due to the time proceedings research. The research took place in the summer semester, when the students of FEM already passed the first tests of the winter semester. The obtained values show a beneficial effect of economic education on the ability of students to solve financial tasks. The $I-SR$ of the respondents from secondary grammar school (SGS) was 51 %. The second best result of the success was measured at graduated of business colleges (62.6 %). It should be noted that all graduated of business colleges (BC) were students of FEM (see table 1). The $I-SR$ of the respondents from secondary vocational colleges (SVC) was 54.3 %. We noticed only minimal differences in correct answers of women and men similar as the Slovak Banking Association when examined the financial literacy of the population of Slovakia [13]. However Bhushan & Medury [1], Ivančová [6], Krechovská [7], Poliaková [12], Tóth et al. [16,17] described in their research higher financial literacy of men as women.

Tab. 3 Contingency table of faculties and questions

Table of faculty by question

Frequency Expected Percent Row Pct Col Pct	faculty	question 1c	1w	2c	2w	3c	3w	4c	4w	5c	5w	6c	6w	7c	7w	Total
Frequency	FEM	20	30	38	12	20	30	41	9	36	14	36	14	37	13	350
Expected		10.194	39.806	28.155	21.845	25.728	24.272	42.233	7.767	28.641	21.359	25.728	24.272	32.039	17.961	
Percent		2.77	4.16	5.27	1.66	2.77	4.16	5.69	1.25	4.99	1.94	4.99	1.94	5.13	1.80	48.54
Row Pct		5.71	8.57	10.86	3.43	5.71	8.57	11.71	2.57	10.29	4.00	10.29	4.00	10.57	3.71	
Col Pct		95.24	36.59	65.52	26.67	37.74	60.00	47.13	56.25	61.02	31.82	67.92	28.00	56.06	35.14	
Frequency	FBFS	1	52	20	33	33	20	46	7	23	30	17	36	29	24	371
Expected		10.806	42.194	29.845	23.155	27.272	25.728	44.767	8.233	30.359	22.641	27.272	25.728	33.961	19.039	
Percent		0.14	7.21	2.77	4.58	4.58	2.77	6.38	0.97	3.19	4.16	2.36	4.99	4.02	3.33	51.46
Row Pct		0.27	14.02	5.39	8.89	8.89	5.39	12.40	1.89	6.20	8.09	4.58	9.70	7.82	6.47	
Col Pct		4.76	63.41	34.48	73.33	62.26	40.00	52.87	43.75	38.98	68.18	32.08	72.00	43.94	64.86	
Frequency	Total	21	82	58	45	53	50	87	16	59	44	53	50	66	37	721
Percent		2.91	11.37	8.04	6.24	7.35	6.93	12.07	2.22	8.18	6.10	7.35	6.93	9.15	5.13	100.00

c - correct answers,   w - wrong answers

We wanted to find interactions between education focus and correct answers, therefore we created contingency tables by software SAS. Values in table 3 present expected frequency, the table percentage, row percentage, and column percentage according of faculties by questions. The differences in answers of students of FEM and FBFS we can see in table 2 and table 3. The students of FBFS were the most successful at question 3 and 4 (use of exchange list, compare the offers of insurance companies). These students could not calculate value added tax, there is only one correct answer to first task. The students of FEM could not calculate value added tax (question 1) neither correct use of exchange list (question 3). The students of FEM solved all other tasks with a success rate more than 72 % (table 2). According to our research in 2014 [2], researched students were also not very successful in calculating value added tax. Maximum score of solved tasks was obtained by only 7 % students in grade 10, by 47 % students in grade 12 of Piarist High School in Nitra and by 4 % first-year students of FEM in Nitra.

We tested an association between faculties and answers (correct, wrong). Using the chi-square test we verified the differences between actual and expected frequencies. The chi-square statistic is 73.0694 with 13 degree of freedom. The associated p-value is < 0.0001, which means that there is significant association between faculties and answers to questions. The measures of association (Phi Coefficient, Contingency Coefficient, and Cramer's V) have a value 0.3183, it is a mediate association. The focus of university education (whether it is economic or not) has statistically significant influence on the correctness of the respondents' answers. Authors Ivančová [6], Krechovská [7], Tóth et al. [16, 17] studied the effect of economic education focus on financial literacy of university students. They found as we do, that economic focus of education has an impact on financial literacy. But differences between economic and non-economic samples of students were not as great as ours.

Tab. 4 Contingency table of secondary schools and questions

Table of sschool by question (Frequency / Expected / Percent / Row Pct / Col Pct)

sschool	Statistic	1c	1w	2c	2w	3c	3w	4c	4w	5c	5w	6c	6w	7c	7w	Total
SGS	Frequency	5	37	18	24	26	16	40	2	18	24	17	25	26	16	294
	Expected	8.5631	33.437	23.65	18.35	21.612	20.388	35.476	6.5243	24.058	17.942	21.612	20.388	26.913	15.087	
	Percent	0.69	5.13	2.50	3.33	3.61	2.22	5.55	0.28	2.50	3.33	2.36	3.47	3.61	2.22	40.78
	Row Pct	1.70	12.59	6.12	8.16	8.84	5.44	13.61	0.68	6.12	8.16	5.78	8.50	8.84	5.44	
	Col Pct	23.81	45.12	31.03	53.33	49.06	32.00	45.98	12.50	30.51	54.55	32.08	50.00	39.39	43.24	
BC	Frequency	13	13	22	4	9	17	17	9	19	7	16	10	18	8	182
	Expected	5.301	20.699	14.641	11.359	13.379	12.621	21.961	4.0388	14.893	11.107	13.379	12.621	16.66	9.3398	
	Percent	1.80	1.80	3.05	0.55	1.25	2.36	2.36	1.25	2.64	0.97	2.22	1.39	2.50	1.11	25.24
	Row Pct	7.14	7.14	12.09	2.20	4.95	9.34	9.34	4.95	10.44	3.85	8.79	5.49	9.89	4.40	
	Col Pct	61.90	15.85	37.93	8.89	16.98	34.00	19.54	56.25	32.20	15.91	30.19	20.00	27.27	21.62	
SVC	Frequency	3	32	18	17	18	17	30	5	22	13	20	15	22	13	245
	Expected	7.1359	27.864	19.709	15.291	18.01	16.99	29.563	5.4369	20.049	14.951	18.01	16.99	22.427	12.573	
	Percent	0.42	4.44	2.50	2.36	2.50	2.36	4.16	0.69	3.05	1.80	2.77	2.08	3.05	1.80	33.98
	Row Pct	1.22	13.06	7.35	6.94	7.35	6.94	12.24	2.04	8.98	5.31	8.16	6.12	8.98	5.31	
	Col Pct	14.29	39.02	31.03	37.78	33.96	34.00	34.48	31.25	37.29	29.55	37.74	30.00	33.33	35.14	
Total	Frequency	21	82	58	45	53	50	87	16	59	44	53	50	66	37	721
	Percent	2.91	11.37	8.04	6.24	7.35	6.93	12.07	2.22	8.18	6.10	7.35	6.93	9.15	5.13	100.00

c - correct answers, w - wrong answers

Table 4 presents the contingency table of graduated secondary schools by answers (correct, wrong). The differences in answers of graduated of secondary grammar schools, graduated of business colleges and graduated of secondary vocational colleges we can see in table 2 and table 4. We tested an association between secondary schools and answers to questions. The chi-square statistic (a value of 57.1874 with 26 DF) provides evidence of an association between secondary school and students answers (p = 0.0004). The measures of association (Phi Coefficient, Contingency Coefficient, and Cramer's V) have a value between 0.1855 - 0.2623, it is weak association.

CONCLUSIONS

The most problematic area of the test was to calculate value added tax. This fact is troubling, because everybody applies the VAT every day during shopping in stores. The obtained values of successfulness show a beneficial effect of economic focus of university studies on the ability of students to solve financial tasks. There were statistically significant differences in the answers of respondents according to their education focus. The overall performance of FEM students was 65.1%, in average each respondent answered correctly more than 4.5 answers out of 7. We verified the relationship between type of secondary school and answers to questions, but did not found statistically significant influence secondary schools on the correctness of the respondents' answers.

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Original Paper

Integral calculus at technical universities

Tomáš, Pechočiak*, **Norbert Kecskés**

Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic

ABSTRACT

Faculty of Engineering of the Slovak University of Agriculture belongs among the six of its faculties. This faculty is attended by students from different types of secondary schools. The study outlines of the faculty contain two semesters in mathematics which should provide students with a theoretical base for further study of major subjects. Students learn here the basics of differential and integral calculus. In the subject of Mathematics 2 they learn to evaluate indefinite and definite integrals. In our paper we focus on how students cope with this part of mathematics and what causes most problems in this area.

KEYWORDS: integral calculus, indefinite and definite integral, correlation coefficient

JEL CLASSIFICATION: C02, C11, I210

INTRODUCTION

Slovak University of Agriculture in Nitra has six faculties. One of them is the Faculty of Engineering. This faculty is attended by students from different types of secondary schools. This is the reason why their level of mathematical knowledge and proficiency is very different. In the first semester in subject of Mathematics 1 they learn the fundamentals of linear algebra (vectors, matrices, determinants, solution of systems of linear equations), functions of one variable (their definition, properties, graphs), differential calculus (limits, derivatives and their use), functions of two variables and basics of the theory of complex numbers. The second semester in subject of Mathematics 2 starts with the integral calculus, i.e. indefinite and definite integral. The origin of this part of mathematics, integral or infinitesimal calculus is associated with scientists of the 17th and 18th century, especially Isaac Newton and Gottfried Wilhelm Leibniz [1]. Students should learn to evaluate the

* Corresponding author: Tomas Pechočiak, Slovak University of Agriculture, Faculty of Economics and Management, Department of Mathematics, Tr. A. Hlinku 2, 949 76 Nitra, Slovak Republic, e-mail: tomas.pechociak@uniag.sk

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selected types of indefinite and definite integrals by various methods: using formulas and theorems on integration, decomposition, substitution method and by parts method.

MATERIAL AND METHODS

As we mentioned above, in the summer semester students of the Faculty of Engineering learn the integral calculus of functions of one variable. The task is to find a function whose derivative is equal to the given function, i.e. to evaluate the indefinite integral of the given function. We say that who has learned to differentiate, should be able to learn to integrate, because the integration formulas can be derived from the differentiation formulas.

After completion of the subject and its practicing at seminars students were submitted to testing. These tests were taken by 158 students. We divided them into three groups, with each group having different problems to solve. The first group consisted of 60 students, the second of 52 and the third of 46 students. Each group of students solved three problems, where one could be solved directly, one by substitution and one by by-parts method, and wherein one of the three problems was a definite integral. Students were informed about the types of test problems without indication which method was suitable for a respective problem. Students could use integration formulas, which were on a separate paper in front of them.

Students in the first group solved these problems:

1. $\int \sin^5 x \cdot \cos^2 x \, dx$,
2. $\int \frac{5x}{\sin^2 x} dx$,
3. $\int_1^2 \left(\frac{x-3}{x} \right)^2 dx$.

In the second group students solved:

1. $\int 4x \cdot \ln x \, dx$,
2. $\int \frac{8}{4-x^2} dx$,
3. $\int_0^{\pi} \sin^3 x \cdot \cos^2 x \, dx$.

The third group was given these problems:

1. $\int \frac{\ln^2 x}{2x} dx$,
2. $\int_{-1}^{2} (3x \quad - 2x + 5) \, dx$,
3. $\int 3x \cdot \sin x \, dx$.

Each example was evaluated by points from 0 to 10 (we took into account the solution steps or partial results), so the student could get together 30 points. After checking of results we decided to find out how students solved individual problems. We used mathematical methods of descriptive statistics. In Excel we created a database in a way that we arranged the results of individual problems for each student from each group in a column. From these data we calculated the mean scores from individual problems and the average number of points for each student. We also examined the correlation coefficients among the problems within the group. Thus, we wanted to verify the assumption that if a student can solve one problem, he/she should be able to solve the next, provided there is a strong correlation between them. Strong correlation is if the correlation coefficient falls into $\left(0.\overline{6}, 1.0\right\rangle$, mean for the values from $\left(0.\overline{3}, 0.\overline{6}\right\rangle$ and weak for the values from $\left\langle 0.0, 0.\overline{3}\right\rangle$. We assumed that if students had been preparing for the test systematically, the correlation should be higher, i.e. students should cope with all three problems comparably. On the other hand, those students who had not prepared or do not have a "talent for mathematics" probably fail to solve all three problems and the correlation coefficient will be again of higher value.

RESULTS AND DISCUSSION

After processing the obtained data, we came to following results. In the first group (consisting of 60 students), the average percentage of problem solving is shown in Table. 1:

Tab. 1 Average number of points and percentage for the problems in the first group

	Average number of points	Percentage
1. problem	$5.1\overline{3}$	$51.\overline{3}$
2. problem	4.00	40.0
3. problem	3.40	34.0

Students from this group best solved the first problem (integral of trigonometric functions, solved by substitution), and worst the third one. The most of the difficulties were caused by powering of $(x-3)^2$ and its subsequent separation into 3 fractions and their modification.
The second group had 52 students and the results are listed in Table 2:

Tab. 2 Average number of points and percentage for the problems in the second group

	Average number of points	Percentage
1. problem	4.07	40.7
2. problem	5.15	51.5
3. problem	3.00	30.0

Students from this group best solved the second problem, which could be solved by means of a formula after some simplification. The worst, in comparison with the first group (the same type of problem) was the solution of the third problem.

The scores from third group (46 students) are listed in Tab.3:

Tab. 3 Average number of points and percentage for the problems in the third group

	Average number of points	Percentage
1. problem	7.00	70.0
2. problem	5.43	54.3
3. problem	6.04	60.4

The third group of students solved the given problems relatively best. Here the students best solved the first problem (by substitution), the worst was the solution of the third problem, which we considered to be the easiest one.

In Tab. 4 we listed the average number of points and percentage of students in individual groups.

Tab. 4 Average number of points and percentage of students in individual groups

	Average number of points	Percentage
1. group	12.53	41.77
2. group	12.23	40.77
3. group	18.48	61.16

As it is clear from this table, the most successful students were in the third group, in the first and the second they were on about the same level. The overall average percentage per student was 14.41 points, representing only 48.04 % success rate. In the next part of the research we investigated the correlation coefficients among the problems in the group. We wanted to determine that if a student knows (or doesn't know) how to solve one problem, if he/she can (or cannot) solve the other, provided there is a strong correlation between these two problems.

Tab. 5 Correlation coefficients among the problems in the first group

	1. problem	2. problem	3. problem
1. problem		0.636568	0.670681
2. problem	0.636568		0.551653
3. problem	0.670681	0.551653	

Tab. 6 Correlation coefficients among the problems in the second group

	1. problem	2. problem	3. problem
1. problem		0.335681	0.70796
2. problem	0.335681		0.555459
3. problem	0.70796	0.555459	

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Tab. 7 Correlation coefficients among the problems in the third group

	1. problem	2. problem	3. problem
1. problem		0.631043	0.579669
2. problem	0.631043		0.650371
3. problem	0.579669	0.650371	

Weak correlation is, if the correlation coefficient falls into $\langle 0.0,\, 0.\overline{3} \rangle$, mean for the values from $\left(0.\overline{3},\, 0.\overline{6} \right\rangle$ and strong for the values from $\left(0.\overline{6},\, 1.0 \right\rangle$. Table 5 shows the correlation coefficients among the problems in the first group, table 6 for the second group and table 7 reveals the relationships among the problems in the third group.

As it is shown in Tab. 5, there is a strong correlation between the first and third problem in the first group. It means that if a student knew (or didn't know) how to solve the first problem, he/she knew (or didn't know) how to solve the third one. There is a mean correlation between the 1. and 2. problem and between the 2. and 3. problem. In the second group there was a strong correlation between the 1. and 3. problem, mean correlation between the 2. and 3. problem and between the 1. and 2. problem there was also a mean correlation but on the lower bound. Finally, in the third group there was a correlation between the problems almost balanced on the middle level.

CONCLUSIONS

In the paper we tried to find out how students understood the chapter on integral calculus, that is, if they were able to evaluate indefinite and definite integrals of a function of one variable by various methods: using formulas and theorems on integration, decomposition, substitution method and by parts method. Some students could not decide, or incorrectly chose the integration method. It was obvious that some of them did not prepare for the test or failed to master the curriculum. One of the reasons could be the fact that these students came from secondary schools, where mathematics is taught only in a limited amount. There were 22 students who scored between 0 and 5 points, while the first and second group it was 8 students, respectively and in the third group it was only 6 students. In contrast, those who earned 25 or more points were 28. In the first group 6 students, in the second 4 and 18 in the third group, while 8 students scored full. The total percentage of problems solving was just a little over 48%, which we consider to be a very low level. The success rate of solution of individual problems by students is on the middle level, the mean correlation coefficient is 0.6, which is close to the upper bound of the interval $\left(0.\overline{3},\, 0.\overline{6} \right\rangle$ and this has confirmed our assumptions. In conclusion, we can state that we will have to turn closer attention to this part of mathematics in order to considerably increase the level of knowledge and skills of students.

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