On topological soft sets

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Abstract

In this paper, we have established topological soft sets over generalized topological spaces and topological spaces, and studied its structural properties. We have derived a topological soft set in any given topological space, and from this point of view, we have given necessary and sufficient condition for homeomorphic Alexandroff spaces using topological soft set technique. At last, we have derived a topological soft set using closed sets in any topological space and we have given necessary and sufficient condition for arbitrary homeomorphic topological spaces using them.

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1. INTRODUCTION

Topology is a major area of mathematics concerned with the most properties of space. Topological spaces show up naturally in almost every branch of mathematics and other sciences. This makes topology one of the great unifying ideas of mathematics. In 2002, Császár defined the concept of generalized topology which is the family closed under arbitrary union of subsets of a set in [6]. At the same time, many scientists use the topology or the generalized topology to understand and model the real world. But this is not always easy. Each phenomenon in the real world can not always be modeled by classical methods. Many theories have been developed for dealing with uncertainties. One of them is the soft set theory which has a rich potential for applications in several directions. The notion of soft set theory was initiated by Molodtsov [16] in 1999. As in [2, 18], this theory have been applied to many area in mathematics, information science and computer science. Of course, most of the mathematicians studied the topological structure on soft sets. Firstly, Shabir and Naz established the concept of soft topological space which is defined over an initial universe with a fixed set of parameters in [20]. They introduced soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point and soft separation axioms. In [8], Ge et al. gave some relations between topology and soft set theory. They presented some characterizations of trivial

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(discrete, T_0, T_1, T_D, and R_0) topological spaces by the null soft set, the absolute soft set, the identical soft set and so on. In [11], Li et al. prepared new characterization between topology and soft set theory. They defined the concept of topological soft set which is a soft set over initial universe such that for each parameter this is topology over initial universe, and studied relations between approximation spaces. But this definition of topological soft set is very specific and restricted. If we have a topological space or more general generalized topological space, then we can get a more convenient way types of topological soft sets. For this reason, in this paper, we re-define the concept of topological soft set in any generalized topological space without any restriction and study its properties.

2. PRELIMINARIES

As the preliminary information, which is necessary to study, give some definitions and properties.

2.1. Soft Set Theory

Let U be an initial universe, E be a set of parameters, $\mathscr{P}(U)$ be the power set of U, and $A \subseteq E$. Molodtsov [16] defined the soft set in the following manner:

DEFINITION 2.1. [16] A pair (F,A) is called a soft set over U where F is a mapping given by $F : A \to \mathscr{P}(U)$.

Some set-theoretic operations defined by [12, 18]

DEFINITION 2.2. [18] For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G,B) and is denoted by $(F,A)\widetilde{\subset}(G,B)$ if

(i) $A \subset B$ and,

(ii) $\forall a \in A$, $F(a) \subset G(a)$.

DEFINITION 2.3. [18] Two soft sets (F,A) and (G,B) over a common universe U are said soft equal if (F,A) is a soft subset of (G,B), and (G,B) is a soft subset of (F,A).

DEFINITION 2.4. [18] Let (F,A) and (G,B) be two soft sets over a common universe U such that $A \cap B \neq \varnothing$. The intersection of (F,A) and (G,B) is denoted by

$(F,A)\widetilde{\cap}(G,B)$, and is defined as $(F,A)\widetilde{\cap}(G,B) = (H,C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

DEFINITION 2.5. [12] The union of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,C) , denoted by $(F,A)\widetilde{\cup}(G,B) = (H,C)$, where $C = A \cup B$, and $\forall c \in C$,

$$H(c) = \begin{cases} F(c) & ,\text{if } c \in A - B \\ G(c) & ,\text{if } c \in B - A \\ F(c) \cup G(c) & ,\text{if } c \in A \cap B \end{cases}$$

DEFINITION 2.6. [4] Let U be an initial universe set, E be the universe set of parameters, and $A \subset E$.

(i) (F,A) is called a relative null soft set (with respect to the parameter set A), denoted by Φ_A, if $F(a) = \varnothing$ for all $a \in A$.

(ii) (F,A) called a relative whole soft set (with respect to the parameter set A), denoted by $\mathscr{U}_A$, if $F(a) = U$ for all $a \in A$.

The relative whole soft set $\mathscr{U}_E$ with respect to the universe set of parameters E is called the absolute soft set over U.

DEFINITION 2.7. [12] Let (F,A) and (G,B) be two soft sets over the common universe U. Then (F,A) **AND** (G,B) denoted by $(F,A) \wedge (G,B)$ and is defined by $(F,A) \wedge (G,B) = (H, A \times B)$ where $H((a,b)) = F(a) \cap G(b)$, for all $(a,b) \in A \times B$.

DEFINITION 2.8. [12] Let (F,A) and (G,B) be two soft sets over the common universe U. Then (F,A) **OR** (G,B) denoted by $(F,A) \vee (G,B)$ and is defined by $(F,A) \vee (G,B) = (H, A \times B)$ where $H((a,b)) = F(a) \cup G(b)$, for all $(a,b) \in A \times B$.

DEFINITION 2.9. [18] The complement of a soft set (F,A) is denoted by $(F,A)^c$ and is defined by $(F,A)^c = (F^c, A)$, where $F^c : A \to \mathscr{P}(U)$ is a mapping given by $F^c(a) = U - F(a)$ for all $a \in A$.

In [14], Min has introduced the concept of similarity between soft sets and investigated some properties. He defined the concept of similarity between soft sets as follows:

DEFINITION 2.10. [14] Let (F,A) and (G,B) be soft sets over a common universe set U. Then (F,A) is similar to (G,B) (simply $(F,A) \cong (G,B)$) if there exists a bijection function $\phi : A \to B$ such that $F(x) = (G \circ \phi)(x)$ for every $x \in A$, where $(G \circ \phi)(x) = G(\phi(x))$.

2.2. Soft Mappings

Kharal and Ahmad [9], defined the notion of a mapping on soft classes and studied several properties of images and inverse images of soft sets supported by examples and counterexamples. They defined that image and inverse image of a soft set under a soft functions as follows:

DEFINITION 2.11. [9] Let U_1, U_2 be initial universes, E_1, E_2 be parameters sets, φ be a function from U_1 to U_2 and ψ be a function from E_1 to E_2. Then the pair (φ, ψ) is called *soft function* from $S(U_1, E_1)$ to $S(U_2, E_2)$. The *image* of each $(F,A) \in S(U_1, E_1)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)(F,A) = (\varphi F, \psi(A))$ and is defined as following;

$$(\varphi F)(\beta) = \begin{cases} \varphi\left(\bigcup_{\alpha \in \psi^{-1}(\beta)\cap A} F(\alpha)\right) & , \psi^{-1}(\beta) \cap A \neq \varnothing \\ \varnothing & , \text{otherwise} \end{cases}$$

for each $\beta \in \psi(A)$.

Similarly, the *inverse image* of each $(G,B) \in S(U_2, E_2)$ under the soft function (φ, ψ) is denoted by $(\varphi, \psi)^{-1}(G,B) = (\varphi^{-1}G, \psi^{-1}(B))$ and is defined as following;

$$(\varphi^{-1}G)(\alpha) = \begin{cases} \varphi^{-1}(G(\psi(\alpha))) & , \psi(\alpha) \in B \\ \varnothing & , \text{otherwise} \end{cases}$$

for each $\alpha \in \psi^{-1}(B)$.

2.3. Soft Equality Relations

Qin and Hong introduced the concept of soft equality relations $\approx_S$ and $\approx^S$ in [19]. Definition of soft equalities are given as follows;

DEFINITION 2.12. [19] Let (F,A), (G,B) be two soft sets over U.

($\approx_S$). (F,A) is called soft equal to (G,B), denoted by $(F,A) \approx_S (G,B)$, if for all $e \in A \cup B$, $e \in A \cap B$ implies $F(e) = G(e)$, $e \in A - B$ implies $F(e) = \varnothing$, and $e \in B - A$ implies $G(e) = \varnothing$.

($\approx^S$). (F,A) is called soft equal to (G,B), denoted by $(F,A) \approx^S (G,B)$, if for all $e \in A \cup B$, $e \in A \cap B$ implies $F(e) = G(e)$, $e \in A - B$ implies $F(e) = U$, and $e \in B - A$ implies $G(e) = U$.

Note that these relations are equivalence relations on the family of all soft sets over U.

3. TOPOLOGICAL SOFT SETS

In topology, topologists defined some generalization of open sets such as semi-openness [10], pre-openness [13], α-openness [17], β-openness [1] etc. In [5] and [6], Császár gave a frame of generalized openness as follows;

Let U be a set and γ be a map from $\mathscr{P}(U)$ into itself. Suppose that γ is monotonic, i.e. $A \subseteq B \subseteq U$ implies $\gamma(A) \subseteq \gamma(B)$. A set $A \subseteq U$ is called *γ-open* if and only if $A \subseteq \gamma(A)$, and $B \subseteq U$ is called *γ-closed* if its complement is γ-open. He showed that any union of γ-open sets is γ-open in [5]. So, he defined the notion of generalized topology which is constituted by γ-open sets in [6]. The collection $\mathfrak{g}$ of subsets of U is called *generalized topology* on U if and only if $\varnothing \in \mathfrak{g}$ and $G_i \in \mathfrak{g}$ for $i \in I \neq \varnothing$ implies $G = \bigcup_{i\in I} G_i \in \mathfrak{g}$. If $\mathfrak{o}$ is a topology on U in the usual sense and denote by iA the $\mathfrak{o}$-interior $intA$, by cA the $\mathfrak{o}$-closure clA, we obtain as important particular cases the collection $\mathfrak{o}$ of all open sets ($\gamma = i$), $\mathfrak{so}$ of all semi-open sets ($\gamma = ci$), $\mathfrak{po}$ of all preopen sets ($\gamma = ic$), $\beta\mathfrak{o}$ of all β-open sets ($\gamma = cic$), $\alpha\mathfrak{o}$ of all α-open sets ($\gamma = ici$) [6]. Therefore, we obtain the relationship

$$\mathfrak{o} \subset \alpha\mathfrak{o} \subset \mathfrak{so} \subset \beta\mathfrak{o} \subset \mathfrak{g} \tag{1}$$

and

$$\mathfrak{o} \subset \alpha\mathfrak{o} \subset \mathfrak{po} \subset \mathfrak{g} \tag{2}$$

from [5–7]. In a similar manner, we obtain that K is closed iff $c(K) \subset K$, K is semi-closed iff $ic(K) \subset K$, K is pre-closed iff $ci(K) \subset K$, K is α-closed iff $cic(K) \subset K$ and K is β-closed iff $ici(K) \subset K$. Then we denote the families of all type closed sets as $\mathfrak{c}$, $\mathfrak{sc}$, $\mathfrak{pc}$, $\alpha\mathfrak{c}$ and $\beta\mathfrak{c}$, respectively. We also denote the family of generalized closed set in the sense of Császár by $\mathfrak{gc}$.

We noted that $\mathfrak{o}$ and $\alpha\mathfrak{o}$ are topologies on U and we obtain that $\mathfrak{o}$ is coarser than $\alpha\mathfrak{o}$ from the above relationships.

In [6], Császár define the concept of generalized continuity of any function from a generalized topological space to another. So, given two generalized topological spaces $(U_1, \mathfrak{g}_1)$ and $(U_2, \mathfrak{g}_2)$ and a mapping $f : U_1 \to U_2$, f is $(\mathfrak{g}_1, \mathfrak{g}_2)$-*continuous* iff $G \in \mathfrak{g}_2$ implies that $f^{-1}(G) \in \mathfrak{g}_1$ [6]. We shortly say that f is $\mathfrak{g}$-continuous if f is generalized continuous. We call that f is $\mathfrak{g}$-*open* ($\mathfrak{g}$-*closed*, respectively) if $G \in \mathfrak{g}$ implies that $f(G) \in \mathfrak{g}'$ (or $G \in \mathfrak{gc} \Rightarrow f(G) \in \mathfrak{g}'\mathfrak{c}$). We also call that f is $\mathfrak{g}$-*homeomorphism* if f is bijective, $\mathfrak{g}$-continuous and $\mathfrak{g}$-open (or $\mathfrak{g}$-closed).

Now, we can define γ-topological soft set over an initial universe space as the follows;

DEFINITION 3.1. Let U be an initial universe and γ be a monotonic map from $\mathscr{P}(U)$ into itself. (F,A) be a soft set over U where $A \subseteq E$. (F,A) is called *γ-topological soft set* if $F(e)$ is γ-open set in U i.e. $F(e) \subseteq \gamma(F(e))$ for all $e \in A$.

In Definition 3.1, in the most general sense, we can conclude that there is a generalized topology on U because of that obtained by γ-open sets. Therefore, we can explain this definition as "(F,A) *is a* $\mathfrak{g}$*-topological soft set over U if and only if* $F(e) \in \mathfrak{g}$ *for all* $e \in A$ *where* $\mathfrak{g}$ *is a generalized topology on* U". If we take the γ as interior operator, then we get the classical topology on U. In [15], W. K. Min defined that (F,A) is *open soft set* over the topological space U such that $F(e)$ is open for all $e \in A$ i.e. $F(e) \subseteq i(F(e))$ for all $e \in A$. From his definition, we obtain that (F,A) is an $\mathfrak{o}$-topological soft set over U in our sense. Clearly, every open soft set is an $\mathfrak{o}$-topological soft set. Furthermore, we can characterize given any γ-topological soft set according to type of openness. For this, we can give following definition:

DEFINITION 3.2. Let U be an initial universe, E be a parameters set, $A \subseteq E$ and (F,A) be a soft set over U.

(1) (F,A) is called $\mathfrak{g}$*-topological soft set* if $F(e) \in \mathfrak{g}$ for all $e \in A$.

(2) (F,A) is called $\mathfrak{o}$*-topological soft set* if $F(e) \in \mathfrak{o}$ for all $e \in A$.

(3) (F,A) is called $\mathfrak{so}$*-topological soft set* if $F(e) \in \mathfrak{so}$ for all $e \in A$.

(4) (F,A) is called $\mathfrak{po}$*-topological soft set* if $F(e) \in \mathfrak{po}$ for all $e \in A$.

(5) (F,A) is called $\alpha\mathfrak{o}$*-topological soft set* if $F(e) \in \alpha\mathfrak{o}$ for all $e \in A$.

(6) (F,A) is called $\beta\mathfrak{o}$*-topological soft set* if $F(e) \in \beta\mathfrak{o}$ for all $e \in A$.

We also can characterize the topological soft sets according to type of closedness as follows.

DEFINITION 3.3. Let U be an initial universe, E be a parameters set, $A \subseteq E$ and (F,A) be a soft set over U.

(1) (F,A) is called $\mathfrak{gc}$*-topological soft set* if $F(e) \in \mathfrak{gc}$ for all $e \in A$.

(2) (F,A) is called $\mathfrak{c}$*-topological soft set* if $F(e) \in \mathfrak{c}$ for all $e \in A$.

(3) (F,A) is called $\mathfrak{sc}$*-topological soft set* if $F(e) \in \mathfrak{sc}$ for all $e \in A$.

(4) (F,A) is called $\mathfrak{pc}$-*topological soft set* if $F(e) \in \mathfrak{pc}$ for all $e \in A$.

(5) (F,A) is called $\alpha\mathfrak{c}$-*topological soft set* if $F(e) \in \alpha\mathfrak{c}$ for all $e \in A$.

(6) (F,A) is called $\beta\mathfrak{c}$-*topological soft set* if $F(e) \in \beta\mathfrak{c}$ for all $e \in A$.

EXAMPLE 3.4. Let $\mathbb{R}$ be the set of real numbers with the usual topology and $\mathbb{N}$ be the parameters set. Define the mappings $F : \mathbb{N} \to \mathscr{P}(\mathbb{R})$ such that $F(n) = (n-1, n+1)$ and $G : \mathbb{N} \to \mathscr{P}(\mathbb{R})$ such that $G(n) = (n-1, n+1]$. Then $(F,\mathbb{N})$ is an $\mathfrak{o}$-topological soft set and $(G,\mathbb{N})$ is a $\mathfrak{so}$-topological soft sets over $\mathbb{R}$, respectively. Moreover, if we define the mapping $H : \mathbb{N} \to \mathscr{P}(\mathbb{R})$ such that $H(n) = [n-1, n+1]$, then we obtain that $(H,\mathbb{N})$ is a $\mathfrak{c}$-topological soft set over $\mathbb{R}$.

EXAMPLE 3.5. Let $U = \{a,b,c\}$ and $\mathfrak{o} = \{\varnothing, U, \{a,b\}, \{b,c\}, \{b\}\}$ be a topology on U. Let $E = \{1,2,3,4,5\}$ and $(F,A) = \{1 = \{b\}, 2 = \{b,c\}\}$ be a soft set over U with $A \subseteq E$. Then (F,A) is a $\mathfrak{o}$-topological soft set over U.

Now, we discuss the obtained results.

Firstly, soft subset of a γ-topological soft set may not be γ-topological in general. If we specifically give the following example for $\mathfrak{o}$-topological soft sets, we can generalize other type.

EXAMPLE 3.6. From Example 3.5, we have the $\mathfrak{o}$-topological soft set (F,A). Although, $(G,B) = \{2 = \{c\}\}$ is a soft subset of (F,A), it is not $\mathfrak{o}$-topological.

THEOREM 3.7. Let (F,A) be a soft set over U, where $A \subseteq E$. Then we have

$$\begin{aligned} (F,A) \text{ is } \mathfrak{o}\text{-topological} &\Rightarrow (F,A) \text{ is } \alpha\mathfrak{o}\text{-topological} \\ &\Rightarrow (F,A) \text{ is } \mathfrak{so}\text{-topological} \\ &\Rightarrow (F,A) \text{ is } \beta\mathfrak{o}\text{-topological} \\ &\Rightarrow (F,A) \text{ is } \mathfrak{g}\text{-topological} \end{aligned}$$

and

$$\begin{aligned} (F,A) \text{ is } \mathfrak{o}\text{-topological} &\Rightarrow (F,A) \text{ is } \alpha\mathfrak{o}\text{-topological} \\ &\Rightarrow (F,A) \text{ is } \mathfrak{po}\text{-topological} \\ &\Rightarrow (F,A) \text{ is } \mathfrak{g}\text{-topological} \end{aligned}$$

PROOF. From Equation (1) and Equation (2), it is obvious.

THEOREM 3.8. If (F,A) is a $\mathfrak{g}$-topological soft set, then its complement $(F,A)^c$ is a $\mathfrak{gc}$-topological soft set.

PROOF. Obvious.

THEOREM 3.9. If (F,A) and (G,B) are $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological), then $(F,A)\widetilde{\cap}(G,B)$ is also $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological).

PROOF. From Definition 2.4, we obtain the soft set $(H,C) = (F,A)\widetilde{\cap}(G,B)$, such that $C = A\cap B$ and $H(e) = F(e)\cap G(e)$ for all $e\in C$. Since (F,A) and (G,B) are topological, $F(e)$ and $G(e)$ in $\mathfrak{o}$ (or $\alpha\mathfrak{o}$) for all $e\in C$, respectively. So, $H(e)\in\mathfrak{o}$ $(H(e)\in\alpha\mathfrak{o})$ for all $e\in A\cap B$. Thus (H,C) is an $\mathfrak{o}$-topological soft set (or $\alpha\mathfrak{o}$-topological soft set) over U .

THEOREM 3.10. If (F,A) and (G,B) are $\mathfrak{g}$-topological, then $(F,A)\widetilde{\cup}(G,B)$ is also $\mathfrak{g}-$topological.

PROOF. Suppose that, $(F,A)\widetilde{\cup}(G,B) = (H,C)$. Since (F,A) and (G,B) are $\mathfrak{g}$-topological soft sets, we have $H(c) = F(c)\in\mathfrak{g}$ for each $c\in A-B$, $H(c) = G(c)\in\mathfrak{g}$ for each $c\in B-A$ and $H(c) = F(c)\cup G(c)\in\mathfrak{g}$ for each $c\in A\cap B$ from Definition 2.5. Thus, (H,C) is a $\mathfrak{g}$-topological soft set.

THEOREM 3.11. Null and absolute soft sets are $\mathfrak{g}$-topological.

PROOF. From Definition 2.6 and Definition 3.2, it is obvious.

THEOREM 3.12. If (F,A) and (G,B) are $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological), then $(F,A)\wedge(G,B)$ is also $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological).

PROOF. From Definition 2.7, we have the soft set $(H,A\times B) = (F,A)\wedge(G,B)$, such that $C = A\times B$ and $H(a,b) = F(a)\cap G(b)$ for all $(a,b)\in A\times B$. Since (F,A) and (G,B) are $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological), then $(H,A\times B)$ is also $\mathfrak{o}$-topological (or $\alpha\mathfrak{o}$-topological).

THEOREM 3.13. If (F,A) and (G,B) are $\mathfrak{g}$-topological, then $(F,A)\vee(G,B)$ is also $\mathfrak{g}$-topological.

PROOF. From Definition 2.8 and Definition 3.2, the proof of this theorem is similar to the proof of above theorem.

THEOREM 3.14. If (F,A) and (G,B) are $\mathfrak{gc}$-topological, then $(F,A)\widetilde{\cap}(G,B)$ is also $\mathfrak{gc}$-topological.

PROOF. Suppose that (F,A) and (G,B) are $\mathfrak{gc}$-topological soft set over U and

$(F,A)\widetilde{\cap}(G,B) = (H,C)$. Then $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$. Since (F,A) and (G,B) are $\mathfrak{gc}$-topological, $F(a), G(b) \in \mathfrak{gc}$ for all $a \in A$ and $b \in B$. So we obtain that $(F(a))^c, (G(b))^c \in \mathfrak{g}$ for all $a \in A$ and $b \in B$. Since $\mathfrak{g}$ is a generalized topology on U. For all $c \in C$, $(F(c))^c \cup (G(c))^c \in \mathfrak{g}$. Then, from the rules of de Morgan, we obtain that $[F(c) \cap G(c)]^c \in \mathfrak{g}$. Hence $F(c) \cap G(c) \in \mathfrak{gc}$ for all $c \in C$. Thus (H,C) is a $\mathfrak{gc}$-topological soft set over U.

THEOREM 3.15. If (F,A) similar to (G,B) and (F,A) is a $\mathfrak{g}$-topological soft set, then (G,B) is also $\mathfrak{g}$-topological.

PROOF. By the definition of similarity (Definition 2.10), we have a bijection $\phi : A \to B$ such that $F(a) = (G \circ \phi)(a)$ for all $a \in A$. Now, we need to see $G(b) \in \mathfrak{g}$ for all $b \in B$. Since ϕ is a bijection, then for all $b \in B$, there exist $a \in A$ such that $\phi(a) = b$. So, for all $b \in B$, we obtain

$$G(b) = G(\phi(a)) = (G \circ \phi)(a) = F(a).$$

Since $F(a) \in \mathfrak{g}$ for all $a \in A$, then $G(b) \in \mathfrak{g}$ for all $b \in B$. Thus (G,B) is a topological soft set over U.

THEOREM 3.16. Let (F,A) and (G,B) be two soft sets over (U,τ).

(a) If $(F,A) \approx_S (G,B)$ and (F,A) is $\mathfrak{g}$-topological, then (G,B) is also $\mathfrak{g}$-topological.

(b) If $(F,A) \approx^S (G,B)$ and (F,A) is $\mathfrak{g}$-topological, then (G,B) is also $\mathfrak{g}$-topological.

PROOF. Proof of (a) and (b) is obvious from Definition 2.12 and Definition 3.2.

THEOREM 3.17. Let $(U_1, \mathfrak{g}_1)$ and $(U_2, \mathfrak{g}_2)$ be generalized topological spaces, $\varphi : U_1 \to U_2$ and $\psi : E_1 \to E_2$ be functions and (F,A) be a soft set over U_1. If φ is a $\mathfrak{g}$-open function and (F,A) is a $\mathfrak{g}$-topological soft set, then its image $(\varphi,\psi)(F,A)$ is a $\mathfrak{g}$-topological soft set over U_2.

PROOF. Since $\varnothing \in \mathfrak{g}_2$. If $(\varphi F)(\beta) = \varnothing$ for each $\beta \in \psi(A)$, then $(\varphi F, \psi(A))$ is $\mathfrak{g}$-topological.

Suppose that $(\varphi F)(\beta) \neq \varnothing$. So, $(\varphi F)(\beta) = \varphi\left(\bigcup_{\alpha \in \psi^{-1}(\beta) \cap A} F(\alpha)\right)$ for each $\beta \in \psi(A)$. Since (F,A) is $\mathfrak{g}$-topological and φ is a $\mathfrak{g}$-open function, then $\bigcup_{\alpha \in \psi^{-1}(\beta) \cap A} F(\alpha) \in \mathfrak{g}_2$. Thus $(\varphi,\psi)(F,A)$ is a $\mathfrak{g}$-topological soft set over U_2.

THEOREM 3.18. Let $(U_1, \mathfrak{g}_1)$ and $(U_2, \mathfrak{g}_2)$ be generalized topological spaces, $\varphi : U_1 \to U_2$ and $\psi : E_1 \to E_2$ be functions and (G,B) be a soft set over U_2. If φ is a $\mathfrak{g}$-continuous function and (G,B) is a $\mathfrak{g}$-topological soft set, then its inverse image

$(\varphi,\psi)^{-1}(G,B)$ is a $\mathfrak{g}$-topological soft set over U_1.

PROOF. Since $\varnothing \in \mathfrak{g}_1$. If $(\varphi^{-1}G)(\alpha) = \varnothing$ for each $\alpha \in \psi^{-1}(B)$, then $(\varphi^{-1}G, \psi^{-1}(B))$ is $\mathfrak{g}$-topological. Otherwise, $(\varphi^{-1}G)(\alpha) = \varphi^{-1}(G(\psi(\alpha)))$ for each $\alpha \in \psi^{-1}(B)$. Since (G,B) is $\mathfrak{g}$-topological and φ is $\mathfrak{g}$-continuous, we obtain that $(\varphi,\psi)^{-1}(G,B)$ is a $\mathfrak{g}$-topological soft set over U_1.

Min [14] has established similarity between two soft sets over fixed initial universe U as the above-mentioned. We can give following definition as generalization of similarity relation between soft sets over two different generalized topological spaces using $\mathfrak{g}$-homeomorphism.

DEFINITION 3.19. Let $(U_1,\mathfrak{g}_1)$ and $(U_2,\mathfrak{g}_2)$ be two generalized topological spaces, (F,A) be a $\mathfrak{g}$-topological soft set over U_1 with $A \subseteq E_1$ and (G,B) be a $\mathfrak{g}$-topological soft set over U_2 with $B \subseteq E_2$ and $\varphi : U_1 \to U_2$ be a $\mathfrak{g}$-homeomorphism. We called that (F,A) is homeomorphically similar to (G,B) if there exists a bijection $\phi : A \to B$ such that $(\varphi \circ F)(e) = (G \circ \phi)(e)$ for each $e \in A$, and denoted by $(F,A) \approxeq (G,B)$.

It is easily seen that if we take the identity function $1_U : (U,\mathfrak{g}) \to (U,\mathfrak{g})$, then we have Definition 2.10.

EXAMPLE 3.20. Let $U_1 = \{a,b,c\}$ and $U_2 = \{x,y,z\}$ be initial universes, and $\mathfrak{o}_1 = \{\varnothing, U_1, \{a\}, \{b\}, \{c\}\}$ and $\mathfrak{o}_2 = \{\varnothing, U_2, \{x\}, \{y\}, \{z\}\}$ be topologies on U_1 and U_2, respectively. Let $\varphi : U_1 \to U_2$ be an $\mathfrak{o}$-homeomorphism such that $\varphi = \{(a,y),(b,x),(c,z)\}$. And let $(F,A) = \{1 = \{a\}, 2 = \{c\}\}$ and $(G,B) = \{9 = \{y\}, 10 = \{x\}\}$ be topological soft sets over U_1 and U_2 respectively. So if we define the bijection $\phi : A \to B$ such that $\phi = \{(1,9),(2,10)\}$, then we obtain;

$$(\varphi \circ F)(1) = \varphi(F(1)) = \varphi(\{a\}) = \{y\} = G(9) = G(\phi(1)) = (G \circ \phi)(1)$$

and

$$(\varphi \circ F)(2) = \varphi(F(2)) = \varphi(\{c\}) = \{z\} = G(10) = G(\phi(2)) = (G \circ \phi)(2).$$

Consequently, $(F,A) \approxeq (G,B)$.

THEOREM 3.21. Let U_1 and U_2 be generalized topological spaces.

(a) If any (F,A) $\mathfrak{g}$-homeomorphically similar to the null soft set Φ_B with respect to B, then (F,A) is also null.

(b) If any (F,A) $\mathfrak{g}$-homeomorphically similar to the absolute soft set $\mathscr{U}_B$ with respect to B, then (F,A) is also absolute.

PROOF. (a) Assume that $\Phi_B = (G,B)$. Since $(F,A) \approxeq \Phi_B = (G,B)$, there exists a bijection $\phi : A \to B$ such that $(\varphi \circ F)(a) = (G \circ \phi)(a)$ for each $a \in A$ where φ is a homeomorphism. Since (G,B) is null, we have

$$\begin{aligned} (\varphi \circ F)(a) &= (G \circ \phi)(a) \\ (\varphi \circ F)(a) &= \varnothing \\ \varphi(F(a)) &= \varnothing \\ F(a) &= \varnothing \end{aligned}$$

for each $a \in A$. Thus (F,A) is null soft set over U_1.

Proof of (b) is done in a similar way to (a).

In [8], Ge et al. gave some notation for soft sets in a topological space. Their investigations are given as follows;

Let $(U,\mathfrak{o})$ be a topological space, then

(1) (I,U) denotes an identical soft set, where $I(x) = \{x\}$ for each $x \in U$.

(2) (N,U) denotes a neighborhood soft set, where $N(x) = B_x$ for each $x \in U$.

(3) (D,U) denotes a derived soft set, where $D(x) = \{x\}'$ for each $x \in U$.

(4) (C,U) denotes a closure soft set, where $C(x) = \overline{\{x\}}$ for each $x \in U$.

(5) (D^c,U) denotes a derived-complement soft set, where $D^c(x) = (\{x\}^c)'$ for each $x \in U$.

(6) (C^c,U) denotes a closure-complement soft set, where $C^c(x) = \overline{(\{x\}^c)}$ for each $x \in U$.

(7) (D^2,U) denotes a bi-derived soft set, where $D^2(x) = (\{x\}')'$ for each $x \in U$.

(8) (CD,U) denotes a closure-derived soft set, where $CD(x) = \overline{(\{x\}')}$ for each $x \in U$.

We can obtain some results using above notations.

Note that, the neighborhood soft set (N,U) is obviously topological if $N(x) = B_x \in \mathfrak{o}$ for each $x \in U$.

THEOREM 3.22. Let $(U,\mathfrak{o})$ be a topological space and (I,U) be the identical soft set over U. If (I,U) $\mathfrak{o}$-topological soft set over U, then $\mathfrak{o} = \mathscr{P}(U)$.

PROOF. Obvious.

THEOREM 3.23. Let $(U,\mathfrak{o})$ be a topological space. If (I,U) similar to (C,U), then ϕ is identity and $\mathfrak{o} = \mathscr{P}(U)$.

PROOF. If (I,U) similar to (C,U), then there exists a bijection $\phi : U \to U$ such that $(C \circ \phi)(x) = I(x)$ for each $x \in U$. Therefore $C(\phi(x)) = \{x\}$ and $\overline{\{\phi(x)\}} = \{x\}$ for each $x \in U$, so $\overline{\{x\}} = \{x\}$. Consequently ϕ is identity and $\mathfrak{o} = \mathscr{P}(U)$.

From Definition 3.19 and Ge et al.'s construction for topological spaces in [8], we obtain following theorem.

THEOREM 3.24. Let $(U_1,\mathfrak{o}_1)$ and $(U_2,\mathfrak{o}_2)$ be topological spaces. If U_1 homeomorphic to U_2, then

(1) $(I,U_1) \approxeq (I,U_2)$,

(2) $(N,U_1) \approxeq (N,U_2)$,

(3) $(D,U_1) \approxeq (D,U_2)$,

(4) $(C,U_1) \approxeq (C,U_2)$,

(5) $(D^c,U_1) \approxeq (D^c,U_2)$,

(6) $(C^c,U_1) \approxeq (C^c,U_2)$,

(7) $(D^2,U_1) \approxeq (D^2,U_2)$,

(8) $(CD,U_1) \approxeq (CD,U_2)$.

PROOF. We prove only (4). Other implications is proved by similar way. If U_1 homeomorphic to U_2, then there exists a homeomorphism φ from the topological space U_1 to the topological space U_2. Therefore, we have $\varphi(\overline{X}) = \overline{\varphi(X)}$ for each $X \subseteq U_1$. Hence, we obtain

$$(\varphi^* \circ C)(x) = \varphi^*(C(x)) = \varphi^*(\overline{\{x\}}) = \overline{\{\varphi(x)\}} = C(\varphi(x)) = (C \circ \varphi)(x)$$

for all $x \in U_1$, where φ^* is a function induced by φ from $\mathscr{P}(U_1)$ to $\mathscr{P}(U_2)$.

In [11], Li et al. gave a relation between topological spaces and soft sets. They defined the concept of topological soft set, restricted form than our definition, as follows;

DEFINITION 3.25. [11] Let (F,A) be a soft set over U. Then (F,A) is called topological, if $\{F(a) \mid a \in A\}$ is a topology on U.

We can give some soft set theoretic results using this definition.

THEOREM 3.26. If (F,A) and (G,B) topological soft set in the sense of Definition 3.25 and (F,A) is similar to (G,B), then the topologies induced by (F,A) and (G,B) are same.

PROOF. It is obvious from Definition 2.10.

In [20], Shabir and Naz gave the concept of soft topology over an initial universe U as follows;

DEFINITION 3.27. [20] Let $\widetilde{\mathfrak{o}}$ be the collection of soft sets over U, then $\widetilde{\mathfrak{o}}$ is said to be a soft topology on U if

(1) $\Phi, \widetilde{U}$ belong to $\widetilde{\mathfrak{o}}$

(2) the union of any number of soft sets in $\widetilde{\mathfrak{o}}$ belongs to $\widetilde{\mathfrak{o}}$

(3) the intersection of any two soft sets in $\widetilde{\mathfrak{o}}$ belongs to $\widetilde{\mathfrak{o}}$.

The triplet $(U, \widetilde{\mathfrak{o}}, E)$ is called a soft topological space over U. The members of $\widetilde{\mathfrak{o}}$ are said to be soft open sets in U.

In [20], Shabir and Naz demonstrated that if $(U, \widetilde{\mathfrak{o}}, E)$ is a soft topological space over U, then the family $\mathfrak{o}_e = \{F(e) \mid (F,E) \in \widetilde{\mathfrak{o}}\}$ for each $e \in E$, defines a topology on U. Therefore if we have a soft topology $\widetilde{\mathfrak{o}}$ over U, then every member of $\widetilde{\mathfrak{o}}$ is a $\mathfrak{o}$-topological soft set over U.

Note that, if we have a γ-topological soft set (F,A) over U then $F(e) \subseteq \gamma(F(e))$ for all $e \in A$ from Definition 3.1. So, this implies that $(F,A) \widetilde{\subset} \gamma(F,A)$. Therefore, we can say that (F,A) is γ-open soft set in Császár' s sense. Now, let $(U, \mathfrak{g})$ be generalized topological spaces and E be fixed parameter set. Consider the family

$$\widetilde{\mathfrak{g}} = \{(F,E) \mid (\forall e \in E)(F(e) \in \mathfrak{g})\}.$$

Then we have following theorem.

THEOREM 3.28. $(U, \widetilde{\mathfrak{g}}, E)$ is a generalized soft topological space.

PROOF. Since $\mathfrak{g}$ is a generalized topology on U, then $\varnothing \in \mathfrak{g}$. At that case, for all $e \in E$, $F(e) = \varnothing \in \mathfrak{g}$. So, we obtain $(F,E) = \widetilde{\Phi} \in \widetilde{\mathfrak{g}}$. Therewithal, consider the subfamily $\{(F_i,E) \mid i \in I\} \subset \widetilde{\mathfrak{g}}$. For all $i \in I$ and $e \in E$, $F_i(e) \in \mathfrak{g}$. Since $\mathfrak{g}$ is a generalized topology on U, then for all $e \in E$, $\bigcup_{i \in I} F_i(e) \in \mathfrak{g}$. Thus, we obtain $\widetilde{\bigcup}_{i \in I}(F_i, E) \in \widetilde{\mathfrak{g}}$. Hence, $\widetilde{\mathfrak{g}}$ is a generalized soft topology on U.

Particularly, if we take topological space instead of generalized topological space, we get soft topology on this way i.e.

$$\widetilde{\mathfrak{o}} = \{(F,E) \mid (\forall e \in E)(F(e) \in \mathfrak{o})\}$$

is a soft topology on U. As a result, we can achieve a soft topology from topological soft sets on any given topological space.

In [16], Molodtsov pointed out that every topological space is considered a soft set. Let $(U,\mathfrak{o})$ be a topological space. Define the mapping $T : U \to \mathscr{P}(\mathfrak{o})$ such that $\forall u \in U$, $T(u) = \{V \in \mathfrak{o} \mid u \in V\}$ i.e. $T(u)$ is the family of open neighborhoods of u. Thus, (T,U) is a soft set over $\mathfrak{o}$.

THEOREM 3.29. Let $\mathfrak{o}$ and $\mathfrak{o}'$ be topologies on U, and (T,U) and (T',U) be $\mathfrak{o}$-topological soft sets over $\mathfrak{o}$ and $\mathfrak{o}'$ respectively. If $(T,U) \cong (T',U)$, then $\mathfrak{o} = \mathfrak{o}'$.

PROOF. Suppose that $A \in \mathfrak{o}$. There is $x \in A \in \mathfrak{o}$. So, A is a open neighborhood of x i.e. $A \in T(x)$. Since $(T,U) \cong (T',U)$, there exist the bijective function $\phi : U \to U$ such that $T(x) = (T' \circ \phi)(x)$ for each $x \in X$. Therefore, $A \in T(x) = (T' \circ \phi)(x)$. We obtain that A is a open neighborhood of $\phi(x)$. Thus $A \in \mathfrak{o}'$.

Similar to the other inclusion is shown. Thus $\mathfrak{o} = \mathfrak{o}'$

COROLLARY 3.30. If $(T,U) \cong (T',U)$, then $\phi : U \to U$ is an identical homeomorphism.

In [3], P. Alexandroff introduced the *Alexandroff space* which is a topological space such that the intersection of every family of open sets is open (or equivalently every point has a minimal neighborhood). These spaces are also called *finitely generated spaces* since their topology is uniquely determined by the family of all finite subspaces. Alexandroff spaces can be viewed as a generalization of finite topological spaces.

With the above expression, we can obtain a soft set over a universe if we have a Alexandroff topology on this universe. At the same time, starting from the above expression over the universe, we can get an $\mathfrak{o}$-topological soft set over the universe as follows. We know that if $T : U \to \mathscr{P}(\mathfrak{o})$ such that $\forall u \in U$, $T(u) = \{A \in \mathfrak{o} \mid u \in A\}$ then (T,U) is a soft set over $\mathfrak{o}$. If we define the mapping $T^* : U \to \mathscr{P}(U)$ such that $T^*(u) = \bigcap\{A \in \mathfrak{o} \mid u \in A\}$, we obtain a soft set over U, and it is an $\mathfrak{o}$-topological soft set since the intersection of open neighborhoods is open in Alexandroff spaces. Thus for all soft sets (T,U) over a Alexandroff space U, we have an $\mathfrak{o}$-topological soft set over U.

EXAMPLE 3.31. Let $U = \{a,b,c\}$, $\mathfrak{o} = \{\varnothing, U, \{a\}, \{a,b\}\}$ be a topology over U. Since U is finite, then $\mathfrak{o}$ is an Alexandroff space. For the mapping $T : U \to \mathscr{P}(\mathfrak{o})$, let

$$(T,U) = \{a = \{\{a\},\{a,b\},U\}, b = \{\{a,b\},U\}, c = \{U\}\}.$$

T^* is induced by T as following.

$$T^*(a) = \bigcap\{\{a\},\{a,b\},U\} = \{a\},$$

$$T^*(b) = \bigcap\{\{a,b\},U\} = \{a,b\}$$

and

$$T^*(c) = \bigcap\{U\} = U.$$

Thus we obtain the soft set $(T^*,U) = \{a = \{a\}, b = \{a,b\}, c = U\}$ over U.

So, we obtain the following theorem for Alexandroff spaces.

THEOREM 3.32. Let $(U,\mathfrak{o})$ and $(V,\mathfrak{o}')$ be Alexandroff topological spaces. $(U,\mathfrak{o})$ is homeomorphic to $(V,\mathfrak{o}')$ if and only if $(T_U^*,U) \approxeq (T_V^*,V)$.

PROOF. Suppose that the space U is homeomorphic to the space V. Then there exist a homeomorphism $\phi : U \to V$. We must show that

$$(\phi \circ T_U^*)(u) = (T_V^* \circ \phi)(u)$$

for all $u \in U$. Since ϕ is a homeomorphism, ϕ is bijective, continuous and open function. In that case, for all $u \in U$,

$$\begin{aligned}
(\phi \circ T_U^*)(u) &= \phi(T_U^*(u)) \\
&= \phi(\bigcap\{A \in \mathfrak{o} \mid u \in A\}) \\
&= \bigcap\{\phi(A) \in \mathfrak{o}' \mid \phi(u) \in \phi(A)\} \\
&= \bigcap\{B \in \mathfrak{o}' \mid \phi(u) \in B\} \\
&= T_V^*(\phi(u)) = (T_V^* \circ \phi)(u)
\end{aligned}$$

Thus we achieve that $(T^*,U) \approxeq (T^*,V)$.

On the other hand, suppose that $(T^*,U) \approxeq (T^*,V)$. Then we have a bijective function

$\phi : U \to V$ such that $(\phi^* \circ T^*)(u) = (T^* \circ \phi)(u)$ [1] for all $u \in U$. We need to show that ϕ is a homeomorphism from U to V. We know that ϕ is bijection. Since for all $u \in U$, $T^*(u) = \bigcap\{A \in \mathfrak{o} \mid u \in A\}$ is an open set in U and $(\phi^* \circ T^*)(u) = (T^* \circ \phi)(u)$, we obtain that ϕ is an open function. Afterwards, since ϕ is a bijection, there exist its inverse $\phi^{-1} : V \to U$. Then we have $(\phi^{-1}(T^*(v)) = T^*(\phi^{-1}(v))$ for all $v \in V$. Since $T^*(v)$ is an open set for all $v \in V$ in the space V, ϕ^{-1} is an open function i.e. ϕ is continuous. Consequently, ϕ is a homeomorphism from U to V.

Of course, the above theorem is not valid for arbitrary topological spaces which is defined by $\mathfrak{o}$-topological soft set in the above manner, since the intersection of arbitrary number of open sets may not be open. However, the intersection of every closed sets is closed in arbitrary topological space. We can define the soft set using closed sets in given any topological space. Let' s consider a topological space $(U, \mathfrak{o})$. Inspired from Molodtsov, define the mapping $K : U \to \mathscr{P}(\mathfrak{c})$ such that $\forall u \in U$, $K(u) = \{A \in \mathfrak{c} \mid u \in A\}$ i.e. $K(u)$ is the family of closed neighborhoods of u. Thus, (K, U) is a soft set over $\mathfrak{c}$. If we define the mapping $K^* : U \to \mathscr{P}(U)$ such that $K^*(u) = \bigcap\{A \in \mathfrak{c} \mid u \in A\}$ which is induced by K. So, (K^*, U) is a $\mathfrak{c}$-topological soft set over U.

We can obtain following theorem for arbitrary topological spaces.

THEOREM 3.33. Let $(U, \mathfrak{o})$ and $(V, \mathfrak{o}')$ be topological spaces. Then $(U, \mathfrak{o})$ is homeomorphic to $(V, \mathfrak{o}')$ if and only if $(K_U^*, U) \approxeq (K_V^*, V)$ for each $u \in U$.

PROOF. Similar to proof of Theorem 3.32.

COROLLARY 3.34. Let $(U, \mathfrak{g})$ and $(V, \mathfrak{g}')$ be generalized topological spaces. $(U, \mathfrak{g})$ is generalized homeomorphic to $(V, \mathfrak{g}')$ if and only if $(K_U^*, U) \approxeq (K_V^*, V)$ where $K^*(u) = \bigcap\{A \in \mathfrak{gc} \mid u \in A\}$.

4. CONCLUSION

In this paper, we have introduced the concept of topological soft set without any restrictions, and we have examined the relationship between them. We have been achieved some results given the notations by Ge et al [8]. We have pointed out that each element of soft topology given by Shabir and Naz in [20] is a topological soft set over a relevant universe and conversely we have pointed out that soft topology and generalized soft topology is obtained when we have topological soft sets. Finally,

[1] ϕ^* is a mapping from $\mathscr{P}(U)$ to $\mathscr{P}(V)$ induced by $\phi : U \to V$.

we have obtained topological soft set for a topological space using Molodtsov's deduction in [16], and we have proved that two space is homeomorphic if and only if topological soft sets derived from them are similar.

This study can be a useful source for researchers working in this area. For the future work, we can examine structures of topological soft sets using topological properties such as separation axioms, connectedness, compactness etc.

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Resolution of system of Volterra integral equations of the first kind by derivation technique and modified decomposition methods

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Abstract

A solution method for various systems of integral equations of the first kind is presented in this paper. The method starts off by transforming the systems via the application of the Leibnitz's derivation technique and then employs three different decomposition methods based on the Standard Adomian decomposition method (SADM) for solutions. To demonstrate the efficiency of the proposed method, some illustrative examples are considered and the obtained results indicate that the approach is indeed of practical interest.

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Keywords: Integral equations; Volterra integral equations; Leibnitz's derivation technique; Adomian decomposition method.

1. INTRODUCTION

Volterra integral equations and their systems are important class of integral equations that arise in many engineering applications. Such equations have been treated by various numerical and analytical techniques [1-4]. However, despite the fact that analytical solutions are the most desired ones, still numerical methods are considered powerful since they solve many problems that analytical solutions do not exist. Methods like the iterative methods, perturbation methods, series expansion in form of certain functions are the usual numerical or approximation methods. Regarding some numerical techniques for systems of integral equations; system of Fredholm integral equations was numerically solved using the wavelet technique [5,6] while the Standard Adomian decomposition method (SADM) was applied to the various systems of Volterra and Fredholm integral equations, respectively, [7-11]. In addition, further application of the ADM in solving integral equations can be seen in the works of Cherrualt and Seng [12] for certain integral equations of the first kind, Babolian et al. [13] for solving linear and nonlinear systems of Volterra integral equations of the second kind and Biazar et al. [14] for Volterra integral equations of the first kind among others.

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Now that the SADM has been proven to be an efficient semi-analytical method for solving various integral equations since its inception in 1980's [15] leading to undergoing several modifications and improvements by many researchers, see [16-20] and also [21-26]; we therefore aim in this paper to consider the certain modifications of the ADM to study the solution of certain systems of Volterra integral equations of the first kind. To achieve our set goal, we first employ the Leibnitz's derivation technique [1] to transform our system to a canonical system suitable for the ADM based modification procedures. We will also establish some comparative study between the classical ADM and our modification procedures.

2. VOLTERRA INTEGRAL EQUATIONS

A system of Volterra integral equations of the first kind can be written as follows:

$$\int_a^{b(x)} k_i(x,t)g_i(u_1(t),u_2(t),\cdots,u_n(t))dt = f_i(x), \quad i=1,2,\cdots,n; \tag{1}$$

where f_i are known functions, $k_i(x,t)$ are the kernels of the ith integral equation, g_i are linear or nonlinear functional of the unknown functions u_i.

We suppose that the system (1) has unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of system (1) could be found in [21,22].

2.1. Linear case

The standard form of the system of Volterra integral equations of the first kind is given by

$$\begin{aligned} f_1(x) &= \int_0^x (K_1(x,t)u_1(t)+\tilde{K}_1(x,t)u_2(t))dt, \\ f_2(x) &= \int_0^x (K_2(x,t)u_1(t)+\tilde{K}_2(x,t)u_2(t))dt, \end{aligned} \tag{2}$$

where the kernels are $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the functions $f_i(x)$, $i=1,2$ are given real-valued functions, and $u_i(x)$, $i=1,2$ are the unknown functions that will be determined.

Now, differentiating both sides of each equation in (2), and using Leibnitz's derivation technique [1], we obtain

$$\begin{aligned} f_1'(x) &= K_1(x,x)u_1(x) + \tilde{K}_1(x,x)u_2(x) \\ &+ \int_0^x (K_{1_x}(x,t)u_1(t) + \tilde{K}_{1_x}(x,t)u_2(t))dt, \\ f_2'(x) &= K_2(x,x)u_1(x) + \tilde{K}_2(x,x)u_2(x) \\ &+ \int_0^x (K_{2_x}(x,t)u_1(t) + \tilde{K}_{2_x}(x,t)u_2(t))dt. \end{aligned}$$

REMARK 2.1. Three remarks can be made here:

1. If at least one of $K_i(x,x)$ and $\tilde{K}_i(x,x)$, $i = 1,2$ in each of the above equations does not vanish, then the system is reduced to a system of Volterra integral equations of the second kind.

2. If $K_i(x,x) = 0$ and $\tilde{K}_i(x,x) = 0$, $i = 1,2$ for any equation, and if $K_{i_x}(x,x) \neq 0$ and $K_{i_x}(x,x) \neq 0$, then we differentiate again that equation.

3. The functions $f_i(x)$ must satisfy specic conditions to guarantee a unique continuous solution for each of the unknown solutions.

2.2. Nonlinear case

We will study a specific case of the systems of nonlinear Volterra integral equations of the first kind given by

$$\begin{aligned} f_1(x) &= \int_0^x (K_1(x,t)u_1(t) + \tilde{K}_1(x,t)F_1(u_2(t)))dt, \\ f_2(x) &= \int_0^x (K_2(x,t)F_2(u_1(t)) + \tilde{K}_2(x,t)u_2(t))dt, \end{aligned} \quad (3)$$

where the kernels are $K_i(x,t)$ and $\tilde{K}_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions and $u_i(x)$, $i = 1,2$ are the unknown functions that will be determined. Recall that the unknown functions $u_i(x)$ appear inside the integral sign for the Volterra integral equations of the first kind. We first need to convert this system to a system of nonlinear Volterra integral equation of the second kind. This can be achieved by differentiating both sides of each part of the system. The conversion technique works effectively by using Leibnitz's technique or rule. Differentiating both sides of each equation in (3) and using Leibnitz's rule, we obtain

$$\begin{aligned} f_1'(x) &= K_1(x,x)u_1(x) + \tilde{K}_1(x,x)F_1(u_2(x)) \\ &+ \int_0^x (K_{1_x}(x,t)u_1(t) + \tilde{K}_{1_x}(x,t)F_1(u_2(t)))dt, \\ f_2'(x) &= K_2(x,x)F_2(u_1(x)) + \tilde{K}_2(x,x)u_2(x) \\ &+ \int_0^x (K_{2_x}(x,t)F_2(u_1(t)) + \tilde{K}_{2_x}(x,t)u_2(t))dt. \end{aligned}$$

$$\begin{aligned} u_1(x) &= \frac{f_1'(x) - \tilde{K}_1(x,x)F_1(u_2(x))}{K_1(x,x)} \\ &- \frac{1}{K_1(x,x)} \int_0^x (K_{1_x}(x,t)u_1(t) + \tilde{K}_{1_x}(x,t)F_1(u_2(t)))dt, \\ u_2(x) &= \frac{f_2'(x) - K_2(x,x)F_2(u_1(x))}{\tilde{K}_2(x,x)} \\ &- \frac{1}{\tilde{K}_2(x,x)} \int_0^x (K_{2_x}(x,t)F_2(u_1(t)) + \tilde{K}_{2_x}(x,t)u_2(t))dt. \end{aligned} \tag{4}$$

It is obvious that the last system is a system of nonlinear Volterra integral equations of the second kind. Notice that the non-homogeneous terms and the kernels have changed to

$$\begin{aligned} g_1(x) &= \frac{f_1'(x) - \tilde{K}_1(x,x)F_1(u_2(x))}{K_1(x,x)}, \\ g_2(x) &= \frac{f_2'(x) - K_2(x,x)F_2(u_1(x))}{\tilde{K}_2(x,x)}, \\ G_1(x,t) &= \frac{K_{1_x}(x,t)}{K_1(x,x)}, \quad \tilde{G}_1(x,t) = \frac{\tilde{K}_{1_x}(x,t)}{K_1(x,x)}, \\ G_2(x,t) &= \frac{K_{2_x}(x,t)}{K_2(x,x)}, \quad \tilde{G}_2(x,t) = \frac{\tilde{K}_{2_x}(x,t)}{K_2(x,x)}. \end{aligned}$$

Then, the form of the system of nonlinear Volterra integral equation of the second kind becomes

$$\begin{cases} u_1(x) = g_1(x) - \int_0^x (G_1(x,t)u_1(t) + \tilde{G}_1(x,t)F_1(u_2(t)))dt, \\ u_2(x) = g_2(x) - \int_0^x (G_2(x,t)F_2(u_1(t)) + \tilde{G}_2(x,t)u_2(t))dt; \end{cases} \tag{5}$$

which is in the Adomian's canonical form with the following recursive scheme:

$$\begin{aligned} u_{1,0}(x) &= g_1(x), \\ u_{1,n+1}(x) &= -\int_0^x (G_1(x,t)u_{1,n}(t) + \tilde{G}_1(x,t)A_{2,n}(t)dt, \\ u_{2,0}(x) &= g_2(x) \\ u_{2,n+1}(x) &= -\int_0^x (G_2(x,t)A_{1,n}(t) + \tilde{G}_2(x,t)u_{2,n}(t)),dt \end{aligned} \quad (6)$$

where $A_{i,n}$, $i = 1,2$ are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[F(\sum_{i=0}^{n}\lambda^i u_i)]_{\lambda=0}, \quad n = 0,1,2,\cdots$$

3. SOME MODIFICATIONS OF THE ADM

3.1. Reliable modification (M1)

The standard decomposition method (SADM) by Adomian [15] was modified by Wazwaz [23]. The modification is based on the assumption that the function $g_i(x),\ i = 1,2$ can be divided into two parts, namely, $g_{i,o}(x)$ and $g_{i,1}(x)$. Under this assumption we set

$$g_i(x) = g_{i,0}(x) + g_{i,1}(x). \quad (7)$$

Accordingly, a slight variation was proposed only on the components $u_{i,0}$ and $u_{i,1}$. The suggestion was that only the part $g_{0,i}$ be assigned to the zeroth component $u_{i,0}$; whereas the remaining part $g_{1,i}$ should be combined with the other terms to define $u_{i,1}$. Consequently the modified recursive relation

$$\begin{aligned} u_{1,0}(x) &= g_{1,0}(x), \\ u_{1,1}(x) &= g_{1,1}(x) - \int_0^x (G_1(x,t)u_{1,0}(t) + \tilde{G}_1(x,t)A_{2,0}(t)dt, \\ u_{1,n+1}(x) &= -\int_0^x (G_1(x,t)u_{1,n}(t) + \tilde{G}_1(x,t)A_{2,n}(t)dt, \\ \\ u_{2,0}(x) &= g_{2,0}(x), \\ u_{2,1}(x) &= g_{2,1}(x) - \int_0^x (G_2(x,t)A_{1,0}(t) + \tilde{G}_2(x,t)u_{2,0}(t))dt, \\ u_{2,n+1}(x) &= -\int_0^x (G_2(x,t)A_{1,n}(t) + \tilde{G}_2(x,t)u_{2,n}(t))dt. \end{aligned} \quad (8)$$

The choice of $g_{i,0}$ such that $u_{i,0}$ contains the minimal number of terms has a strong influence in accelerating the convergence rate of the solution. This means that the success of this method depends mainly on the proper choice of $g_{i,0}$ and $g_{i,1}$ and it is mainly trials. The modification demonstrates a rapid convergence of the series solution if compared with the SADM, and it may give the exact solution for nonlinear equations by using two iterations only and without using the so-called Adomian polynomials.

3.2. New modification (M2)

The modified decomposition method in (3.1) depends entirely on the proper selection of the functions $g_{0,i}$ and $g_{1,i}$. It appears that trials are the only criteria that can be applied so far. In the new modification by Wazwaz [24] we can replace the process of dividing g_i into two components by a series of infinite components. We therefore suggest that g_i be expressed in Taylor series

$$g_i(x) = \sum_{n=0}^{\infty} g_{i,n}(x). \tag{9}$$

A new recursive relationship is then expressed as

$$\begin{aligned} u_{1,0}(x) &= g_{1,0}(x), \\ u_{1,n+1}(x) &= g_{1,n} - \int_0^x (G_1(x,t)u_{1,n}(t) + \tilde{G}_1(x,t)A_{2,n}(t)dt, \\ u_{2,0}(x) &= g_{2,0}(x). \\ u_{2,n+1}(x) &= g_{2,n} - \int_0^x (G_2(x,t)A_{1,n}(t) + \tilde{G}_2(x,t)u_{2,n}(t))dt. \end{aligned} \tag{10}$$

It is important to note that if g_i consists of one term only, then scheme in (10) reduces to relation (7). Moreover if g_i consists of two terms, then relation (10) reduces to the modified relation (8).

3.3. Restarted Adomian decomposition method (M3)

The restarted Adomian method (RADM) by Babolian et al. [20] was based on the standard ADM for algebraic equations, see also Sadeghi et al. [25] for the application. The method modified the SADM by a slight variation in $u_{i,0}$ and $u_{i,1}$ components thereby accelerating the rate of convergence better than the SADM.

3.3.1. Description of the method. We introduce the algorithm as the following

(1) Choose small natural numbers n and m.

(2) Apply the Adomian method on equations (5) to calculate

$$(u_{1,0},u_{2,0}),(u_{1,1},u_{2,1}),\cdots,(u_{1,m},u_{2,m}).$$

set

$$w_1^1 = u_{1,0}+u_{1,1}+\cdots+u_{1,m},$$

$$w_2^1 = u_{2,0}+u_{2,1}+\cdots+u_{2,m}.$$

(3) Let z_1 and z_2 be the proper functions which will be determined next.

For $j=2:n$ do

$$\begin{cases} w_1^{j-1} = z_1, \\ w_2^{j-1} = z_2, \end{cases}$$

$$\begin{cases} u_{1,0} = z_1, \\ u_{2,0} = z_2, \end{cases}$$

$$\begin{cases} u_{1,1} = g_1 - z_1 + A_{1,0}, \\ u_{2,1} = g_2 - z_2 + A_{2,0}, \end{cases}$$

$$\begin{cases} u_{1,m+1} = A_{1,m}, \\ u_{2,m+1} = A_{2,m}. \end{cases}$$

set

$$w_1^j = u_{1,0}+u_{1,1}+\cdots+u_{1,m},$$

$$w_2^j = u_{2,0}+u_{2,1}+\cdots+u_{2,m}.$$

end.

Remarks

(1) w_i^n can be considered as the approximate solution of Eq. (1).

(2) The Adomian method usually gives sum of some first terms as an approximation of u_i. Thus, in this algorithm we can update $u_{i,0}$ in each step with the exception of the terms with large index; so n and m are considered small.

4. NUMERICAL EXAMPLE

EXAMPLE 4.1. Consider the following system of linear Volterra integral equations with the exact solutions: $u_1(x) = x^2$ and $u_2(x) = x$,

$$\int_0^x ((1-x^2+t^2)u_1(t) - (2x-t)u_2(t))dt = -\tfrac{1}{3}x^3 - \tfrac{2}{15}x^5,$$
$$\int_0^x ((x+t^2)u_1(t) + (2-x+t)u_2(t))dt = x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{3}x^4 + \tfrac{1}{5}x^5.$$

Differentiating the above equations with respect to x we have

$$u_1(x) - xu_2(x) - 2\int_0^x (xu_1(t) + u_2(t))dt = -x^2 - \tfrac{2}{3}x^4,$$
$$(x+x^2)u_1(x) + 2u_2(x) + \int_0^x (u_1(t) - u_2(t))dt = 2x - \tfrac{1}{2}x^2 + \tfrac{4}{3}x^3 + x^4.$$

Or

$$u_1(x) = -x^2 - \tfrac{2}{3}x^4 + xu_2(x) + 2\int_0^x (xu_1(t) + u_2(t))dt,$$
$$u_2(x) = x - \tfrac{1}{4}x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4 - \tfrac{1}{2}(x+x^2)u_1(x) - \tfrac{1}{2}\int_0^x (u_1(t) - u_2(t))dt.$$

Thus, SADM offers the following recursive scheme

$$u_{1,0}(x) = -x^2 - \tfrac{2}{3}x^4,$$
$$u_{2,0}(x) = x - \tfrac{1}{4}x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4,$$

$$u_{1,n+1}(x) = xu_{2,n}(x) + 2\int_0^x (xu_{1,n}(t) + u_{2,n}(t))dt,$$
$$u_{2,n+1}(x) = -\tfrac{1}{2}(x+x^2)u_{1,n}(x) - \tfrac{1}{2}\int_0^x (u_{1,n}(t) - u_{2,n}(t))d, \quad n = 0, 1, 2, \cdots$$

(i) Relible Modification (M1)

The M1 methods gives the following recursive scheme from our problem as follows

$$u_{1,0}(x) = -\tfrac{2}{3}x^4,$$
$$u_{2,0}(x) = x,$$

$$u_{1,1}(x) = -x^2 + xu_{2,0}(x) + 2\int_0^x (xu_{1,0}(t) + u_{2,0}(t))dt,$$
$$u_{2,1}(x) = -\tfrac{1}{4}x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4 - \tfrac{1}{2}(x+x^2)u_{1,0}(x) - \tfrac{1}{2}\int_0^x (u_{1,0}(t) - u_{2,0}(t))dt,$$

$$u_{1,n+1}(x) = xu_{2,n}(x) + 2\int_0^x (xu_{1,n}(t) + u_{2,n}(t))dt,$$
$$u_{2,n+1}(x) = -\tfrac{1}{2}(x+x^2)u_{1,n}(x) - \tfrac{1}{2}\int_0^x (u_{1,n}(t) - u_{2,n}(t))dt, \quad n = 1,2,\cdots$$

(ii) Restarted ADM (M3)

Considering the small indexes n and m, say $n = 3$, $m = 4$;

step1:

$$\begin{cases} u_{1,0}(x) = -x^2 - \tfrac{2}{3}x^4, \\ u_{2,0}(x) = x - \tfrac{1}{4}x^2 + \tfrac{2}{3}x^3 + \tfrac{1}{2}x^4, \\ u_{1,n+1}(x) = xu_{2,n}(x) + 2\int_0^x (xu_{1,n}(t) + u_{2,n}(t))dt, \\ u_{2,n+1}(x) = -\tfrac{1}{2}(x+x^2)u_{1,n}(x) - \tfrac{1}{2}\int_0^x (u_{1,n}(t) - u_{2,n}(t))dt, \ \ n = 0,1,2,\cdots \end{cases}$$

$$\begin{cases} u_{1,1}(x) = 2x^2 - \tfrac{5}{12}x^3 + \tfrac{1}{3}x^4 + \tfrac{7}{10}x^5 - \tfrac{4}{15}x^6, \\ u_{2,1}(x) = \tfrac{1}{4}x^2 + \tfrac{5}{8}x^3 + \tfrac{7}{12}x^4 + \tfrac{9}{20}x^5 + \tfrac{1}{3}x^6, \\ u_{1,2}(x) = \tfrac{5}{12}x^3 + \tfrac{109}{48}x^4 + \tfrac{73}{120}x^5 + \tfrac{11}{15}x^6 + \cdots \\ u_{2,2}(x) = -\tfrac{31}{24}x^3 - \tfrac{127}{192}x^4 + \tfrac{1}{15}x^5 - \tfrac{43}{80}x^6 + \cdots \\ u_{1,3}(x) = -\tfrac{31}{16}x^4 - \tfrac{689}{960}x^5 + \tfrac{359}{360}x^6 + \cdots \\ u_{2,3}(x) = -\tfrac{27}{64} - \tfrac{3143}{1920}x^5 - \tfrac{1069}{720}x^6 + \cdots \end{cases}$$

$$w_1^1 = u_{1,0} + u_{1,1} + u_{1,2} + u_{1,3} = x^2 + \frac{189}{320}x^5 + \frac{527}{360}x^6 + \cdots$$
$$w_2^1 = u_{2,0} + u_{2,1} + u_{2,2} + g_{2,3} = x - \frac{717}{640}x^5 - \frac{76}{45}x^6 + \cdots$$

step2:

$$\begin{cases} u_{1,0}(x) = x^2 + \frac{189}{320}x^5 + \frac{527}{360}x^6 + \cdots \\ u_{2,0}(x) = x - \frac{717}{640}x^5 - \frac{76}{45}x^6 + \cdots \\ u_{1,1}(x) = (x^2 - \frac{2}{3}x^4) - (x^2 + \frac{189}{320}x^5 + \frac{527}{360}x^6 + \cdots) + xu_{2,0}(x) + 2\int_0^x (xu_{1,0}(t) + u_{2,0}(t))dt, \\ u_{2,1}(x) = (x - \frac{1}{4}x^2 + \frac{2}{3}x^3 + \frac{1}{2}x^4) - (x - \frac{717}{640}x^5 - \frac{76}{45}x^6 + \cdots) - \frac{1}{2}(x+x^2)u_{1,0}(x) \\ -\frac{1}{2}\int_0^x (u_{1,0}(t) - u_{2,0}(t))dt, \\ u_{1,n+1}(x) = xu_{2,n}(x) + 2\int_0^x (xu_{1,n}(t) + u_{2,n}(t))dt, \\ u_{2,n+1}(x) = -\frac{1}{2}(x+x^2)u_{1,n}(x) - \frac{1}{2}\int_0^x (u_{1,n}(t) - u_{2,n}(t))dt, \quad n = 1, 2, \cdots \end{cases}$$

$$\begin{cases} u_{1,1}(x) = -\frac{189}{320}x^5 - \frac{4259}{1440}x^6 - \frac{43307}{20160}x^7 + \cdots \\ u_{2,1}(x) = \frac{717}{640}x^5 + \frac{28823}{23040}x^6 - \frac{4261}{13440}x^7 + \cdots \\ u_{1,2}(x) = \frac{239}{160}x^6 + \frac{5059}{3584}x^7 - \frac{200203}{161280}x^8 + \cdots \\ u_{2,2}(x) = \frac{1121}{2560}x^6 + \frac{74359}{35840}x^7 + \frac{1720759}{645120}x^8 + \cdots \\ u_{1,3}(x) = \frac{10089}{17920}x^7 + \frac{432979}{143360}x^8 + \cdots \\ u_{2,3}(x) = -\frac{29471}{35840}x^7 - \frac{809239}{573440}x^8 + \cdots \end{cases}$$

$$w_1^2 = x^2 + \frac{29471}{2872}x^8 - \frac{1139}{560}x^9 + \cdots$$
$$w_2^2 = x + \frac{211073}{573440}x^8 - \frac{213207}{143360}x^9 + \cdots$$

step3:

$$w_1^3 = x^2 - \frac{1487022307}{851558400}x^{12} - \frac{81990701}{252313600}x^{11} + \cdots$$
$$w_2^3 = x + \frac{519896129}{425779200}x^{12} + \frac{175553087}{504627200}x^{11} + \cdots$$

step4:

$$w_1^3 = x^2 - \frac{162739797083}{166985728000}x^{15} - \frac{66509498791}{275526451200}x^{14} + \cdots$$
$$w_2^4 = x - \frac{28883455355929}{33063174144000}x^{15} - \frac{100436326331}{801531494400}x^{14} + \cdots$$

We therefore establish the comparisons between the exact solution and the solutions obtained by SADM, M1 and M3 methods for $u_1(x)$ and $u_2(x)$ and reported in Tables 1 and 2, and Figures 1 and 2, respectively.

x	$u_1(x)$	SADM	error	M1	error	M3	error
0.1	0.01	0.009998	0.000002	0.009999	0.000001	0.01	0
0.2	0.04	0.039877	0.000123	0.039935	0.000065	0.04	0
0.3	0.09	0.088414	0.001586	0.089162	0.000838	0.09	0
0.4	0.16	0.149935	0.010065	0.154715	0.005285	0.159998	0.000002
0.5	0.25	0.206619	0.043381	0.227417	0.022583	0.249929	0.000071

Table I: Comparison between exact solution $u_1(x)$ and approximate solutions using SADM, M1 and M3 methods

x	$u_2(x)$	SADM	error	M1	error	M3	error
0.1	0.1	0.099999	0.000001	0.1	0	0.1	0
0.2	0.2	0.199954	0.000046	0.199981	0.000019	0.2	0
0.3	0.3	0.299346	0.000654	0.299690	0.000310	0.3	0
0.4	0.4	0.395531	0.004469	0.397601	0.002399	0.399997	0.000003
0.5	0.5	0.479537	0.020463	0.487737	0.012263	0.499886	0.000114

Table II: Comparison between exact solution $u_2(x)$ and approximate solutions using SADM, M1 and M3 methods

Figure1: The Exact solution u_1(x) and Approximate solution using methods (SADM), (M1) and (M3)

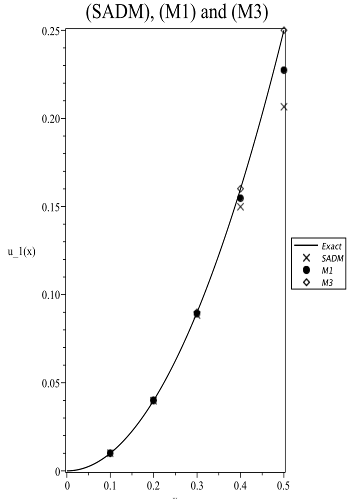
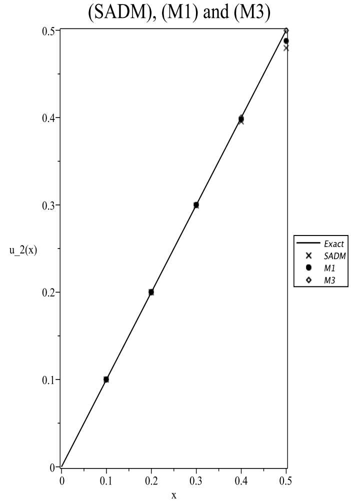


Figure2: The Exact solution u_2(x) and Approximate solution using methods (SADM), (M1) and (M3)



EXAMPLE 4.2. Consider the following system of nonlinear Volterra integral equations given the exact solutions: $u_1(x) = x + e^x$, and $u_2(x) = x - e^x$;

$$\int_0^x (u_1(t) + (x-t)u_1(t)u_2(t))dt = -\frac{3}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x},$$

$$\int_0^x (u_1(t) + (x-t)u_1(t)u_2(t))dt = \frac{5}{4} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x}.$$

Differentiating the above equation with respect to x we get:

$$u_1(x)+\int_0^x u_1(t)u_2(t)dt=\tfrac{1}{2}+x+\tfrac{1}{3}x^3+e^x-\tfrac{1}{2}e^{2x} \quad ,$$
$$u_2(x)+\int_0^x u_1(t)u_2(t)dt=\tfrac{1}{2}+x+\tfrac{1}{3}x^3-e^x-\tfrac{1}{2}e^{2x}.$$

Or

$$u_1(x)=\tfrac{1}{2}+x+\tfrac{1}{3}x^3+e^x-\tfrac{1}{2}e^{2x}-\int_0^x u_1(t)u_2(t)dt,$$
$$u_2(x)=\tfrac{1}{2}+x+\tfrac{1}{3}x^3-e^x-\tfrac{1}{2}e^{2x}-\int_0^x u_1(t)u_2(t)dt.$$

Applying the SADM to above system yields the following recursive scheme:

$$u_{1,0}(x)=\tfrac{1}{2}+x+\tfrac{1}{3}x^3+e^x-\tfrac{1}{2}e^{2x},$$
$$u_{2,0}(x)=\tfrac{1}{2}+x+\tfrac{1}{3}x^3-e^x-\tfrac{1}{2}e^{2x},$$

$$u_{1,n+1}(x)=-\int_0^x A_n(t)dt,$$
$$u_{2,n+1}(x)=-\int_0^x A_n(t)dt, \quad n=0,1,2,\cdots$$

where A_n are the Adomian polynomials given by:

$$\begin{aligned}A_0(t) &= u_{1,0}(t)u_{2,0}(t),\\ A_1(t) &= u_{1,0}(t)u_{2,1}(t)+u_{1,1}(t)u_{2,0}(t),\\ A_2(t) &= u_{1,0}(t)u_{2,2}(t)+u_{1,2}(t)u_{2,0}(t)+u_{1,1}(t)u_{2,1}(t),\end{aligned}$$

and so on.

(i) Relible Modification (M1)

On using M1, the following recursive scheme to the problem is obtained:

$$u_{1,0}(x)=x+e^x,$$
$$u_{2,0}(x)=x-e^x,$$

$$u_{1,1}(x)=\tfrac{1}{2}+\tfrac{1}{3}x^3-\tfrac{1}{2}e^{2x}-\int_0^x A_0(t)dt,$$
$$u_{2,1}(x)=\tfrac{1}{2}+\tfrac{1}{3}x^3-\tfrac{1}{2}e^{2x}-\int_0^x A_0(t)dt,$$

$$u_{1,n+1}(x)=-\int_0^x A_n(t)dt,$$
$$u_{2,n+1}(x)=-\int_0^x A_n(t)dt, \quad n=1,2,\cdots$$

(ii) New Modification (M2)

Using M2 method, we first set the Taylor's expansion for $f_i(x)$ as

$$f_1(x) = 1 + x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + o[x]^4,$$
$$f_2(x) = -1 - x - \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 + o[x]^4.$$

Thus, we get the following recursive relation from (10);

$$u_{1,0}(x) = 1,$$
$$u_{2,0}(x) = -1,$$
$$u_{1,1}(x) = x - \int_0^x A_0(t)dt = 2x,$$
$$u_{2,1}(x) = -x - \int_0^x A_0(t)dt = 0,$$
$$u_{1,2}(x) = -\tfrac{1}{2}x^2 - \int_0^x A_1(t)dt = \tfrac{1}{2}x^2,$$
$$u_{2,2}(x) = -\tfrac{3}{2}x^2 - \int_0^x A_1(t)dt = -\tfrac{1}{2}x^2,$$
$$u_{1,3}(x) = -\tfrac{1}{6}x^3 - \int_0^x A_2(t)dt = \tfrac{1}{6}x^3,$$
$$u_{2,3}(x) = -\tfrac{1}{2}x^3 - \int_0^x A_2(t)dt = -\tfrac{1}{6}x^3,$$

and so on. We establish the comparisons between the exact solution and the solutions obtained by M1 and M2 methods for $u_1(x)$ and $u_2(x)$ and reported in Tables 3 and 4, and Figures 3 and 4, respectively.

x	$u_1(x)$	SADM	error	M1	error	M2	error
0.1	1.205171	1.205169	0.000002	1.205171	0	1.205167	0.000004
0.2	1.421403	1.421337	0.000065	1.421403	0	1.421333	0.000069
0.3	1.649859	1.649245	0.000614	1.649859	0	1.649462	0.000359
0.4	1.891825	1.888585	0.003239	1.891825	0	1.890667	0.001158
0.5	2.148721	2.136289	0.012432	2.148721	0	2.145833	0.002888

Table III: Comparison between exact solution $u_1(x)$ and approximate solutions using SADM, M1 and M2 methods

x	$u_2(x)$	SADM	error	M1	error	M2	error
0.1	-1.005171	-1.005173	0.000002	-1.005171	0	-1.005167	0.000004
0.2	-1.021403	-1.021468	0.000066	-1.021403	0	-1.021333	0.000064
0.3	-1.049859	-1.050473	0.000614	-1.049859	0	-1.049500	0.000359
0.4	-1.091825	-1.095064	0.003239	-1.091825	0	-1.090667	0.001158
0.5	-1.148721	-1.161153	0.012432	-1.148721	0	-1.145833	0.002888

Table IV: Comparison between exact solution $u_2(x)$ and approximate solutions using SADM, M1 and M2 methods

Figure3: The Exact solution u_1(x) and Approximate solution using methods (SADM), (M1) and (M2)

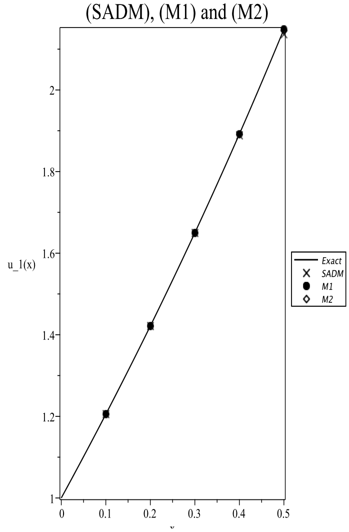
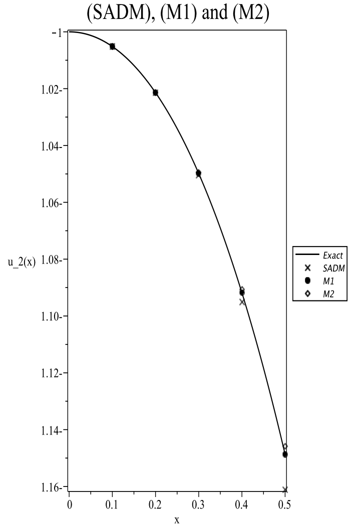


Figure4: The Exact solution u_2(x) and Approximate solution using methods (SADM), (M1) and (M2)



5. CONCLUSIONS

In conclusion, we have presented three recursive schemes based on the modifications of the Standard Adomian decomposition method (SADM) for solving system of Volterra integral equations of the first kind. However in doing that, we successfully utilized the Leibniz' derivation technique to transform the integral equations to the conical forms where Adomian technique is applicable. We further applied the presented schemes to some test problems and found remarkable approximates solutions as reported in the given tables and graphs. The computations associated with the test problems were performed using a Maple.

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A new subclass of meromorphic functions with positive coefficients defining by linear operator

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Abstract

In this paper, we introduce and study a new class $\sigma,(\alpha,\lambda)$ of meromorphic univalent functions defined in $E=\{z : z\in\mathbb{C} \text{ and } 0<|z|<1\}=E\setminus\{0\}$. We obtain coefficient inequalities, distortion theorems, extreme points, closure theorems, radius of convexity estimates and integral operators. Finally, we obtained neighbourhood result for the class $\sigma_p(\gamma,\lambda)$.

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1. INTRODUCTION

Let Σ denote the class of meromorphic functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,\ (a_n\geq 0) \tag{1}$$

which are analytic in the punctured unit disc $E=\{z : z\in C \text{ and } 0<|z|<1\}$. A function $f\in\sum$ is meromorphic starlike of order α $(0\leq\alpha<1)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\ \ (z\in E). \tag{2}$$

The class of all such functions is denoted by $\Sigma^*(\alpha)$. A function $f\in\Sigma$ is meromorphic convex of order α $(0\leq\alpha<1)$ if

$$-Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\quad(z\in E). \tag{3}$$

The class of such functions is denoted by $\Sigma_k^*(\alpha)$. The class $\Sigma(\alpha)$ and $\Sigma_k^*(\alpha)$ were introduced and studied by Clunie [1], Pommerenke [6], Miller [4] and Mogra et al. [5]. Let Σ_p be the class of functions $f\in\Sigma$ with $a_n\geq 0$. The subclass of Σ_p consisting of starlike function of order α is denoted by $\Sigma_p^*(\alpha)$. For a function $f(z)\in\Sigma$, Frasin

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and Darus [2] defined an operator $I^k : \Sigma \to \Sigma$ as follows

$$I^0 f(z) = f(z)$$
$$I^1 f(z) = zf'(z) + \frac{2}{z}$$
$$I^2 f(z) = z(I^1 f(z))' + \frac{2}{z}$$

and for $k = 1, 2, 3, \cdots$, we have

$$\begin{aligned} I^k f(z) &= z(I^{k-1} f(z))' + \frac{2}{z} \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} n^k a_n z^k \quad (k \in N_0 = N \cup \{0\}, z \in E). \end{aligned}$$

Definition 1.1. Let $0 \leq \alpha < 1$. Further let $f(z) \in \Sigma_p$ be given by (1), $0 \leq \lambda < 1$. The class $\sigma_p(\alpha, \lambda)$ is defined by

$$\sigma_p(\alpha, \lambda) = \left\{ f \in \Sigma_p : Re\left[\frac{zf'(z)}{(\lambda - 1)f(z) + \lambda z f'(z)} \right] > \alpha \right\}. \tag{4}$$

Clearly $\sigma_p(\alpha, 0)$ reduces to $\Sigma_p^*(\alpha)$.

In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $\sigma_p(\alpha, \lambda)$. Properties of certain integral operators and neighbourhood properties are also discussed for the class. The subclass of Σ_p consisting of starlike functions of order α is denoted by $\Sigma_p^*(\alpha)$.

2. COEFFICIENTS INEQUALITIES

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $\Sigma_p(\lambda, \alpha)$.

THEOREM 2.1. Let $f(z) \in \Sigma_p$ be given by (1). Then $f \in \sigma_p(\lambda, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} n^k [n + \alpha - \alpha\lambda(1 + n)] a_n \leq 1 - \alpha. \tag{5}$$

PROOF. If $f \in \sigma_p(\lambda, \alpha)$ then

$$Re\left\{ \frac{z(I^k f(z))'}{(\lambda - 1)(I^k f(z)) + \lambda z (I^k f(z))'} \right\} = Re\left\{ \frac{-1 + \sum\limits_{n=1}^{\infty} n^{k+1} a_n z^{n+1}}{-1 + \sum\limits_{n=1}^{\infty} (\lambda - 1 + \lambda n) n^k a_n z^{n+1}} \right\} > \alpha.$$

By letting $z \to 1^-$, we have $\left\{ \frac{-1+\sum\limits_{n=1}^{\infty} n^{k+1} a_n}{-1+\sum\limits_{n=1}^{\infty} n^k(\lambda-1+\lambda n)a_n} \right\} > \alpha$.
This shows that (5) holds.

Conversely assume that (5) holds. Since
$Re(w) > \alpha$ if and only if $|w-1| < |w+1-2\alpha|$.
It is sufficient to show that

$$\left| \frac{z(I^k f(z))' - [(\lambda-1)I^k f(z) + \lambda z(I^k f(z))']}{z(I^k f(z))' + (1-2\alpha)[(\lambda-1)I^k f(z) + \lambda z(I^k f(z))']} \right| < 1.$$

Using (5), we see that

$$\left| \frac{z(I^k f(z))' - [(\lambda-1)I^k f(z) + \lambda z(I^k f(z))']}{z(I^k f(z))' + (1-2\alpha)[(\lambda-1)I^k f(z) + \lambda z(I^k f(z))']} \right|$$
$$= \left| \frac{\sum\limits_{n=1}^{\infty} n^k(1-\lambda)(n+1)a_n z^{n+1}}{-2(1-\alpha) + \sum\limits_{n=1}^{\infty} n^k[n(1+(1-2\alpha)\lambda) + (1-2\alpha)(\lambda-1)]a_n z^{n+1}} \right|$$
$$\leq \frac{\sum\limits_{n=1}^{\infty} n^k(1-\lambda)(n+1)a_n}{2(1-\alpha) - \sum\limits_{n=1}^{\infty} n^k[n(1+(1-2\alpha)\lambda) + (1-2\alpha)(\lambda-1)]a_n}$$
$$\leq 1.$$

Thus we have $f \in \sigma_p(\lambda, \alpha)$.

COROLLARY 2.2. If $f \in \sigma_p(\lambda, \alpha)$ then

$$a_n \leq \frac{(1-\alpha)}{n^k[n+\alpha-\alpha\lambda(1+n)]}.$$

PROOF. The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z} + \frac{(1-\alpha)}{n^k[n+\alpha-\alpha\lambda(1+n)]} z^n, \quad n = 1,2,3\cdots.$$

3. GROWTH AND DISTORTION THEOREM

THEOREM 3.1. If $f \in \sigma_p(\lambda, \alpha)$, then

$$\frac{1}{r} - \frac{(1-\alpha)}{(1+\alpha-2\alpha\lambda)} r \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)}{(1+\alpha-2\alpha\lambda)} r. \tag{6}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda}\, z. \tag{7}$$

PROOF. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n$$

since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{1+\alpha-2\alpha\lambda}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} r.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{1-\alpha}{1+\alpha-2\alpha\lambda} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} z$.

THEOREM 3.2. If $f \in \sigma_p(\alpha,\lambda)$ then

$$\frac{1}{r^2} - \frac{1-\alpha}{1+\alpha-2\alpha\lambda} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{1+\alpha-2\alpha\lambda} \quad (|z| = r).$$

The result is sharp for the function given by (7).

4. CLOSURE THEOREMS

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,k} z^n, \quad k = 1,2,3,\cdots. \tag{8}$$

We shall prove the following closure theorems for the class $\sigma_p(\lambda,\alpha)$.

THEOREM 4.1. Let the function defined by (8) be in the class $\sigma_p(\lambda,\alpha)$ for every $k = 1,2,3,\cdots$. Then the function $f(z)$ defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0)$$

belongs to the class $\sigma_p(\lambda,\alpha)$, where $a_n = \frac{1}{m}\sum_{n=1}^{m} a_{n,k}$ $(n = 1,2,\cdots)$.

PROOF. Since $F_k(z) \in \sigma_p(\lambda,\alpha)$, it follows from Theorem 2.1 that,

$$\sum_{n=1}^{\infty} n^k[n+\alpha-\alpha\lambda(1+n)]a_{n,k} \leq 1-\alpha, \text{ for all } k = 1,2,\cdots,m. \tag{9}$$

Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^k[n+\alpha-\alpha\lambda(1+n)]a_n \\
=&\sum_{n=1}^{\infty} n^k[n+\alpha-\alpha\lambda(1+n)]\Big(\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\Big) \\
=&\frac{1}{m}\sum_{n=1}^{m}\left(\sum_{n=1}^{\infty} n^k[n+\alpha-\alpha\lambda(1+n)]a_{n,k}\right) \\
&\leq 1-\alpha.
\end{aligned}$$

By Theorem 2.1, we have $f(z) \in \sigma_p(\lambda,\alpha)$.

THEOREM 4.2. The class $\sigma_p(\alpha,\lambda)$ is closed under convex linear combination.

PROOF. Let the function $F_k(z)$ given by (8) be in the class $\sigma_p(\alpha,\lambda)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1-\lambda)F_2(z), \ \ (0 \leq \lambda \leq 1)$$

is also in the class $\sigma_p(\alpha,\lambda)$. Since for $(0 \leq \lambda \leq 1)$,

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty}[\lambda a_{n,1} + (1-\lambda)a_{n,2}]z^n.$$

We observe that

$$\sum_{n=1}^{\infty}\{n^k[n+\alpha-\alpha\lambda(1+n)]\}[\lambda a_{n,1} + (1-\lambda)a_{n,2}]$$

$$\begin{aligned}
&= \lambda\sum_{n=1}^{\infty}\{n^k[n+\alpha-\alpha\lambda(1+n)]\}a_{n,1} + (1-\lambda)\sum_{n=1}^{\infty}\{n^k[n+\alpha-\alpha\lambda(1+n)]\}a_{n,2} \\
&\leq 1-\alpha.
\end{aligned}$$

By Theorem 2.1, we have $H(z) \in \sigma_p(\alpha,\lambda)$.

THEOREM 4.3. Let $F_0(z) = 1$ and

$$F_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{1-\alpha}{n^k[n+\alpha-\alpha\lambda(1+n)]}z^n, \ \ for \ \ n = 1,2,\cdots.$$

Then $f(z) \in \sigma_p(\alpha,\lambda)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$, where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.

PROOF. Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_n F_n(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\lambda_n(1-\alpha)}{n^k[n+\alpha-\alpha\lambda(1+n)]} z^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\lambda_n(1-\alpha)}{n^k[n+\alpha-\alpha\lambda(1+n)]} \frac{n^k[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)} \\ &= \sum_{n=1}^{\infty} \lambda_n = 1-\lambda_0 \leq 1. \end{aligned}$$

By Theorem 2.1, we have $H(z) \in \sigma_p(\alpha,\lambda)$.

Conversely, let $f(z) \in \sigma_p(\alpha,\lambda)$. From Theorem 2.1, we have

$$a_n \leq \frac{1-\alpha}{n^k[n+\alpha-\alpha\lambda(1+n)]} \quad for\; n = 1,2,\cdots.$$

We may take $\lambda_n = \frac{n^k[n+\alpha-\alpha\lambda(1+n)]}{1-\alpha} a_n \;\; for\;\; n = 1,2,\cdots$ and $\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n$.
Then $f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z)$.

5. RADIUS OF MEROMORPHIC STARLIKENESS AND MEROMORPHIC CONVEXITY

THEOREM 5.1. Let the function f be in the class $\sigma_p(\alpha,\lambda)$. Then f is meromorphically starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1(\alpha,\lambda,\rho)$, where

$$r_1(\alpha,\lambda,\rho) = \inf_{n\geq 1} \left[\frac{(1-\rho)(1-\alpha)}{n^k(n+2-\rho)[n+\alpha-\alpha\lambda(1+n)]} \right]^{\frac{1}{n+1}}. \tag{10}$$

PROOF. Let $f(z)$ is in $\sigma_p(\alpha,\lambda)$. Then by Theorem 2.1, we have

$$n^k[n+\alpha-\alpha\lambda(1+n)]\, a_n \leq (1-\alpha). \tag{11}$$

It is sufficient to show that

$$\left| 1 + \frac{zf'(z)}{f(z} \right| \leq 1-\rho. \tag{12}$$

Or equivalently

$$\left|1+\frac{zf'(z)}{f(z)}\right| = \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{\frac{1}{z}-\sum_{n=1}^{\infty}a_n z^n}\right| \leq 1-\rho.$$

or

$$\sum_{n=1}^{\infty}\frac{(n+2-\rho)}{1-\rho}a_n|z|^{n+1} \leq 1, \text{ for } 0 \leq \rho < 1 \text{ and } |z| < r_1(\alpha,\lambda,\rho).$$

By theorem 2.1, (12) will be true if

$$\left(\frac{n+2-\rho}{1-\rho}\right)|z|^{n+1} \leq \frac{1-\alpha}{n^k[n+\alpha-\alpha\lambda(1+n)]}$$

or if

$$|z| \leq \left[\frac{1-\alpha}{(n+2-\rho)\{n^k[n+\alpha-\alpha\lambda(1+n)]\}}\right]^{\frac{1}{n+1}}, \; n \geq 1.$$

This completes the proof of Theorem.

THEOREM 5.2. Let the function $f(z)$ be in the class $\sigma_p(\alpha,\lambda)$. Then f is meromorphically convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_2(\alpha,\lambda,\rho)$, where

$$r_2(\alpha,\lambda,\rho) = \inf_{n\geq 1}\left[\frac{(1-\rho)\{n^k[n+\alpha-\alpha\lambda(n+1)]\}}{n(n+2-\rho)(1-\alpha)}\right]^{\frac{1}{n+1}}, \; n \geq 1. \tag{13}$$

PROOF. Let $f(z)$ be in $\sigma_p(\alpha,\lambda)$. Then by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} n^k[n+\alpha-\alpha\lambda(1+n)]\; a_n \leq (1-\alpha). \tag{14}$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z}\right| \leq 1-\delta.$$

for $|z| < r_2 = r_2(\alpha,\lambda,\rho)$, where $r_2(\alpha,\lambda,\rho)$ is specified in the statement of the Theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{\frac{-1}{z^2}+\sum_{n=1}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty} na_n|z|^{n+1}}.$$

This will be bounded by $(1-\rho)$ if

$$\sum_{n=1}^{\infty}\frac{n(n+2-\delta)}{1-\delta}a_n|z|^{n+1} \leq 1, \tag{15}$$

By (14), it follows that (15) is true if

$$\frac{n(n+2-\rho)}{1-\rho}|z|^{n+1} \leq \frac{n^k[n+\alpha-\alpha\lambda(1+n)]}{1-\alpha},\ n \geq 1$$

or

$$|z| \leq \left[\frac{(1-\rho)\{n^k[n+\alpha-\alpha\lambda(1+n)]\}}{n(n+2-\rho)(1-\alpha)}\right]^{\frac{1}{n+1}},\ n \geq 1. \tag{16}$$

This completes the proof of Theorem.

6. INTEGRAL OPERATORS

In this section, we consider integral transform of functions in the class $\sigma_p(\alpha,\lambda)$.

THEOREM 6.1. Let the function $f(z)$ given by (1) be in $\sigma_p(\alpha,\lambda)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du\ (0 \leq u \leq 1, 0 < c < \infty)$$

is in $\sigma_p(\delta,\lambda)$, where

$$\delta = \frac{(c+2)(1+\alpha-2\alpha\lambda)-c(1-\alpha)}{c(1-\alpha)(1-2\lambda)+(1+\alpha)(1-2\lambda)(c+2)}.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)}z.$$

PROOF. Let $f(z) \in \sigma_p(\alpha,\lambda)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^c f(uz)du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c}{c+n+1} a_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\{n^k[n+\delta-\delta\lambda(1+n)]\}}{(c+n+1)(1-\delta)} a_n \leq 1. \tag{17}$$

Since $f \in \sigma_p(\alpha,\lambda)$, we have

$$\sum_{n=1}^{\infty} \frac{n^k[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)} a_n \leq 1. \tag{18}$$

Note that (17) is satisfied if

$$\frac{c[n+\delta-\delta\lambda(1+n)]}{(c+n+1)(1-\delta)} \leq \frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)}.$$

Solving for δ, we have

$$\delta \leq \frac{(c+n+1)[n+\alpha-\alpha\lambda(1+n)]-cn(1-\alpha)}{c(1-\alpha)[1-\lambda(1+n)]+[n+\alpha-\alpha\lambda(1+n)](c+n+1)} = G(n).$$

A simple computation will show that $G(n)$ is increasing and $G(n) \geq G(1)$. Using this, the result follows.

For the choice of $\lambda = 0$, we have the following result of Uralegaddi and Gangi [8].

Remark 6.2. Let the function $f(z)$ defined by (1) be in $\Sigma_p^*(\alpha)$. Then integral operator

$$F(z) = c\int_0^1 u^c f(uz)du \quad (0 < u \leq 1, 0 < c < \infty)$$

is in $\Sigma_p^*(\alpha)$, where $\delta = \frac{1+\alpha+c\alpha}{1+\alpha+c}$.

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha}z.$$

7. NEIGHBOURHOODS FOR THE CLASS $\Sigma_P^\gamma(\alpha,\lambda)$

In this section, we determine the neighbourhood for the class $\Sigma_p^\gamma(\alpha,\lambda)$, which we define as follows.

Definition 7.1. A function $f \in \Sigma_p$ is said to be in the class $\sigma_p^\gamma(\alpha,\lambda)$ if there exists a function $g \in \sigma_p^\gamma(\alpha,\lambda)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\gamma, \quad (z \in E, 0 \leq \gamma < 1). \tag{19}$$

Following the earlier works on neighbourhoods of analytic functions by Goodman [3] and Ruscheweyh [7], we define the $\delta-$neighbourhood of function $f \in \Sigma_p$ by

$$N_\delta(f) = \left\{g \in \Sigma_p : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \ \ and \ \ \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta\right\}. \tag{20}$$

THEOREM 7.2. If $g \in \sigma_p(\alpha, \lambda)$ and

$$\gamma = 1 - \frac{\delta(1+\alpha-2\alpha\lambda)}{2\alpha - 2\alpha\lambda} \tag{21}$$

then $N_\delta(g) \subset \sigma_p^\gamma(\alpha, \lambda)$.

PROOF. Let $f \in N_\delta(g)$. Then we find from (20) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \tag{22}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \geq \delta \quad (n \in N). \tag{23}$$

Since $g \in \sigma_p(\alpha, \lambda)$, we have

$$\sum_{n=1}^{\infty} b_n \leq \frac{1-\alpha}{1+\alpha-2\alpha\lambda}. \tag{24}$$

So that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \\ &= \frac{\delta(1+\alpha-2\alpha\lambda)}{2\alpha - 2\alpha\lambda} \\ &= 1 - \gamma \end{aligned}$$

provided γ is given by (21). Hence by definition, $f \in \sigma_p^\gamma(\alpha, \lambda)$.

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A new generalization of Aradhana distribution: Properties and applications

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Abstract

In this paper, we introduce a new generalization of Aradhana distribution called as Weighted Aradhana Distribution (WID). The statistical properties of this distribution are derived and the model parameters are estimated by maximum likelihood estimation. Simulation study of ML estimates of the parameters is carried out in R software. Finally, an application to real data set is presented to examine the significance of newly introduced model.

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Keywords: Aradhana Distribution, Weighting Technique, Structural Properties and Maximum Likelihood Estimation, Simulation.

1. INTRODUCTION

Aradhana distribution is a newly proposed lifetime model formulated by Rama Shanker (2016) for several engineering applications and calculated its various characteristics including stochastic ordering, moments, order statistics, Renyi entropy, Stress-Strength reliability and ML estimation.

Probability density function (pdf) of Aradhana Distribution (AD) is given by

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+2}(1+x)^2e^{-\theta x}\quad x>0,\theta>0 \tag{1.1}$$

The corresponding cdf of (1.1) is given by

$$F(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2\theta+2)}{\theta^2+2\theta+2}\right]e^{-\theta x}\ ,\ x>0,\theta>0 \tag{1.2}$$

2. WEIGHTED ARADHANA DISTRIBUTION (WAD)

Often scientists cannot select sampling units in observational studies with equal probability. Well defined sampling frames often do not exist for human, wildlife, insect, plant, or fish populations. Recorded observations on individuals in these

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populations are biased and will not have the original distribution unless every observation is given an equal chance of being recorded. Weighted distribution theory gives a unified approach for modeling these biased data. The concept of weighted distributions can be traced to the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher (1934). Rao (1965) pointed out that in many situations the recorded observations cannot be considered as a random sample from the original distribution due to one or the other reason. Various weighted distributions have been discussed in Blumenthal (1963), Patil and Rao (1977,1978), Para and Jan (2018), Mahfoud and Patil (1982), Gupta and Kirmani (1990) and Hassan, Wani and Para (2018) among others.

Assume X is a non negative random variable with probability density function (pdf) $f(x)$. Let $W(x)$ be the weight function which is a non negative function, then the probability density function of the weighted random variable X_w is given by:

$$f_W(x) = \frac{W(x) f(x)}{E(w(x))}, \quad x > 0,$$

where $w(x)$ be a non-negative weight function and $E(w(x)) = \int w(x) f(x) dx < \infty$.

In this paper, we have considered the weight function as $w(x) = x^c$ to obtain the weighted Aradhana model. The probability density of weighted Aradhana distribution is given as:

$$f_w(x,c,\theta) = \frac{x^c f(x,\theta)}{E[x^c]},$$

$$f_w(x,c,\theta) = \frac{x^c \theta^{(c+3)} (1+x)^2 e^{-\theta x}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))}, \quad x > 0, \theta > 0, c > 0,$$

(2.1)

where $E(x^c) = \dfrac{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))}{\theta^c(\theta^2 + 2\theta + 2)}$.

The corresponding cdf of weighted Aradhana Distribution (AID) is obtained as

$$F_w(x;c,\theta) = \int_0^x f_w(x;c,\theta) dx$$

$$= \int_0^x \frac{x^c \theta^{(c+3)} (1+x)^2 e^{-\theta x}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))} dx \qquad \text{, put } \theta x = t \text{ ,}$$

$\theta\, dx = dt$,

$as\, x \to 0, t \to 0 \;\; and \;\; x \to x, t \to \theta\, x$, after simplification

$$F_w(x;c,\theta) = \frac{\theta^2}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))} \left(\gamma(c+1,\theta x) + \frac{1}{\theta^2}\gamma(c+3,\theta x) + \frac{2}{\theta}\gamma(c+2,\theta x) \right) ,$$

$x > 0, \theta > 0, c > 0,$ (2.2)

where θ and c are positive parameters and $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is a lower incomplete gamma function.

The graphs of probability density function and cumulative distribution function are plotted for different values of parameters θ and c given in fig.1 and fig. 2 respectively.

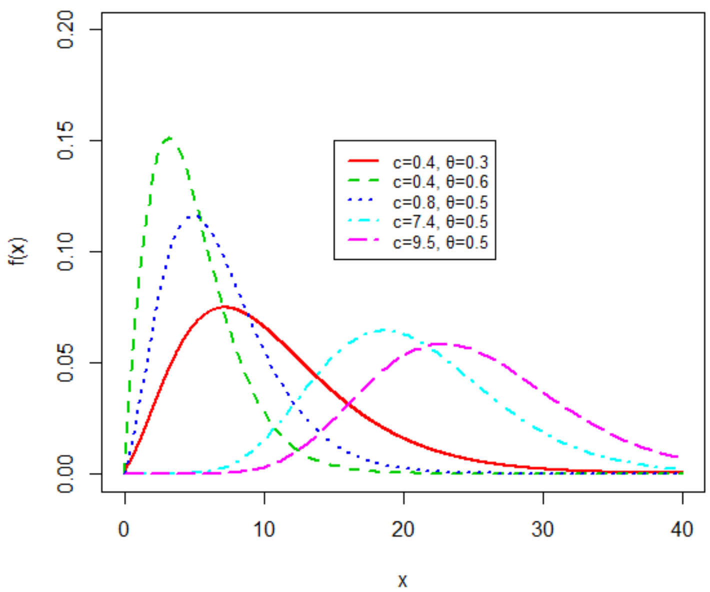

Fig. 1: Probability density function plot of Weighted Aradhana Distribution

Fig. 1 gives the description of some of the possible shapes of weighed Aradhana distribution for different values of the parameters θ and c. It illustrates that the density function of weighted Aradhana distribution is positively skewed. For fixed θ it becomes more and more flatter as the value of c is increased. Fig. 2 shows the graph of distribution function which is an increasing function.

3. SPECIAL CASES

Case 1: If we put $c=0$, then weighted Aradhana distribution (2.1) reduces to Aradhana distribution with probability density function as:

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2\theta+3}(1+x)^2 e^{-\theta x}\ x>0,\theta>0,$$

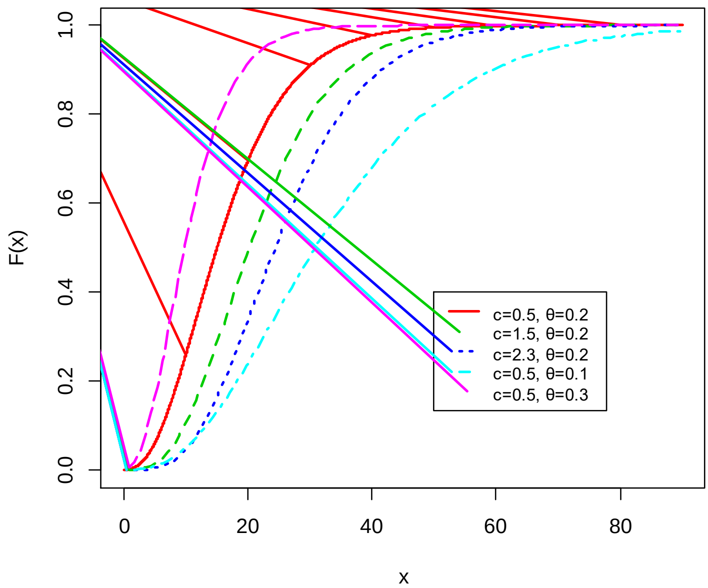

Fig.2: CDF plot of weighted Aradhana Distribution

4. RELIABILITY ANALYSIS

In this section, we have obtained the reliability, hazard rate, reverse hazard rate of the proposed weighted Aradhana Distribution.

4.1. Reliability function R(x)

The reliability function is defined as the probability that a system survives beyond a specified time. It is also referred to as survival or survivor function of the distribution. It can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Aradhana distribution is calculated as:

$$R_w(x,c,\theta)=1-\frac{\theta^2}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}\left(\gamma(c+1,\theta x)+\frac{1}{\theta^2}\gamma(c+3,\theta x)+\frac{2}{\theta}\gamma(c+2,\theta x)\right)$$

$$x>0,\theta>0,c>0, \qquad (4.1)$$

The graphical representation of the reliability function for the weighted Aradhana distribution is shown in fig. 3.

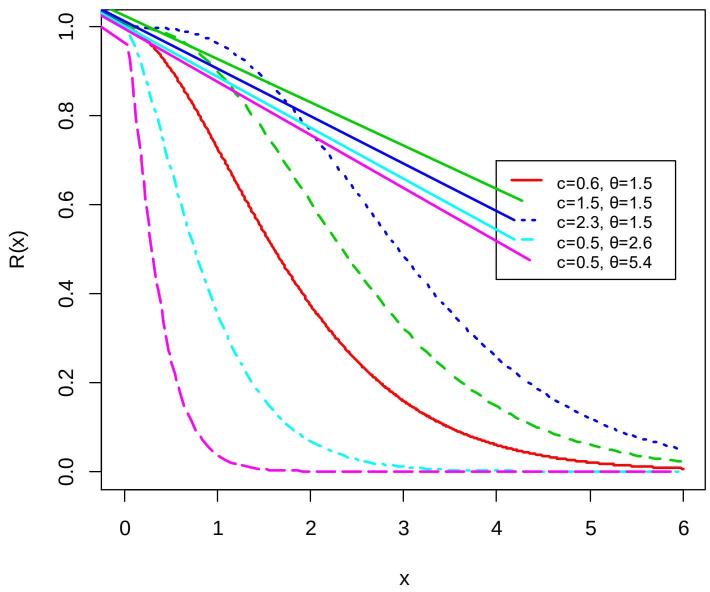

Fig. 3: Reliability function plot of weighted Aradhana Distribution

4.2. Hazard Function

The hazard function is also known as hazard rate, instantaneous failure rate or force of mortality and is given as:

$$H.R = h(x;c,\theta) = \frac{f_w(x,\theta)}{R_w(x,\theta)}.$$

$$h(x;c,\theta) = \frac{x^c \theta^{c+3}(1+x)^2 e^{-\theta x}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2)) - \theta^2(\gamma(c+1,\theta x) + \frac{1}{\theta^2}\gamma(c+3,\theta x) + \frac{2}{\theta}\gamma(c+2,\theta x))}$$

4.3. Reverse Hazard Rate

The reverse hazard rate of the weighted Aradhana distribution are respectively given as:

$$R.H.R = h_r(x,c,\theta) = \frac{f_w(x,\theta)}{F_w(x,\theta)} = \frac{x^c \theta^{c+3}(1+x)^2 e^{-\theta x}}{\theta^2(\gamma(c+1,\theta x) + \frac{1}{\theta^2}\gamma(c+3,\theta x) + \frac{2}{\theta}\gamma(c+2,\theta,x))},$$

$x > 0, \theta > 0, c > 0,$

5. STATISTICAL PROPERTIES

In this section, the different structural properties of the proposed weighted Aradhana model have been evaluated. These include moments, harmonic mean, moment generating function and characteristic function

5.1. Moments

Suppose X is a random variable following weighted Aradhana distribution with parameter θ, and then the rth moment for a given probability distribution is given by

$$\mu_r' = E(X_w{}^r) = \int_0^\infty x^r f_w(x,c,\theta)dx$$

$$= \int_0^\infty x^r \frac{x^c \theta^{(c+3)}(1+x)^2 e^{-\theta x}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))} dx$$

$$\mu_r' = \frac{(c+r)!(\theta^2+(c+r+1)(c+r+2)+2\theta(c+r+1))}{\theta^r\, c!(\theta^2+2\theta(c+1)+(c+1)(c+2)}$$

$$\mu_1' = \frac{(c+1)(\theta^2+(c+2)(c+3)+2\theta(c+2))}{\theta(\theta^2+2\theta(c+1)+(c+1)(c+2))}$$

$$V(X) = \frac{(c+1)(c+2)(\theta^2+(c+3)(c+4)+2\theta(c+3))}{\theta^2(\theta^2+2\theta(c+1)+(c+1)(c+2))} - \left[\frac{(c+1)(\theta^2+(c+2)(c+3)+2\theta(c+2)}{\theta(\theta^2+2\theta(c+1)+(c+1)(c+2))}\right]^2$$

5.2. Harmonic mean

The harmonic mean for the proposed model is computed as:

$$H.M = E\left[\frac{1}{X}\right] = \int_0^\infty \frac{1}{x} f_w(x;c,\theta)dx$$

$$= \int_0^\infty \frac{1}{x} \frac{x^c\theta^{c+3}(1+x)^2 e^{-\theta x}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}dx$$

$$H.M = \frac{c(\theta^2+2\theta(c+1)+(c+1)(c+2))}{\theta(\theta^2+c(c+1)+2c\theta)}, \qquad \theta>0, c>0,$$

5.3. Moment generating function and Characteristic function of Weighted Aradhana Distribution (WAD)

We will derive moment generating function and characteristic function of WAD in this section.

THEOREM 1.1. If X has the WAD (c,θ), then the moment generating function $M_X(t)$ and the characteristic function $\psi_X(t)$ has the following form

$$M_X(t) = \frac{\theta^{c+3}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2)}\left[\frac{(\theta-t)^2+(c+1)(c+2)+2(c+1)(\theta-t)}{(\theta-t)^2}\right]$$

and $$\psi_X(t) = \frac{\theta^{c+3}}{\theta^2+2\theta(c+1)+(c+1)(c+2)}\left\{\frac{(\theta-it)^2(c+1)(c+2)+2(c+1)(\theta-it)}{(\theta-it)^2}\right\}$$

respectively.

PROOF. We begin with the well known definition of the moment generating function given by

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx}\ f(x;\theta,c)\,dx$$

$$= \frac{\theta^{c+3}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2)}\int_0^\infty \left[x^c(1+x^2+2x)e^{-x(\theta-t)}\right]dx$$

$$= \frac{\theta^{c+3}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2)}\int_0^\infty [\frac{c!}{(\theta-t)^{c+1}} + \frac{(c+2)!}{(\theta-t)^{c+3}} + \frac{2(c+1)!}{(\theta-t)^{c+2}}]$$

$$\Rightarrow M_X(t) = \frac{\theta^{c+3}}{(\theta^2 + 2\theta(c+1) + (c+1)(c+2)}\left[\frac{(\theta-t)^2 + (c+1)(c+2) + 2(c+1)(\theta-t)}{(\theta-t)^2}\right]$$

Also we know that $\psi_X(t) = M_X(it)$

Therefore,

$$\psi_X(t) = \frac{\theta^{c+3}}{(\theta^2 + 2\theta(c+1) + (c+1) + (c+2))}\left\{\frac{(\theta-it)^2 + (c+1)(c+2) + 2(c+1)(\theta-it)}{(\theta-it)^2}\right\}$$

6. ORDER STATISTICS

Let $X_{(1)}, X_{(2)}, X_{(3)}...., X_{(n)}$ be the ordered statistics of the random sample $X_1, X_2, X_3,....X_n$ drawn from the continuous distribution with cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$, then the probability density function of rth order statistics $X_{(r)}$ is given by:

$$f_{x(r)}(x,c,\theta) = \frac{n!}{(r-1)!(n-r)!}\,f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}. \qquad r{=}1, 2, 3,\ldots,n$$

Using the equations (2.1) and (2.2), the probability density function of rth order statistics of weighted Aradhana distribution is given by:

$$f_{w(r)}(x,c,\theta)=\frac{n!}{(r-1)!(n-r)!}\frac{x^c\theta^{c+3}(1+x)^2e^{-\theta x}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}$$
$$\left[\frac{\theta^2}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}(\gamma(c+1,\theta x)+\frac{1}{\theta^2}\gamma(c+3,\theta x)+\frac{2}{\theta}\gamma(c+2,\theta x))\right]^{r-1}$$

$$\left[1-\frac{\theta^2}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}(\gamma(c+1,\theta x)+\frac{1}{\theta^2}\gamma(c+3,\theta x)+\frac{2}{\theta}\gamma(c+2,\theta x))\right]^{n-r}.$$

Then, the pdf of first order $X_{(1)}$ weighted Aradhana distribution is given by:

$$f_{w(1)}(x,c,\theta)=n\frac{x^c\theta^{c+3}(1+x)^2e^{-\theta x}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}$$
$$\left[1-\frac{\theta^2}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}(\gamma(c+1,\theta x)+\frac{1}{\theta^2}\gamma(c+3,\theta x)+\frac{2}{\theta}\gamma(c+2,\theta x)\right]^{n-1}.$$

and the pdf of nth order $X_{(n)}$ weighted Aradhana model is given as:

$$f_{w(n)}(x,c,\theta)=n\frac{x^c\theta^{c+3}(1+x)^2e^{-\theta x}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}$$
$$\left[\frac{\theta^2}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}(\gamma(c+1,\theta x)+\frac{1}{\theta^2}\gamma(c+3,\theta x)+\frac{2}{\theta}\gamma(c+2,\theta x))\right]^{n-1}.$$

7. METHOD OF MAXIMUM LIKELIHOOD ESTIMATION OF WEIGHTED ARADHANA DISTRIBUTION

This is one of the most useful method for estimating the different parameters of the distribution. Let $x_1, x_2, x_3.....x_n$ be the random sample of size n drawn from weighted Aradhana distribution, then the likelihood function of weighted Aradhana distribution is given as:

$$L(x\mid c,\theta)=\prod_{i=1}^{n} f(x;c,\theta)=\prod_{i=1}^{n}\frac{x^c\theta^{c+3}(1+x)^2e^{-\theta x}}{c!(\theta^2+2\theta(c+1)+(c+1)(c+2))}$$

The log likelihood function becomes:

$$\log L=c\log\sum_{i=1}^{n}x_i+n(c+3)\log(\theta)+2\log\sum_{i=1}^{n}(1+x_i)-\theta\sum_{i=1}^{n}x_i-n\log(c!)-n\log(\theta^2+2\theta(c+1)+(c+1)(c+2)) \tag{7.1}$$

Differentiating the log-likelihood function with respect to θ and c.This is done by partially differentiate (7.1) with respect to θ and c and equating the result to zero, we obtain the following normal equations,

$$\frac{d\log L}{d\theta}=\frac{n(c+3)}{\theta}-\sum_{i=1}^{n}x_i-\frac{n(2(\theta+c+1))}{\theta^2+2\theta(c+1)+(c+1)(c+2)}=0. \tag{7.2}$$

$$\frac{d\log L}{dc}=\log(\sum_{i=1}^{n}x_i)+n\log\theta+\frac{n}{c!}-\frac{n(2\theta+2c+3)}{\log(\theta^2+2\theta(c+1)+(c+1)(c+2))}=0$$

(7.3)

By solving equations (7.2) and (7.3), the maximum likelihood estimators of the parameters of the weighted Aradhana distribution are obtained using the numerical methods like Newton Raphson method.

We can compute the maximized unrestricted and restricted log likelihoods to construct the likelihood ratio (LR) statistics for testing the significance of weighted parameter of the proposed model. For example, we can use LR test to check whether the fitted weighted Aradhana distribution for a given data set is statistically “superior” to the fitted Aradhana distribution. In any case, hypothesis tests of the type $H_0:\Theta=\Theta_0$ versus $H_1:\Theta\neq\Theta_0$ can be performed using LR statistics. In this case, the LR statistic for testing H_0 versus H_1 is $\omega=2(L(\hat{\Theta})-L(\hat{\Theta}_0))$ where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically $(as\ n\to\infty)$ distributed as χ^2_k, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.05 at 95% confidence level.

7.1. Simulation Study of ML estimators of Weighted Aradhana Distribution

Using R statistical software for simulation study of Maximum Likelihood (ML) estimates, we study the performance of ML estimators for different sample sizes (n=25, 75, 100, 200, 400). Using inverse CDF technique for data generation from WAD, the process was repeated 500 times for calculation of bias, variance and Mean Square Error (MSE). For six random parameter combinations of WAD, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size (see table 1). Hence, performance of ML estimators is quite well and consistent in case of WAD.

Table I: Average bias, variance and MSE of ML estimates of WAD for different sample sizes

Parameter	n	c=0.5, θ =0.3			c=0.9, θ =0.7		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.316084	0.106875	0.206784	0.314656	0.398373	0.497381
θ		0.054112	0.004628	0.007556	0.14309	0.058252	0.078726
c	75	0.072541	0.04108	0.046343	0.010907	0.063456	0.063575
θ		0.026196	0.002303	0.002989	0.036994	0.017707	0.019076
c	100	0.082578	0.028625	0.035444	0.051299	0.040644	0.043275
θ		0.029879	0.002561	0.003454	0.047157	0.008902	0.011125
c	200	0.067269	0.019194	0.023719	0.058119	0.026055	0.029433
θ		0.026875	0.002309	0.003031	0.064354	0.004743	0.008885
c	400	0.039515	0.013038	0.014599	0.083429	0.020617	0.027578
θ		0.025358	0.002233	0.002876	0.029984	0.00469	0.005589
Parameter	n	c=1.2, θ =0.9			c=1.8, θ =1.5		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.20579	0.293873	0.336222	0.312345	0.295641	0.393201
θ		0.181872	0.111975	0.145052	0.211940	0.181030	0.225949
c	75	0.17945	0.15103	0.183232	0.203705	0.264116	0.305612
θ		0.112415	0.035996	0.048633	0.161424	0.108448	0.134506
c	100	0.109011	0.088331	0.100214	0.151620	0.147000	0.169988
θ		0.084267	0.034166	0.041267	0.125316	0.049200	0.064904
c	200	0.011496	0.020992	0.021125	0.089401	0.077759	0.085752
θ		0.017037	0.014832	0.015123	0.078205	0.044283	0.050398
c	400	0.001589	0.016442	0.016445	0.082926	0.044597	0.051474
θ		0.019839	0.006001	0.006394	0.047921	0.022859	0.025155

Parameter	n	c=2.5, θ =1.7			c=2.8, θ =2.5		
		Bias	Variance	MSE	Bias	Variance	MSE
c	25	0.342660	0.567742	0.685158	0.325616	1.161363	1.267389
θ		0.327808	0.383191	0.490649	0.295561	0.544709	0.632065
c	75	0.131818	0.210193	0.227569	0.425697	0.452776	0.633994
θ		0.092191	0.111219	0.119718	0.328474	0.239748	0.347643
c	100	-0.028951	0.094912	0.095750	0.286518	0.190818	0.272910
θ		0.058024	0.046838	0.050205	0.259910	0.143187	0.210740
c	200	-0.023804	0.104879	0.105446	0.137308	0.125422	0.144275
θ		0.003426	0.073616	0.073628	0.101188	0.068870	0.079109
c	400	0.042858	0.065325	0.067162	0.039300	0.068572	0.070116
θ		0.057663	0.028284	0.031608	0.030030	0.032064	0.032966

8. LIKELIHOOD RATIO TEST

Let $x_1, x_2, ..., x_n$ be a random sample drawn from Aradhana distribution or weighted Aradhana distribution. We test the hypothesis

$H_0 : f(x) = f(x,\theta)\, v/s\, H_1 : f(x) = f_w(x,\theta)$

In order to test whether the random sample come from Aradhana distribution or weighted Aradhana distribution, we use the following test statistic

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_w(x,\theta)}{f(x,\theta)}$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{\dfrac{x^c \theta^{c+3}(1+x)^2 e^{-\theta x}}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))}}{\dfrac{\theta^3(1+x)^2 e^{-\theta x}}{\theta^2 + 2\theta + 2}}$$

$$\Delta = \frac{\theta^{cn}(\theta^2 + 2\theta + 2)^n \prod_{i=1}^{n} x_i^{\,c}}{(c!)^n (\theta^2 + 2\theta(c+1) + (c+1)(c+2))^n}$$

We reject the null hypothesis if

$$\Delta = \left[\frac{\theta^c (\theta^2 + 2\theta + 2)}{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))} \right]^n \prod_{i=1}^{n} x_i^{\,c} > k$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^* \quad where k^* = k\left[\frac{c!(\theta^2 + 2\theta(c+1) + (c+1)(c+2))}{\theta^c (\theta^2 + 2\theta + 2)} \right]^n$$

For a large sample of size n, $2\log\Delta$ is distributed as chi-squared distribution with one degree of freedom. Also we reject the null hypothesis when probability value is given by $p(\Delta^* > \beta^*), where\, \beta^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the statistic Δ^*.

9. APPLICATIONS OF WEIGHTED ARADHANA DISTRIBUTIONS

To illustrate the significance of the suggested model, a real life example is presented. The goodness of fit result of the suggested model (Weighted Aradhana Distribution) is compared with the base model (Aradhana Distribution). In this case, we analyze the strength data, which was originally reported by Badar and Priest (1982) and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge lengths of 10mm with sample sizes n =63. Surles and Padgett (2001) also studied this data set. The data is given in table 2.

Table II: Strength data set (gauge lengths of 10 mm).

1.901	2.132	2.203	2.228	2.257	2.35	2.361	2.396	2.397	2.445	2.454
2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659
2.675	2.738	2.74	2.856	2.917	2.928	2.937	2.937	2.977	2.996	3.03
3.125	3.139	3.145	3.22	3.223	3.235	3.243	3.264	3.272	3.294	3.332
3.346	3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628	3.852
3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.02			

By fitting Aradhana distribution and weighted Aradhana distribution to this data set, we observed that weighted Aradhana distribution fits statistically well as compared to Aradhana distribution. For comparison of the two distributions, we consider the criteria like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, AICC and BIC values.

$$\text{AIC} = 2k - 2\log L \qquad \text{AICC} = \text{AIC} + \frac{2k(k+1)}{n-k-1} \text{ and BIC} = k\log n - 2\log L$$

where k is the number of parameters in the statistical model, n is the sample size and -log$L$ is the maximized value of the log-likelihood function under the considered model. From Table 3, it has been observed that Weighted Aradhana distribution have the lesser AIC, AICC, -log$L$ and BIC values as compared to Aradhana Distribution. Hence we can conclude that the Weighted Aradhana distribution leads to a better fit than the Aradhana distribution. Also likelihood ratio test reveals that weighted parameter c plays statistically significant role for data set given in table 2.

Table III: ML estimates, AIC, AICC, BIC, -logL Criterion and Likelihood Ratio Test values for strength data set (gauge lengths of 10 mm).

Criteria	$\hat{\theta}$	$\hat{c}$	**-logL**	**AIC**	**AICC**	**BIC**	**Likelihood Ratio Statistic**
Aradhana Distribution	0.766 (0.05)	-	112.075	226.150	226.216	228.293	110.36
Weighted Aradhana Distribution	8.49 (1.49)	23.4 (4.52)	56.894	117.787	117.987	122.074	

10. CONCLUSION

In the present study, we have introduced a new generalization of the Aradhana distribution called as Weighted Aradhana distribution. The subject distribution is generated by using the weighting technique and taking the one parameter Aradhana distribution as the base distribution. Some mathematical properties along with reliability measures are discussed. Model is fitted to real life data for examining its significance.

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A stochastic approach to number of corona virus cases

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Abstract

This paper introduces a stochastic approach to case numbers of a pandemic disease. By defining the stochastic process random walk process is used. Some stochastic aspects for this disease are argued before stochastic study is started. During random walk process modeling new patients, recovering patients and dead conclusions are modelled and probabilities changes in some stages. Let the structure of this study includes vanishing process as a walk step, some wave happenings like big differences about spread speed as a big step in treatment- an effective vaccine or an influential chemical usage- a second corona virus pumping with virus mutation, a second global happening which bumping virus spread are defined as stages. This study only simulates a stochastic process of corona virus effects.

Mathematics Subject Classification: 60J74
Keywords: Random walk; vanishing probability; stochastic process; stationary probability; Markov chain.

1. INTRODUCTION

In industrial engineering, studying in reliability and stochastic processes of systems is very popular. In many situations this issue has vital importance. Besides there are many studies with random walks and many of them include independent increases. In a study a new class is determined by phase type distribution and in [1] bivariate geometric distribution was chosen as a discrete phase type.

Let $X_1, X_2, \ldots, X_N$ be a sequence of independent and identically distributed (iid) random variables independent of the discrete random variable N. Commonly, the main issue is to determine the distribution of the compound random variable $S(X_1, X_2, \ldots, X_N)$ for popular choices of S, to illustrate $S(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{n} x_i$, $S(x_1, x_2, \ldots, x_N) = min(x_1, x_2, \ldots, x_N)$,, $S(x_1, x_2, \ldots, x_N) = min(x_1, x_2, \ldots, x_N)$. Some studies in this area are in [2]-[5] .

The discrete phase type distributions are quite rich in modeling waiting time distributions like geometric, negative binomial, and bivariate geometric distribution.

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In a Markov process a definition for discrete phase type distribution can be defined with d transient states and an absorbing state may be defined as ”0”.

In [6] extreme shock models in phase type was studied and by this optimal component changes determined. At first rules in shock models assumed and later optimization calculated with phase type distribution.

In [7], multivariate Poisson process combined with Markov process by usability of phase type in shock models.

In this study a stochastic approach is introduced and the process of this system is studied. This random walk system includes vanishing process for dead and recovered patients, new patients and the probabilities in probability transition are changing with some stages that define big differences about spread speed as a big step in treatment- an effective vaccine or an influential chemical usage- a second corona virus wave with virus mutation and a second global happening which bumps virus spread.

2. MATERIAL AND METHODS

2.1. Multivariate geometric distribution

The univariate geometric distribution is famous to be the sole discrete distribution which has memoryless property. Multivariate geometric distribution has been studied in many articles [8]-[12]. Especially in shock models this memoryless specialty is studied commonly.

Every multivariate extension in geometric distribution have the same memoryless specialty and there can be widespread seen on studies about Marshall–Olkin multivariate exponential distribution via multivariate geometric compounding of exponentially distributed random variables.

Bivariate geometric distribution was examined in a study and the formulation is as follow [13].

$$Pr(N = n, M = m) = \sum\sum\binom{m+n}{n} {p_1}^n {p_2}^m p_0 \qquad p_0 + p_1 + p_2 = 1 \quad \text{n, m} = 0,1,2,\dots$$

p_0 : probability of no adding,

p_1 : probability of adding to upper part,

p_2 : probability of adding to lower part.

To decide the number of new components in system marginal distribution functions are needed. We reach marginal distributions as follow.

$$Pr(N=n)=\sum_{m=0}^{\infty}\frac{(m+n)!}{m!\,n!}{p_1}^n{p_2}^m p_0=\frac{{p_1}^n p_0}{(1-p_2)^{n+1}}\sum_{m=0}^{\infty}\frac{(m+n)!}{m!\,n!}{p_2}^m(1-p_2)^{n+1}$$

With using negative binomial distribution the sum is 1. So the first marginal distribution is in equation (1).

$$Pr(N=n)=\frac{{p_1}^n p_0}{(1-p_2)^{n+1}} \tag{1}$$

Geometric distribution with success rate $\frac{p_0}{p_0+p_1}$ is gained. Second marginal distribution can be obtained with the same way in equation (2).

$$Pr(M=m)=\frac{{p_2}^m p_0}{(1-p_1)^{m+1}} \tag{2}$$

Therefore geometric distribution with $\frac{p_0}{p_0+p_2}$ success rate is gained.

In [14] applications of reliabilities in bivariate geometric distribution was assessed and gained characteristics of this valuable distribution.

In that study moment generating function and characteristics obtained as below.

$$E(e^{t_1 n+t_2 m})=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{t_1 n+t_2 m}\binom{m+n}{n}(p_1)^n(p_2)^m p_3$$

$$=\frac{p_3}{1-p_1e^{t_1}-p_2e^{t_2}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\binom{m+n}{n}(e^{t_1}p_1)^n(e^{t_2}p_2)^m(1-p_1e^{t_1}-p_2e^{t_2})$$

$$=\frac{p_3}{1-p_1e^{t_1}-p_2e^{t_2}}$$

$$E(N)=\frac{d}{dt_1}\left(\frac{p_3}{1-p_1e^{t_1}-p_2e^{t_2}}\right)\bigg|_{t_1=0,t_2=0}=\frac{p_3p_1}{(1-p_1-p_2)^2}=\frac{p_1}{p_3}$$

$$E(M)=\frac{p_2}{p_3}$$

$$E(NM)=\frac{d^2}{dt_1dt_2}\left(\frac{p_3}{1-p_1e^{t_1}-p_2e^{t_2}}\right)\bigg|_{t_1=0,t_2=0}=\frac{2p_1p_3}{{p_3}^2}$$

$$E(N^2)=\frac{d^2}{d{t_1}^2}\left(\frac{p_3}{1-p_1e^{t_1}-p_2e^{t_2}}\right)\bigg|_{t_1=0,t_2=0}=\frac{p_1}{p_3}\left(1+\frac{2p_1}{p_3}\right)$$

$$E(M^2) = \frac{d^2}{d{t_2}^2}\left(\frac{p_3}{1 - p_1 e^{t_1} - p_2 e^{t_2}}\right)\bigg|_{t_1=0, t_2=0} = \frac{p_2}{p_3}\left(1 + \frac{2p_2}{p_3}\right)$$

2.2. Markov chain

When the next position of a system rely solely on its current position and not on its prior positions, the system mostly be assumed to be a Markovian process. A kind of this model is suggested in [15]. In that process, Markov model was used for describing deterioration.

We pay attention to Markov process by defining a finite state space with either a discrete or continuous time parameter [16]. In literature there have been many studies in which a more general process called a "semi-Markov process" was discussed. This process includes discrete time parameter Markov chain. Some processes in semi-Markov process may be used to describe some classes of systems related to inspection, replacement, and repair.

When we try to find a definite form of steps with using bivariate geometric transition probabilities;

$$ {P_{ij}}^n = P(X_{n+1} = j | X_n = i)$$

We may assume transition probabilities are independent.

When the comings are independent, then there are two different marginal geometric distributions which are available for comings. Their probability values are $\frac{p_0}{p_0+p_1}$ and $\frac{p_0}{p_0+p_2}$.

When the arrivals are dependent, then first marginal distribution is geometric distribution with probability value $\frac{p_0}{p_0+p_1}$ and the other marginal distribution is negative binomial distribution with unsuccessful try number m, success number is n.

2.3. Vanishing process

Assume that the time till vanishing of a Markov sequence which has finite state space is shown with random variable T.

Random variable T has a discrete phase type distribution and probability function is as below,

$$P(T = n) = \underline{a}Q^{n-1}\underline{t'}$$

Q is a matrix which includes transition probabilities between transition states,

$\underline{t}$ is a vector which includes transition probabilities between transition states to vanishing state

$\underline{a}$ is a sub vector of starting probabilities.

Example (Geometric Distribution)

Describing trial numbers till first success. X_n shows the value in n^{th} try. So $X_n \in \{0,1\}$, 1 is vanishing state. The probability transition matrix is as follows,

$$p = \begin{matrix} 0 \\ 1 \end{matrix} \quad \begin{matrix} 0 & 1 \\ \left\| \begin{matrix} 1-p & p \\ 0 & 1 \end{matrix} \right\| \end{matrix}$$

$Q = 1 - p$, $\underline{t'} = p$, $\underline{a} = 1$

$$P(T = n) = (1 - p)^{n-1}p$$

Thus we may say discrete phase type distributions is a generalized version of geometric distribution.

2.4. Random walk process

Assume $\{Y_n, n = 1,2, \dots\}$ is a sequence of random variables with d-dimensional and it is independently identical distributed. X_0 is a constant vector in $\mathbb{R}^d$.

$\{X_n, n = 1,2, \dots\}$ process which is defined with $X_n = X_0 + Y_1 + \cdots + Y_n$, $n = 1,2, \dots$ is called

d-dimensional random walk. If X_0 and Y_n valued in $\mathbb{Z}^d$ space than $\{X_n, n = 1,2, \dots\}$ process is called d-dimensional cage random walk. In cage random walk when $\varepsilon_k = -1$ or 1, $k = 1.2 \dots, d$ and Y_n jumps only from $x = (x_1, x_2, \dots, x_d)$ to $y = (x_1 + \varepsilon_1, x_2 + \varepsilon_2, \dots, x_d + \varepsilon_d)$ than this is called simple random walk.

For a simple random walk for any given jump each jumps from $2d$ moves happens with probability $p = \frac{1}{2d}$ thereby this process is called symmetrical random walk process.

In all these situations when Y_n jumps are only independent but not with identical distribution than $\{X_n, n = 1,2, \dots\}$ process is called non-homogenous random walk.

$$X_1 = X_0 + Y_1$$
$$X_2 = X_1 + Y_2$$
$$\vdots$$
$$X_n = X_{n-1} + Y_n \, , n = 1,2, \dots$$

Thus a random walk is independently incremental and stationary incremental.

3. RESULTS AND DISCUSSIONS

The structure of our approach includes two different options which are increasing and vanishing processes. When the next step goes to increasing the number of patients increases with new additions, on the other hand when the next step goes to vanishing process the number of patients decreases with dead or recover. Our approach assumes some steps which define the rules of this complex structure.

- The process starts after 100000 patients determined.
- The increasing probability in transition is p, and vanishing probability in transition is q.
- X_n shows the conclusion of corona disease after n step.
- In the first 15 days we assume $p = 3q$.
- In the second 15 days we assume $p = q$. Because some precautions may take into account.
- In the third 15 days with more strict precautions $p = \frac{1}{3}q$.
- In the fourth 15 days people may be pay less attention to disease $p = q$.
- After 60 days we assume the probabilities as $p = \frac{1}{3}q$ till the disease is over.

- p_0 represents no change in both ways - increasing patient number and vanishing patients. $p_0 = 0.001$ in every step.
- When no change in both ways occurs, the system halts.
- When a walk goes to increases jump it adds %5 of patient number in the starting point of step adding to the patient number and when a walk goes to decreases jump it goes %5 of patient number in the starting point to reduce down with recover or dead patients.

4. APPLICATION

In every step there are formed many equations that they create path of the process. Some examples are as follows;

$$P(X_2 = 0, X_4 = 0) = P(X_2 = 0)P(X_4 - X_2 = 0) = P(X_2 = 0)P(X_2 = 0)$$

$$P(X_4 = 2, X_{10} = 6) = P(X_4 = 2)P(X_{10} - X_4 = 4) = P(X_4 = 2)P(X_6 = 4)$$

$$P(X_4 = 2, X_{10} = 6, X_{14} = 8) = P(X_4 = 2)P(X_{10} - X_4 = 4)P(X_{14} - X_{10} = 2)$$

$$= P(X_4 = 2)P(X_6 = 4)P(X_4 = 2)$$

Expectation numbers of increases and decreases in the path of the first step process may be find as below;

$$E(N) = \frac{d}{dt_1}\left(\frac{p_3}{1 - p_1 e^{t_1} - p_2 e^{t_2}}\right)\Bigg|_{t_1=0, t_2=0} = \frac{p_3 p_1}{(1 - p_1 - p_2)^2} = \frac{p_1}{p_3}$$

After gaining this expectation we consider that whether this walk does not stop in 15 step, we expect that this walk includes $E(N)$ increases and we will find the other expectation for decreases with

$$E(M) = \frac{d}{dt_2}\left(\frac{p_3}{1 - p_1 e^{t_1} - p_2 e^{t_2}}\right)\Bigg|_{t_1=0, t_2=0} = \frac{p_3 p_2}{(1 - p_1 - p_2)^2} = \frac{p_2}{p_3}$$

And we portion the expectations for gaining 15 step expectations.

$$expectation\ of\ increases = 15\frac{E(N)}{E(M) + E(N)}$$

$$expectation\ of\ decreases = 15\frac{E(M)}{E(M) + E(N)}$$

Every jump position in random walk only relates to previous one not the others so every jump has basic Markov process specialty. Some of the transition probability matrices of the first step of this system are as below. These matrices tell us the jumps' position probability from the beginning.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.001 & 0.74925 & 0.24975 \\ 0.001 & 0.74925 & 0.24975 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.002 & 0.7485 & 0.2495 \\ 0.002 & 0.7485 & 0.2495 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.004 & 0.747 & 0.249 \\ 0.004 & 0.747 & 0.249 \end{bmatrix}$$

There may be many different paths of the same position in the same step. So each path must be included in the calculation of the expectation.

$$E(X_n) = \sum_{i=1}^{n} P(X_i = x_i)\, x_i$$

To illustrate, a graph of a path is in figure 1.

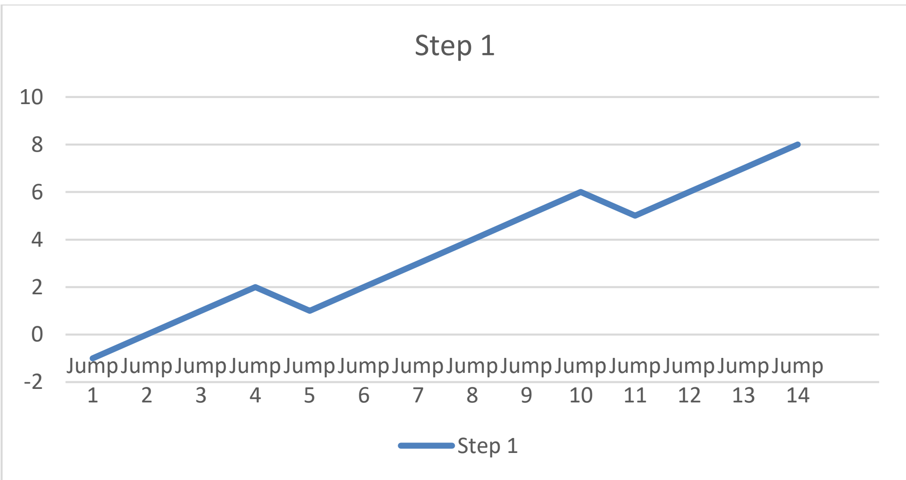

Fig. 1. One path of step 1

Step 1

Some of probability transition matrices are as below;

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.001 & 0.74925 & 0.24975 \\ 0.001 & 0.74925 & 0.24975 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.002 & 0.7485 & 0.2495 \\ 0.002 & 0.7485 & 0.2495 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.004 & 0.747 & 0.249 \\ 0.004 & 0.747 & 0.249 \end{bmatrix}$$

Some of paths and probability calculations are as follow;

$$P(X_2 = 0, X_4 = 0) = P(X_2 = 0)P(X_4 - X_2 = 0) = P(X_2 = 0)P(X_2 = 0)$$

$$= P(X_2 = 0)^2 = \left(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\right)^2$$

$$= \left(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\right)^2$$

$$= \left(P(X_1 = -1)P(X_1 = 1) + P(X_1 = 1)P(X_1 = -1)\right)^2$$

$$= (0.74925 * 0.24975 + 0.24975 * 0.74925)^2 = 0.14$$

$$P(X_4 = 2, X_{10} = 6) = P(X_4 = 2)P(X_{10} - X_4 = 4) = P(X_4 = 2)P(X_6 = 4)$$

$$= 4P(X_1 = -1)P(X_1 = 1)^3 6P(X_1 = -1)P(X_1 = 1)^5 = 0.42 * 0.3538 = 0.1486$$

The probabilities of each situation with including every path is in Table 1.

$$E(X_n) = \sum_{i=1}^{n} P(X_i = x_i)\, x_i$$

$$E(X_{15}) = P(X_1 = 1)^{15} P(X_1 = -1)^0 * \left(100000 * (1 + 15 * 0.05 - 0 * 0.005)\right)$$

$$+P(X_1 = 1)^{14} P(X_1 = -1)^1 * \left(100000 * (1 + 14 * 0.05 - 1 * 0.005)\right)$$

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$$+P(X_1 = 1)^{0} P(X_1 = -1)^{15} * \left(100000 * (1 + 0 * 0.05 - 15 * 0.005)\right)$$

$$= 135451$$

Step 2

Some of probability transition matrices in the second 15 days are as below;

$$P=\begin{bmatrix}1 & 0 & 0\\ 0.001 & 0.4995 & 0.4995\\ 0.001 & 0.4995 & 0.4995\end{bmatrix}$$
$$P^2=\begin{bmatrix}1 & 0 & 0\\ 0.002 & 0.499 & 0.499\\ 0.002 & 0.499 & 0.499\end{bmatrix}$$
$$P^3=\begin{bmatrix}1 & 0 & 0\\ 0.004 & 0.498 & 0.498\\ 0.004 & 0.498 & 0.498\end{bmatrix}$$

Some of paths and probability calculations are as follow;

$$P(X_2=0,X_4=0)=P(X_2=0)P(X_4-X_2=0)=P(X_2=0)P(X_2=0)$$
$$=P(X_2=0)^2=\big(P(X_1=-1),P(X_2=0)+P(X_1=1),P(X_2=0)\big)^2$$
$$=\big(P(X_1=-1),P(X_2=0)+P(X_1=1),P(X_2=0)\big)^2$$
$$=\big(P(X_1=-1)P(X_1=1)+P(X_1=1)P(X_1=-1)\big)^2$$
$$=(0.4995*0.4995+0.4995*0.4995)^2=0.249$$

$$P(X_4=2,X_{10}=6)=P(X_4=2)P(X_{10}-X_4=4)=P(X_4=2)P(X_6=4)$$
$$=4P(X_1=-1)P(X_1=1)^3 6P(X_1=-1)P(X_1=1)^5=0.249*0.0931$$
$$=0.3421$$

The probabilities of each situation with including every path is in Table 1.

$$E(X_n)=\sum_{i=1}^{n}P(X_i=x_i)\,x_i$$
$$E(X_{15})=P(X_1=1)^{15}P(X_1=-1)^0*\big(135451*(1+15*0.05-0*0.005)\big)$$
$$+P(X_1=1)^{14}P(X_1=-1)^1*\big(135451*(1+14*0.05-1*0.005)\big)$$
$$\vdots$$
$$+P(X_1=1)^{0}P(X_1=-1)^{15}*\big(135451*(1+0*0.05-15*0.005)\big)$$
$$=133433$$

Step 3

Some of probability transition matrices are as below;

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.001 & 0.24975 & 0.74925 \\ 0.001 & 0.24975 & 0.74925 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.002 & 0.2495 & 0.7485 \\ 0.002 & 0.2495 & 0.7485 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.004 & 0.249 & 0.747 \\ 0.004 & 0.249 & 0.747 \end{bmatrix}$$

Some of paths and probability calculations are as follow;

$$P(X_2 = 0, X_4 = 0) = P(X_2 = 0)P(X_4 - X_2 = 0) = P(X_2 = 0)P(X_2 = 0)$$
$$= P(X_2 = 0)^2 = \big(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\big)^2$$
$$= \big(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\big)^2$$
$$= \big(P(X_1 = -1)P(X_1 = 1) + P(X_1 = 1)P(X_1 = -1)\big)^2$$
$$= (0.74925 * 0.24975 + 0.24975 * 0.74925)^2 = 0.14$$

$$P(X_4 = 2, X_{10} = 6) = P(X_4 = 2)P(X_{10} - X_4 = 4) = P(X_4 = 2)P(X_6 = 4)$$
$$= 4P(X_1 = -1)P(X_1 = 1)^3 6P(X_1 = -1)P(X_1 = 1)^5 = 0.0155 * 0.0043$$
$$= 0.0198$$

The probabilities of each situation with including every path is in Table 1.

$$E(X_n) = \sum_{i=1}^{n} P(X_i = x_i)\, x_i$$

$$E(X_{15}) = P(X_1 = 1)^{15} P(X_1 = -1)^0 * \big(133433 * (1 + 15 * 0.05 - 0 * 0.005)\big)$$
$$+ P(X_1 = 1)^{14} P(X_1 = -1)^1 * \big(133433 * (1 + 14 * 0.05 - 1 * 0.005)\big)$$
$$\vdots$$
$$+ P(X_1 = 1)^0 P(X_1 = -1)^{15} * \big(133433 * (1 + 0 * 0.05 - 15 * 0.005)\big)$$
$$= 82153$$

Step 4

Some of probability transition matrices in the fourth 15 days are as below;

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.001 & 0.4995 & 0.4995 \\ 0.001 & 0.4995 & 0.4995 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.002 & 0.499 & 0.499 \\ 0.002 & 0.499 & 0.499 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.004 & 0.498 & 0.498 \\ 0.004 & 0.498 & 0.498 \end{bmatrix}$$

Some of paths and probability calculations are as follow;

$$P(X_2 = 0, X_4 = 0) = P(X_2 = 0)P(X_4 - X_2 = 0) = P(X_2 = 0)P(X_2 = 0)$$

$$= P(X_2 = 0)^2 = \left(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\right)^2$$

$$= \left(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\right)^2$$

$$= \left(P(X_1 = -1)P(X_1 = 1) + P(X_1 = 1)P(X_1 = -1)\right)^2$$

$$= (0.4995 * 0.4995 + 0.4995 * 0.4995)^2 = 0.249$$

$$P(X_4 = 2, X_{10} = 6) = P(X_4 = 2)P(X_{10} - X_4 = 4) = P(X_4 = 2)P(X_6 = 4)$$

$$= 4P(X_1 = -1)P(X_1 = 1)^3 6P(X_1 = -1)P(X_1 = 1)^5 = 0.249 * 0.0931$$

$$= 0.3421$$

The probabilities of each situations with including every path is in Table 1.

$$E(X_n) = \sum_{i=1}^{n} P(X_i = x_i)\, x_i$$

$$E(X_{15}) = P(X_1 = 1)^{15} P(X_1 = -1)^0 * \left(82153 * (1 + 15 * 0.05 - 0 * 0.005)\right)$$

$$+P(X_1 = 1)^{14} P(X_1 = -1)^1 * \left(82153 * (1 + 14 * 0.05 - 1 * 0.005)\right)$$

$$\vdots$$

$$+P(X_1 = 1)^0 P(X_1 = -1)^{15} * \left(82153 * (1 + 0 * 0.05 - 15 * 0.005)\right)$$

$$= 80929$$

Step 5

Some of probability transition matrices are as below;

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.001 & 0.24975 & 0.74925 \\ 0.001 & 0.24975 & 0.74925 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0.002 & 0.2495 & 0.7485 \\ 0.002 & 0.2495 & 0.7485 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.004 & 0.249 & 0.747 \\ 0.004 & 0.249 & 0.747 \end{bmatrix}$$

Some of paths and probability calculations are as follow;

$$P(X_2 = 0, X_4 = 0) = P(X_2 = 0)P(X_4 - X_2 = 0) = P(X_2 = 0)P(X_2 = 0)$$

$$= P(X_2 = 0)^2 = \big(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\big)^2$$

$$= \big(P(X_1 = -1), P(X_2 = 0) + P(X_1 = 1), P(X_2 = 0)\big)^2$$

$$= \big(P(X_1 = -1)P(X_1 = 1) + P(X_1 = 1)P(X_1 = -1)\big)^2$$

$$= (0.74925 * 0.24975 + 0.24975 * 0.74925)^2 = 0.14$$

$$P(X_4 = 2, X_{10} = 6) = P(X_4 = 2)P(X_{10} - X_4 = 4) = P(X_4 = 2)P(X_6 = 4)$$

$$= 4P(X_1 = -1)P(X_1 = 1)^3 6P(X_1 = -1)P(X_1 = 1)^5 = 0.0155 * 0.0043$$

$$= 0.0198$$

The probabilities of each situation with including every path is in Table 1.

$$E(X_n) = \sum_{i=1}^{n} P(X_i = x_i)\, x_i$$

$$E(X_{15}) = P(X_1 = 1)^{15} P(X_1 = -1)^0 * \big(80929 * (1 + 15 * 0.05 - 0 * 0.005)\big)$$

$$+P(X_1 = 1)^{14} P(X_1 = -1)^1 * \big(80929 * (1 + 14 * 0.05 - 1 * 0.005)\big)$$

$$\vdots$$

$$+P(X_1 = 1)^{0} P(X_1 = -1)^{15} * \big(80929 * (1 + 0 * 0.05 - 15 * 0.005)\big)$$

$$= 49827$$

But in this step we try to predict the conclusion of this pandemic disease. When we continue with the same transition probabilities, we reach a conclusion that this pandemic disease starts to decrease sharply after 23 days in step 5. After 25 days the expectation is 29598, after 32 days the expectation is 15675. According to expectation of this stochastic process this pandemic is totally wipe out after 36 days.

5. CONCLUSION

Related with corona virus we introduce a stochastic approach to this disease. At first we defined the rules of this pandemics' spread. Later we make a study to recognize the spread of this disease step by step. In the end we try to define the approximate time of vanishing for pandemic according to rules of our stochastic process. And according to stochastic process for this system, we conclude that obeying rules is very important to strict spreading of this pandemic. We conclude that if reliable data sets use in the process for defining probabilities in transition matrices, more exact and reliable conclusions will be reached.

APPENDIX

number of increase	**STEP 1**	**STEP 2**	**STEP 3**	**STEP 4**	**STEP 5**
0	0,013164	3,01E-05	9,17E-10	3,01E-05	9,17E-10
1	0,065822	0,000451	4,13E-08	0,000451	4,13E-08
2	0,153585	0,003157	8,67E-07	0,003157	8,67E-07
3	0,221845	0,013679	1,13E-05	0,013679	1,13E-05
4	0,221845	0,041036	0,000101	0,041036	0,000101
5	0,162686	0,090279	0,000669	0,090279	0,000669
6	0,090381	0,150465	0,003347	0,150465	0,003347
7	0,038735	0,193455	0,012912	0,193455	0,012912
8	0,012912	0,193455	0,038735	0,193455	0,038735
9	0,003347	0,150465	0,090381	0,150465	0,090381
10	0,000669	0,090279	0,162686	0,090279	0,162686
11	0,000101	0,041036	0,221845	0,041036	0,221845
12	1,13E-05	0,013679	0,221845	0,013679	0,221845
13	8,67E-07	0,003157	0,153585	0,003157	0,153585
14	4,13E-08	0,000451	0,065822	0,000451	0,065822
15	9,17E-10	3,01E-05	0,013164	3,01E-05	0,013164

Fig. 2. Probabilities of situations included path combinations.

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CORRECTION*

Exponentiated quasi power Lindley power series distribution with applications in medical science

A. HASSAN, A. RASHID AND N. AKHTAR

Abstract

The present paper introduces an advanced five parameter lifetime model which is obtained by compounding exponentiated quasi power Lindley distribution with power series family of distributions. The EQPLPS family of distributions contains several lifetime sub-classes such as quasi power Lindley power series, power Lindley power series, quasi Lindley power series and Lindley power series. The proposed distribution exhibits decreasing, increasing and bathtub shaped hazard rate functions depending on its parameters. It is more flexible as it can generate new lifetime distributions as well as some existing distributions. Various statistical properties including closed form expressions for density function, cumulative function, limiting behaviour, moment generating function and moments of order statistics are brought forefront. The capability of the quantile measures in terms of Lambert W function is also discussed. Ultimately, the potentiality and the flexibility of the new class of distributions has been demonstrated by taking three real life data sets by comparing its sub-models.

Mathematics Subject Classification 2010: 62E15, 60E05.
Keywords: Exponentiated Quasi Power Lindley distribution, Lambert W function, order statistics, MLE.

1. INTRODUCTION

The modeling of lifetime data has received prominent attention from researchers for the last decade. To predict the ambiguous behaviour of random events as death, appearance of some disease and system failure is a major concern for statisticians. There are diverse lifetime models available for researchers to predict this uncertain behaviour but at times due to complex pattern of data sets, these models do not provide a suitable fit. In order to prevail from this difficulty, researchers have focussed their

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attention on compounding mechanism which is a sound way to develop an appropriate and flexible models to fit the lifetime data of different types.

Keeping this in mind, Adamidis and Loukas (1998), Kus (2007), Tahmasbi (2008) constructed several lifetime distributions through this mechanism that proved to be operative in modeling of lifetime data having different features. Researchers developed many lifetime distributions by this technique which are very flexible and can accommodate different types of data sets. For instance, Chahkandi and Ganjali (2009) obtained a compound class of exponential power series distributions. As Weibull distribution contains the exponential distribution as a special case, Morais and Baretto-Souza (2011) substituted the exponential distribution with a Weibull distribution in this mechanism and obtained a compound class of Weibull power series distributions which contains EPS distribution as a special case. Adil and Jan (2016) introduced a new family of lifetime distributions by compounding a Lindley distribution with power series distribution that contains Lindley Geometric as special case due to Zakerzadeh and Mahmoudi (2012). Moreover, many authors discussed some special cases of the LPS family that are very flexible in terms of density and hazard rate functions. Adil and Jan (2018a, 2018b) obtained a lifetime distribution for series system and generalized version of complementary Lindley power series family of compound lifetime distributions related to parallel system which generalizes most of the lifetime distributions and have versatile properties. Arsalan et al. (2019) introduced the exponential Burr XII power series.

2. EXPONENTIATED QUASI POWER LINDLEY DISTRIBUTION

Manuela Ghica et al. (2017) introduced an Exponentiated Quasi Power Lindley Distribution (EQPLD) defined by its pdf as

$$g(y;\alpha,\theta,\beta,b,)=\frac{\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b};x>0,\alpha,\theta,\beta,b>0$$

This new distribution reduces to the quasi Lindley distribution, the exponential distribution and gamma distribution. In terms of reliability, the various shapes of the EQPL distribution give it a benefit, being more flexible to model many real systems which generally exhibit bath-tub shaped failure rate. The corresponding cdf of the above equation becomes

$$G(y;\alpha,\theta,\beta,b) = \left[1 - \left[1 + \frac{\theta x^{\beta}}{\alpha\theta + 1}\right] e^{-\theta x^{\beta}}\right]^{b}; x > 0, \alpha, \theta, \beta, b > 0$$

2.1. Special cases

Case 1: At $b = 1$, EQPLD reduces to Quasi Power Lindley distribution.

Case 2: At $\beta = 1$, EQPLD reduces to the Power Lindley distribution introduced by ME Ghitany (2013).

Case 3: At $b = 1, \beta = 1$, EQPLD reduces to the Quasi Lindley distribution introduced by Shanker and Mishra (2013).

Case 4: At $b = 1, \beta = 1, \alpha = 1$, EQPLD reduces to the Lindley distribution.

3. THE EQPLPS FAMILY

In this section, we derive the family of EQPLPS distributions by compounding the EQPL class of distributions with the power series distributions.

Let N be a discrete random variable following the power series distribution (truncated at zero) with probability mass function given by

$$P(N = n) = \frac{a_n \gamma^n}{C(\gamma)}, n = 1,2,...$$

Where $a_n \geq 0$ be reliant on n, $C(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$ and $\gamma \in (0, s)$ is chosen in such a way that $C(\gamma)$ is finite. The power series family of distributions, contains Poisson, logarithmic, geometric and binomial distributions as special cases. Valuable extents of above distributions truncated at zero are given in table 1.

Table1: Useful Extents Of Zero Truncated Power Series Distribution

Distribution	a_n	$C(\gamma)$	$C'(\gamma)$	$C''(\gamma)$	$C^{-1}(\gamma)$	γ
Poisson	$n!^{-1}$	$e^{\gamma} - 1$	e^{γ}	e^{γ}	$\log(\gamma + 1)$	$\gamma \in (0, \infty)$

Logarithmic	n^{-1}	$-\log(1-\gamma)$	$(1-\gamma)^{-1}$	$(1-\gamma)^{-2}$	$1-e^{-\gamma}$	$\varphi\in(0,1)$
Geometric	1	$\gamma(1-\gamma)^{-1}$	$(1-\gamma)^{-2}$	$2(1-\gamma^{-3})$	$\gamma(\gamma+1)^{-1}$	$\gamma\in(0,1)$
Binomial	$\binom{m}{n}$	$(\gamma+1)^{m}-1$	$m(\gamma+1)^{m-1}$	$\frac{m(m-1)}{(\gamma-1)^{2-m}}$	$(\gamma-1)^{\frac{1}{m}}-1$	$\gamma\in(0,1)$

Given N, let $X=\max(X_1,X_2,...X_N)$, where $X_i, i=1,2,...N$ are independent and identically distributed (iid) random variables with cdf ().Then the cdf of $X\mid N=n$ is given by

$$F(X\mid N=n)=[G(x;\alpha,\theta,\beta,b)]^n=\left\{\left[1-\left[1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right]e^{-\theta x^{\beta}}\right]^b\right\}^n \quad ,$$
$$x>0,\alpha,\theta,\beta,b>0,n\geq 1$$

The EQPLPS is then defined by the marginal cdf of X, which is given by

$$F(x,\alpha,\theta,\beta,b,\gamma)=\sum_{n=1}^{\infty}\frac{a_n\gamma^n}{C(\gamma)}[G(x;\alpha,\theta,\beta,b,\gamma)]^n$$

$$F(x)=\frac{C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^b\right]}{C(\gamma)}, x>0,\alpha>0,\theta>0,\beta>0,b>0,\gamma>0 \qquad (1)$$

Here, a random variable X following Exponentiated Quasi Power Lindley power series distribution with parameters $\alpha,\theta,\beta,b,\gamma$ will be denoted by $X\sim EQPLPS(\alpha,\theta,\beta,b,\gamma)$. This new class of distributions contains several lifetime distributions as special cases which will be discussed in section (9).

4. DENSITY, SURVIVAL AND HAZARD RATE FUNCTIONS

The pdf, survival and hazard functions are respectively given by

$$f(x)=\frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}(\alpha+x^{\beta})\gamma e^{-\theta x^{\beta}}\frac{C'\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^b\right]}{C(\gamma)}, x>0 \qquad (2)$$

$$S(x)=1-\frac{C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}, x>0$$

And the hazard function is given as

$$h(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\frac{C^{'}\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)-C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}, x>0$$

THEOREM 4.1. The Exponentiated Quasi Power Lindley distribution is a limiting case of EQPLPS distribution when $\gamma\to 0^{+}$.

PROOF. From the cdf of EQPLPS distribution , we have

$$\lim_{\gamma\to 0^{+}}F(x)=\lim_{\gamma\to 0^{+}}\frac{C^{'}\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}$$

We know that

$$C(\gamma)=\sum_{n=1}^{\infty}a_{n}\gamma^{n}$$

$$\lim_{\gamma\to 0^{+}}F(x)=\lim_{\gamma\to 0^{+}}\frac{\sum_{n=1}^{\infty}a_{n}\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]^{n}}{\sum_{n=1}^{\infty}a_{n}\gamma^{n}}$$

Using L' Hospital's rule, it follows that

$$\lim_{\gamma\to 0^{+}}F(x)=\frac{a_{1}\left\{\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}+\sum_{n=1}^{\infty}a_{n}\gamma^{n-1}\left[\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{n}}{a_{1}+\sum_{n=1}^{\infty}a_{n}n\gamma^{n-1}}$$

$$=\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}$$

Which is the cdf of exponentiated quasi power Lindley distribution.

THEOREM 4.2. The densities of the EQPLPS distribution can be expressed as an infinite linear combination of densities of n^{th} order statistics of the exponentiated quasi power Lindley distribution

$$f(x)=\sum_{n=1}^{\infty}P(N=n)g_n(x,n) \qquad (3) \qquad (3)$$

Where $g_n(x,n)=\max(X_1,X_2,...X_n)$ is the n^{th} order statistics of exponentiated quasi power Lindley distribution.

PROOF. As we know that

$$C(\gamma)=\sum_{n=1}^{\infty}na_n\gamma^{n-1}$$

Therefore, the pdf of EQPLPS distribution reduces to the expression after using the above argument as follows

$$f(x)=\frac{\beta\theta^2bx^{\beta-1}}{(\alpha\theta+1)}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\sum_{n=1}^{\infty}\frac{na_n\gamma^n}{C(\gamma)}\left[\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{n-1}$$

$$f(x)=\sum_{n=1}^{\infty}P(N=n)g_n(x,n)$$

Where

$$g_n(x,n)=\frac{n\beta\theta^2bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\left[\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{n-1}$$

is the n^{th} order statistics of exponentiated quasi power Lindley distribution.

Therefore the densities of EQPLPS distribution can be expressed as an infinite linear combination of the n^{th} order statistics of exponentiated quasi power Lindley distribution.

5. MOMENT GENERATING FUNCTION

The moment generating function of EQPLPS distribution can be obtained from (3)

$$M_X(t)=\sum_{n=1}^{\infty}P(N=n)M_{X_{(n)}}(t)$$

Where $M_{X_{(n)}}(t)$ is the moment generating function of nth order statistics of exponentiated quasi power Lindley distribution.

$$M_{X_{(n)}}(t)=\frac{n\beta\theta^2 b}{\alpha\theta+1}\int_0^{\infty}e^{tx}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\left[\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]^{n-1}dx$$

$$=\frac{n\beta\theta^2}{\alpha\theta+1}\sum_{j=0}^{n-1}\binom{n-1}{j}\sum_{k=0}^{\infty}\binom{bj+b-1}{k}(-1)^{j+k}\int_0^{\infty}e^{tx}x^{\beta-1}\left(\alpha+x^{\beta}\right)e^{-(\theta+\theta k)x^{\beta}}\left[1+\left(\frac{\theta x^{\beta}}{\alpha\theta+1}\right)\right]^{k}dx$$

Using $e^{tx}=\sum_{l=0}^{\infty}\frac{t^l x^l}{l!}$

$$=\frac{n\theta^2 b}{(\alpha\theta+1)}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k}\frac{t^l}{l!}\left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i}\times\left[\frac{\alpha\Gamma\left(k-i+\frac{l}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{l}{\beta}\right)}{(\theta+\theta k)^{k-i+\frac{l}{\beta}+1}}\right]$$

And it follows that

$$M_X(t)=\frac{\theta^2 b}{\alpha\theta+1}\sum_{n=1}^{\infty}\frac{na_n\gamma^n}{C(\gamma)}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k}\frac{t^l}{l!}\left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i}\times\left[\frac{\alpha\Gamma\left(k-i+\frac{l}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{l}{\beta}\right)}{(\theta+\theta k)^{k-i+\frac{l}{\beta}+1}}\right]$$

The rth moment of the EQPLPS distribution about origin is

$$E\left(X^r\right)=\sum_{n=1}^{\infty}P(N=n)\int_0^{\infty}x^r\,g_n(x)dx$$

$$E\left(X^r\right)=\frac{\theta^2 b}{\alpha\theta+1}\sum_{n=1}^{\infty}\frac{na_n\gamma^n}{C(\gamma)}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k}\left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i}\times$$
$$\left[\frac{\alpha\Gamma\left(k-i+\frac{r}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{r}{\beta}+2\right)}{(\theta+\theta k)^{k-i+\frac{r}{\beta}+2}}\right] \quad (4)$$

6. QUANTILE FUNCTION

THEOREM 6.1. If $X \sim EQPLPS(\alpha,\theta,\beta,b,\gamma)$, then the quantile function of X is

$$Q(p)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}(vC(\gamma))}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

Where $v\in(0,1)$ and W(.) denotes the Lambert W function (see Corless et al.(1996))

PROOF. The quantile function denoted by Q(p) is the root of the equation

$$\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}=1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}:0<v<1 \quad (5)$$

Setting $Z(v)=-\left(1+\alpha\theta+\theta[Q(v)]^{\beta}\right)$, we may rewrite (5) as

$$Z(v)e^{z(v)}=-(1+\alpha\theta)\left[1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right]e^{-(1+\alpha\theta)}$$

So the solution for Z(v) is

$$Z(v)=W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]$$

Solving the equation

$$W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]=-\left(1+\alpha\theta+\theta[Q(v)]^{\beta}\right)$$

Which upon solving for Q(v) gives

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

Using above equation the quartiles of the EQPLPS distribution can be determined. Median of the exponentiated quasi power Lindley power series distribution is given by

$$Q\left(\frac{1}{2}\right)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}\left[\frac{1}{2}C(\gamma)\right]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

7. ORDER STATISTICS AND THEIR MOMENTS

Let $X_1, X_2, ...X_n$ be a random sample of size n having EQPLPS distribution. The pdf and cdf of ith order statistics say $X_{i::n}$ can be obtained as

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}[F(x)]^{i-1}[1-F(x)]^{n-i}\, f(x) \qquad (6)$$

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\, f(x)\left[\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}}{C(\gamma)}\right]^{i-1}\left[1-\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}}{C(\gamma)}\right]^{n-i}$$

Expression (6) can also be written as

$$f_{i:n}(x)\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{k}\, f(x)[F(x)]^{k+i-1}$$

The associated cdf of $f_{i:n}(x)$ denoted by $F_{i:n}(x)$ becomes

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\left[C\left\{\frac{\gamma\left[1-\left(1+\frac{\theta x^\beta}{1+\alpha\theta}\right)e^{-\theta x^\beta}\right]^b}{C(\gamma)}\right\}\right]^{k+i} \tag{7}$$

Expression for r[th] moment of i[th] order statistics with cdf (3.1) can be obtained by using a well- known result given by Barakat et al. (2004) as follows

$$E\left(X_{i:n}^r\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_0^\infty x^{r-1}\left[1-\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^\beta}{1+\alpha\theta}\right)e^{-\theta x^\beta}\right]^b\right\}}{C(\gamma)}\right]^k dx$$

8. PARAMETER ESTIMATION

Let $X_1, X_2, ...X_n$ be a random sample with observed value $x=(x_1, x_2, ...x_n)$ obtained from EQPLPS distribution with parameters $\alpha, \theta, \beta, b\, and\, \gamma$. Let $\Theta=(\alpha, \theta, \beta, b, \gamma)^T$ be the parameter vector. The log likelihood function is given by

$$l_n=l_n(y,\Theta)=n\log\beta+n\log\gamma+2n\log\theta+n\log b-n\log(\alpha\theta+1)-n\log C(\gamma)+(\beta-1)\sum_{i=1}^n\log x_i-\theta\sum_{i=1}^n x_i^\beta$$
$$+\sum_{i=1}^n\log\left(\alpha+x_i^\beta\right)+(b-1)\sum_{i=1}^n\log\left[1-\left(1+\frac{\theta x_i^\beta}{\alpha\theta+1}\right)e^{-\theta x_i^\beta}\right]+\sum_{i=1}^n\log\left[C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^\beta}{\alpha\theta+1}\right)e^{-\theta x_i^\beta}\right]^b\right\}\right]$$

The corresponding score functions are

$$\frac{\partial l_n}{\partial \theta} = \frac{2n}{\theta} - \frac{n\alpha}{\alpha\theta+1} - \sum_{i=1}^{n} x_i^{\beta} + \frac{\theta(b-1)}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\left[\frac{x_i^{\beta} e^{-\theta x_i^{\beta}}\left(\alpha^2\theta + 2\alpha + \alpha\theta x_i^{\beta} + x_i^{\beta}\right)}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]}\right]$$

$$+\frac{b\gamma\theta}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b-1} x_i^{\beta} e^{-\theta x_i^{\beta}}\left(\alpha^2\theta + 2\alpha + \alpha\theta x_i^{\beta} + x_i^{\beta}\right)$$

$$\frac{\partial l_n}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\frac{x_i^{\beta}\log x_i}{\left(\alpha + x_i^{\beta}\right)} - \theta\sum_{i=1}^{n} x_i^{\beta}\log x_i + \frac{(b-1)\theta^2}{(\alpha\theta+1)}\sum_{i=1}^{n}\left(x_i^{\beta}\log x_i e^{-\theta x_i^{\beta}}\right)\left(\alpha + x_i^{\beta}\right) +$$

$$+\sum_{i=1}^{n}\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\frac{\left(x_i^{\beta}\log x_i e^{-\theta x_i^{\beta}}\right)\left(\alpha + x_i^{\beta}\right)}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]}$$

$$\frac{\partial l_n}{\partial \alpha} = \frac{n\theta}{\alpha\theta+1} + \sum_{i=1}^{n}\frac{1}{\alpha + x_i^{\beta}} + \frac{\theta^2(b-1)}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{x_i^{\beta} e^{-\theta x_i^{\beta}}}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]} + \frac{\gamma b\theta^2}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta y_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}$$

$$\times\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b-1}\left(x_i^{\beta} e^{-\theta x_i^{\beta}}\right)$$

$$\frac{\partial l_n}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right] + \gamma\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

$$\times\log\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

$$\frac{\partial l_n}{\partial \gamma} = \frac{n}{\gamma} - \frac{nC'(\gamma)}{C(\gamma)} + \sum_{i=1}^{n}\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

MLEs of θ, α, β, b & γ cannot be obtained by solving above complex equations as these equations are not in closed form. So we solve the above equations by using iteration method through R software.

9. SPECIAL SUB-MODELS OF THE EQPLPS MODEL

9.1. Exponentiated Quasi Power Lindley Poisson Distribution (EQPLPD)

The corresponding cdf, pdf, survival function and hazard function of EQPLPD can be obtained respectively by using $C(\gamma) = e^{\gamma} - 1$ and $C'(\gamma) = e^{\gamma}$ in (1) & (2).

$$F(x) = \frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}} - 1}{e^{\gamma} - 1}, x > 0$$

$$f(x) = \frac{\beta\theta^{2} b x^{\beta-1}}{\alpha\theta+1}\left(\alpha + x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1} \frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma} - 1}, x > 0$$

$$S(x) = \frac{e^{\gamma} - e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma} - 1}$$

$$h(y) = \frac{\beta\theta^{2} b x^{\beta-1}}{\alpha\theta+1}\left(\alpha + x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1} \frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma} - e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]}}$$

For $x, \theta, \alpha, \beta, b > 0,\ and\ \ 0 < \gamma < \infty$. The expression for r[th] moment of a random variable following EQPLPS distribution becomes by substituting $a_n = n!^{-1}$ and $C(\gamma) = e^{\gamma} - 1$ in (4).

$$E(X^{r}) = \frac{n\theta^{2} b}{(\alpha\theta+1)(e^{\theta}-1)} \sum_{n=1}^{\infty} \frac{\gamma^{n}}{n!} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \binom{n-1}{j} \binom{bj+b-1}{k} \left(\frac{\theta}{\alpha\theta+1}\right)^{k-i} (-1)^{j+k} \times \left[\frac{\alpha\Gamma\left(k-i+\frac{r}{\beta}+1\right) + (\theta+\theta k)\Gamma\left(k-i+\frac{r}{\beta}+2\right)}{(\theta+\theta k)^{k-i+\frac{r}{\beta}+2}}\right]$$

The pdf and cdf of order statistics of EQPLPD can be obtained by using the cdf and pdf of EQPLPD in (6) and (7).

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\frac{\beta\theta^2bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}\times\frac{\left[e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}-1\right]^{k+i-1}}{\left[e^{\gamma}-1\right]^{k+i}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\left[\frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}-1}{e^{\gamma}-1}\right]^{k+i}$$

The quantile function can be obtained by substituting $C(\gamma)=e^{\gamma}-1$ and $C^{-1}(\gamma)=\log(\gamma+1)$ in (5), we have

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{\log\left[v\left(e^{\gamma}-1\right)+1\right]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

9.2. Exponentiated Quasi Power Lindley Logarithmic Distribution (EQPLLD)

The cdf, pdf, survival function and hazard function of EQPLLD is obtained by using $C(\gamma)=-\log(1-\gamma)$ and $C^{'}(\gamma)=(1-\gamma)^{-1}$ in (1) and (2).

$$F(x)=\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)},\ x>0$$

$$f(x)=\frac{\beta\theta^2bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{\log(1-\gamma)},\ x>0$$

$$S(x)=1-\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)},x>0$$

$$h(x)=\frac{\beta\theta^{2}x^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{\log(1-\gamma)-\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]\right]}$$

For $x,\alpha,\theta,\beta,b>0, and\ 0<\gamma<1$.The r^th moment of EQPLLD can be obtained by substituting $a_n=n^{-1}\ \ and\ \ C(\gamma)=-\log(1-\gamma)$ in (4)

The pdf and cdf of EQPLLD can be obtained by substituting its pdf and cdf in (6) and (7).

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{k}\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}$$

$$\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}\times\frac{\left[\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]\right]^{k+i-1}}{\left[\log(1-\gamma)\right]^{k+i}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^{k}}{(k+i)}\left[\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)}\right]^{k+i}$$

By substituting $C(\gamma)=-\log(1-\gamma)$ and $C^{-1}(\gamma)=1-e^{-\gamma}$ in (5), the quantile function of EQPLL distribution is obtained as

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{e^{v\log(1-\gamma)}}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

9.3. Exponentiated Quasi Power Lindley Geometric Distribution (EQPLGD)

The cdf, pdf, survival function and hazard function of EQPLGD can be obtained by using $C(\gamma)=\gamma(1-\gamma)^{-1}$ & $C^{'}(\gamma)=(1-\gamma)^{-2}$ in (1) & (2).

$$F(x)=\frac{(1-\gamma)\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}{1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}},\ x>0$$

$$f(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}(1-\gamma)\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-2}$$

$$S(x)=\frac{1-\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}{1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}},\ x>0$$

$$h(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{(\alpha\theta+1)}(1-\gamma)\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{1-\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}$$

For $x,\alpha,\theta,\beta,b>0$ *and* $0<\gamma<1$. The rth moment of EQPLGD can be obtained by substituting $a_n=1$ and $C(\gamma)=\gamma(1-\gamma)^{-1}$ in (4).

The pdf and cdf of order statistics of EQPLGD can be obtained by using the cdf and pdf of EQPLGD in (6) and (7) ,we have

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\frac{\beta\theta^2bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}(1-\gamma)^{k+i}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}$$

$$\times\frac{\left[\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i-1}}{\left[1-\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i+1}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{k+i}\frac{\left[(1-\gamma)\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i}}{\left[1-\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i}},\ x>0$$

By substituting $C(\gamma)=\gamma(1-\gamma)^{-1}$ and $C^{-1}(\gamma)=\gamma(\gamma+1)^{-1}$ in (5), the quantile function of EQPLG distribution is obtained as

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{v}{\gamma(v-1)+1}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

9.4. Exponentiated Quasi Power Lindley Binomial Distribution (EQPLBD)

The cdf , pdf, survival function and hazard function of EQPLBD can be obtained respectively by taking $C(\gamma)=(\gamma+1)^m-1$ and $C^{'}(\gamma)=m(\gamma+1)^{m-1}$ in (1) and (2).

$$F(x)=\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b+1\right]^m-1}{(\gamma+1)^m-1},\ x>0$$

$$f(x)=\frac{m\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m-1}}{(\gamma+1)^m-1},\ x>0$$

$$S(x)=1-\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}-1}{(\gamma+1)^m-1}$$

$$h(x)=\frac{m\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m-1}}{(\gamma+1)^m-\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}}$$

For $x,\alpha,\theta,\beta,b>0\, and\ 0<\phi<\infty.$ The r[th] moment of a random variable following EQPLBD becomes by taking $a_n=\binom{m}{n}\ and\ C(\gamma)=(\gamma+1)^m-1$ in (4).

The pdf and cdf of order statistics of EQPLBD can be obtained respectively by using the pdf and cdf of EQPLBD in (6) and (7).

$$f_{i:n}(y)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\frac{m\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}$$
$$\times\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m-1}\frac{\left[\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta y^{\alpha}}\right]^{b}+1\right]^{m}-1\right]^{k+i-1}}{\left[(\gamma+1)^m-1\right]^{k+i}}$$

$$F_{i:n}(y)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\frac{\left[\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}-1\right]^{k+i}}{\left[(\gamma+1)^m-1\right]^{k+i}}$$

10. APPLICATION

To show the superiority of the proposed distribution, we compare its sub–models by taking four real life data sets.

Data set 1. The first data set represents the Lifetime of fatigue of Kevlar 373/epoxy, that are subject to constant pressure at the 90% stress level until all had failed. The data set is

0.0251	0.886	0.0891	0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.912	0.9836
1.0483	1.0596	1.0773	1.1733	1.257	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.488	1.5728	1.5733	1.7083	1.7263	1.746	1.763	1.7746	1.8275	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.133	2.21
2.246	2.2878	2.3203	2.347	2.3513	2.4951	2.526	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.096	

Table 10.1: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta} = 0.924, \hat{\gamma} = 0.000000122\ , \hat{\theta} = 1.110, \hat{b} = 1.254, \hat{\alpha} = 0.233$	253.13	260.76
EQPLG	$\hat{\beta} = 1.060, \hat{\gamma} = 0.00000145\ , \hat{\theta} = 0.837, \hat{b} = 1.092, \hat{\alpha} = 0.525$	253.86	261.49
EQPLL	$\hat{\beta} = 1.036, \hat{\gamma} = 0.0000000980\ , \hat{\theta} = 0.914, \hat{b} = 1.159, \hat{\alpha} = 0.483$	253.62	261.25

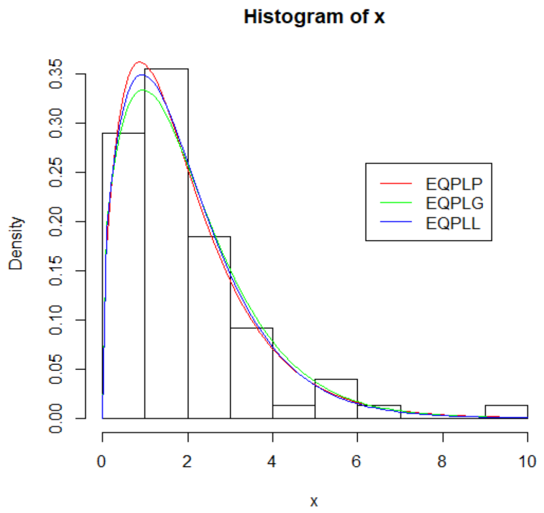

Fig 1: Fitting of EQPLP, EQPLG, EQPLL to the fatigue lifetime data.

Data Set 2. The data set reported by E fron B (1988) and was used by Rama Shanker (2016) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT). The data set is

6.53	7	10.42	14.48	16.10	22.70	34	41.55	42	45.28	49.40	53.62
63	64	83	84	91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165	146	149	154	157
160	160	165	173	176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101	1146	1417		

Table.10.2: Analysis of model fitting

MODEL	**MLE**	**AIC**	**BIC**
EQPLP	$\hat{\beta}=0.372, \hat{\gamma}=2.625, \hat{\theta}=0.605, \hat{b}=4.945, \hat{\alpha}=2.570$	750.37	758.00
EQPLG	$\hat{\beta}=0.542, \hat{\gamma}=0.000000839\ , \hat{\theta}=0.152, \hat{b}=1.749, \hat{\alpha}=0.390$	751.59	759.22
EQPLL	$\hat{\beta}=0.535, \hat{\gamma}=0.00000152\ , \hat{\theta}=0.160, \hat{b}=1.789, \hat{\alpha}=0.384$	751.58	6759.21

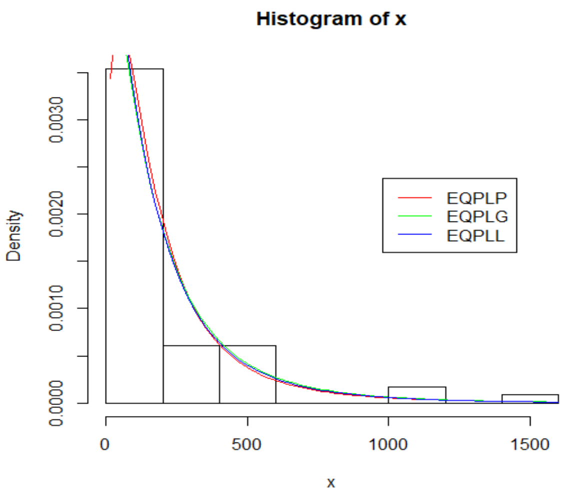

Fig 2: Fitting of EQPLP, EQPLG, EQPLL to the survival data.

Data Set 3. The data set reported by Efron B (1988) and was used by Rama Shanker (2016) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47
68.46	78.26	74.47	81.43	84	92	94	110	112	119	127
130	133	140	146	155	159	173	179	194	195	209
249	281	319	339	432	469	519	633	725	817	1776

Table 10.3: Analysis of model fitting

MODEL	**MLE**	**AIC**	**BIC**
EQPLP	$\hat{\beta}=0.576, \hat{\gamma}=0.00000132\ , \hat{\theta}=0.125, \hat{b}=1.75, \hat{\alpha}=0.381$	568.43	576.06
EQPLG	$\hat{\beta}=0.578, \hat{\gamma}=0.0000000653\ , \hat{\theta}=0.126, \hat{b}=1.480, \hat{\alpha}=0.451$	569.03	576.67
EQPLL	$\hat{\beta}=0.597, \hat{\gamma}=0.00000328\ , \hat{\theta}=0.108, \hat{b}=1.640, \hat{\alpha}=0.411$	568.83	576.47

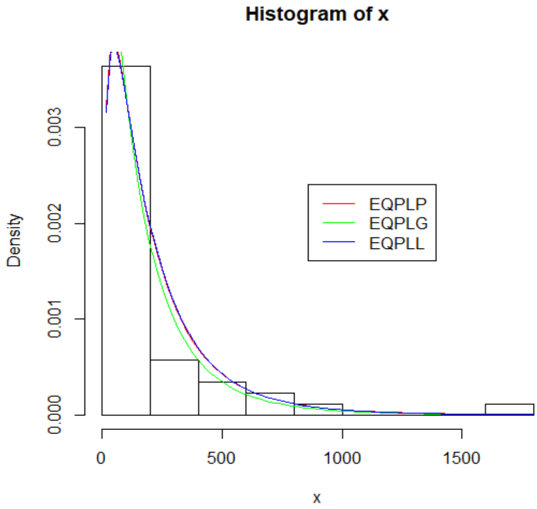

Fig 3: Fitting of EQPLP, EQPLG, EQPLL to the survival data.

Data Set 4. The data set represents remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee & Wang (2003) and was used by Rama Shanker (2016) in modeling of lifetime data .

Table.10.4: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta} = 0.673, \hat{\gamma} = 0.00000279, \hat{\theta} = 0.552, \hat{b} = 1.415, \hat{\alpha} = 0.406$	827.63	835.26
EQPLG	$\hat{\beta} = 0.727, \hat{\gamma} = 0.00000172, \hat{\theta} = 0.458, \hat{b} = 1.384, \hat{\alpha} = 0.429$	828.34	835.98
EQPLL	$\hat{\beta} = 0.713, \hat{\gamma} = 0.000000556, \hat{\theta} = 0.472, \hat{b} = 1.403, \hat{\alpha} = 0.403$	828.21	835.85

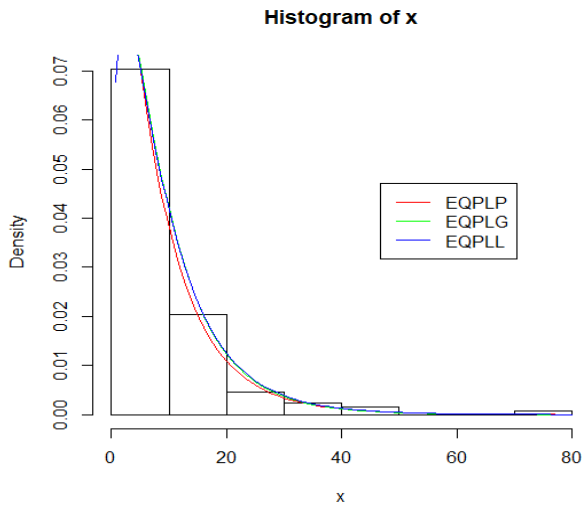

Fig 4: Fitting of EQPLP, EQPLG, EQPLL to the cancer data

All the sub-models fit well but among them EQPLP class of distributions performs excellently well as it possesses the lowest values of AIC and BIC values.

11. CONCLUSION

We have proposed a new five parameter lifetime distribution for parallel system by compounding Exponentiated Quasi Power Lindley distribution with power series distribution. The mathematical properties including density function, moment generating function, order statistics, quantile function have been obtained. The parameters have been estimated by the method of maximum likelihood estimation. The proposed model contains some lifetime sub-classes and has the competency to yield many beneficial and flexible distributions for modelling lifetime data. Ultimately, the sub-models have been compared by applying them to four real life data sets to show the flexibility of the proposed model.

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