{MERAA} Mathematics in Education, Research and Applications

ISSN 2453-6881

Math Educ Res Appl, 2021(7), 1

Received: 2021-01-10
Accepted: 2021-02-15
Online published: 2021-09-30
DOI: https://doi.org/10.15414/meraa.2021.07.01.01-09

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Original Paper

An algorithm for solving of Euler parameters differential equations system

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ABSTRACT

The design an optimal numerical method for solving a system of ordinary differential equations simultaneously is described in this paper. System of differential equations was represented by a system of linear ordinary differential equations of Euler's parameters called quaternions. The components of angular velocity were obtained by the experimental way. The angular velocity of the centre of gravity was determined from sensors of acceleration located in the plane of the centre of gravity of the machine. The used numerical method for solving was a fourth-order Runge-Kutta method. The stability of solving was based on the orthogonality of a direct cosine matrix. The numerical process was controlled on every step in numerical integration. The algorithm was designed in the C# programming language.

KEYWORDS: spatial dislocation, quaternion, numerical integration

JEL CLASSIFICATION: C63

INTRODUCTION

The goal of this contribution is that the published research supplement the missing part of many scientific books and scientific articles dealing with spacecraft attitude dynamics of rigid body movement in three-dimensional space. Determination of the dislocation of a certain moving frame of the system is a very important part of applied dynamics. One of these cases refers to the robotic arms movement as defined in [5]. In even a specific case is the problem to determine the dislocation of the whole moving body in three-dimensional spaces. This case refers to a moving vehicle on the ground with respect to an inertial coordinate system. There are many methods how to determine body dislocation, for example, using GPS. But using GPS giving only coordinates of moving objects with no acceptable precision. On the other hand, the data obtained from sensors of acceleration or gyroscopic gauge, giving the more

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usable data set. Angular velocities can be utilized by the system of quaternion differential equations (SQDE) [10]. Derivation of SQDE was published by [2, 3, 6]. The analytical method of solving the system of simultaneous linear differential equations (SLDE) was published by [4]. The application of the numerical integration method to solve ordinary differential equations was published by [8]. The numerical solution of SLDE was published by [9, 11]. Matrix notation of the quaternion vector space was analysed by [1, 7]. The exact and clear description of numerical solution SQDE is presented in this contribution with the utilization of matrix formalism and C# programming language.

MATERIAL AND METHODS

Measurement system and object

An object for measurement was a municipal services tool carrier Reform Metrac H6X. The machine on duty is depicted in Figure 1. The basic parameters and centre of gravity dislocation of the machine were published in [10].

Figure 1. Metrac H6X on terrain

Figure 2. Dislocations of acceleration sensors on machine

For the measurement of accelerations of the machine were used the ADXL 345 sensors. The sensor measured accelerations in the XYZ axes. The sensors were dislocated in the plane of the center of gravity of the machine where the z-coordinate dimension was zero with respect to the center of gravity. Mounted sensors are depicted in Figure 2. We were provided an experimental ride with a machine with a defined trajectory as depicted in Figure 1. For application, we used the data from the ride down direction along the downhill with turning back to uphill with braking. The relevant accelerations were recorded in real-time [8]. The processed angular velocities of the center of gravity of the machine are depicted in Figure 3.

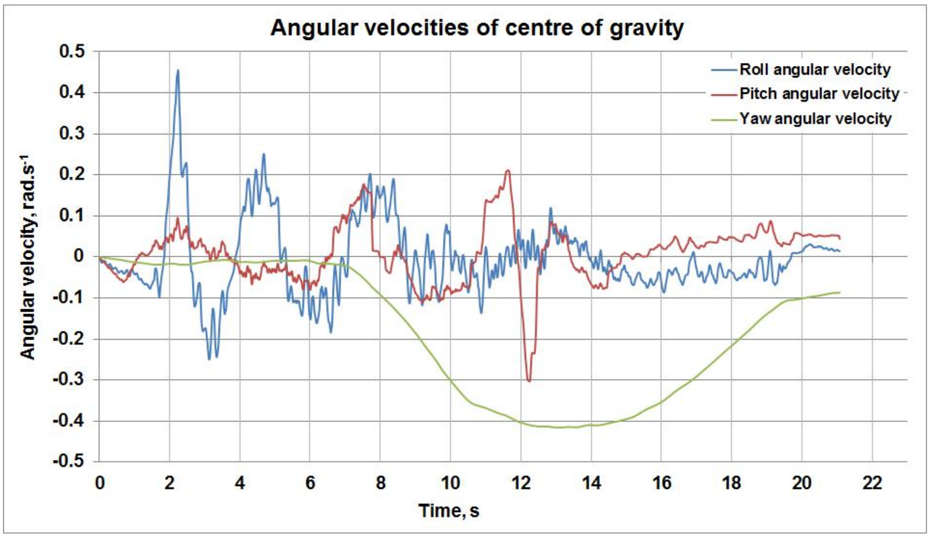

Figure 3. Angular velocities of centre of gravity of the machine

Numerical method

For stability of numerical solving of system of differential equations, we chose the fourth order Runge-Kutta numerical method. The systems of simultaneous differential equations are in the form (01).

$$\frac{d\,{}^1y}{dx} = {}^1f\left(x, {}^1y, \dots, {}^ny\right)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots \qquad (01)$$
$$\frac{d\,{}^ny}{dx} = {}^nf\left(x, {}^1y, \dots, {}^ny\right)$$

Written in the shorter form of system (01) is:

$$\frac{d\,{}^iy}{dx} = {}^if\left(x, {}^iy, \dots, {}^ny\right), i = 1,2,3 \dots n, \qquad (02)$$

with initial conditions:

$${}^1y\left(x_0\right) = {}^1y_0, \dots, {}^ny\left(x_0\right) = {}^ny_0 . \qquad (03)$$

Rewrite:

$${}^iy\left(x_0\right) = {}^iy_0\ , i = 1,2,3 \dots n . \qquad (04)$$

For solving system (01) was used the fourth-order Runge-Kutta method, with constant step size in the next form:

$${}^iY_{(j)} = {}^iY_{(j-1)} + \frac{\Delta t}{6}\left({}^ik_1 + 2\,.{}^ik_2 + 2\,.{}^ik_3 + {}^ik_4\right), \qquad (05)$$

where the coefficients are:

$$\begin{aligned}
{}^{i}k_1 &= {}^{i}f\left(x_{(j-1)},\ {}^{1}Y_{(j-1)},\dots,{}^{n}Y_k\right),\\
{}^{i}k_2 &= {}^{i}f\left(x_{(j-1)}+\frac{\Delta t}{2},\ {}^{1}Y_{(j-1)}+\frac{\Delta t}{2}.{}^{1}k_1,\dots,{}^{n}Y_k+\frac{\Delta t}{2}.{}^{n}k_1\right),\\
{}^{i}k_3 &= {}^{i}f\left(x_{(j-1)}+\frac{\Delta t}{2},\ {}^{1}Y_{(j-1)}+\frac{\Delta t}{2}.{}^{1}k_2,\dots,{}^{n}Y_k+\frac{\Delta t}{2}.{}^{n}k_2\right),\\
{}^{i}k_4 &= {}^{i}f\left(x_{(j-1)}+\Delta t,\ {}^{1}Y_{(j-1)}+\Delta t.{}^{1}k_3,\dots,{}^{n}Y_k+\Delta t.{}^{n}k_3\right)\ .
\end{aligned} \tag{06}$$

Where the variable n is the count of the differential equations in system and k is the count of discrete points of technical functions in the interval $\langle 1, k\rangle$. In the solutions we assume that the function f, respectively functions ${}^{1}f$..... ${}^{n}f$, are the continuous in the interval $\langle j-1, j\rangle$ and satisfied with the solution on all points. Variable Δt is the step size of the method.

Simultaneous quaternion differential equations

In most spacecraft applications occur the SQDE in the next form:

$$\begin{bmatrix}\frac{dq_0}{dt}\\ \frac{dq_1}{dt}\\ \frac{dq_2}{dt}\\ \frac{dq_3}{dt}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0 & -\omega_x & -\omega_y & -\omega_z\\ \omega_x & 0 & \omega_z & -\omega_y\\ \omega_y & -\omega_z & 0 & \omega_x\\ \omega_z & \omega_y & -\omega_x & 0\end{bmatrix}.\begin{bmatrix}q_0\\ q_1\\ q_2\\ q_3\end{bmatrix}, \tag{07}$$

The equations can be rewritten in matrix form:

$$\left[\frac{dq_i}{dt}\right]_{i=0,1,2,3}=\frac{1}{2}\left[\omega_j\right]_{j=x,y,z}.\left[q_i\right]_{i=0,1,2,3}, \tag{08}$$

and without indexation in general form:

$$\left[\frac{dQ}{dt}\right]=\frac{1}{2}[\Omega].[Q], \tag{09}$$

where:

$\left[\frac{dQ}{dt}\right]$ - matrix of quaternion differentials,

$[\Omega]$ - matrix of angular velocities,

$[Q]$ - matrix of quaternions.

RESULTS AND DISCUSSION

To design an efficient algorithm we have to rewrite the matrix form (07) to separate single equations system to the form with the derivatives on the left side and other members placed on the right side of the equation. By these steps, we get the system of simultaneous equations (10) as follows:

$$\frac{dq_0}{dt}=\frac{1}{2}.\left(-\omega_x.q_1-\omega_y.q_2-\omega_z.q_3\right);\frac{dq_2}{dt}=\frac{1}{2}.\left(\omega_y.q_0-\omega_z.q_1+\omega_x.q_3\right),$$
$$\frac{dq_1}{dt}=\frac{1}{2}.\left(\omega_x.q_0+\omega_z.q_2-\omega_y.q_3\right);\frac{dq_3}{dt}=\frac{1}{2}.\left(\omega_z.q_0+\omega_y.q_1-\omega_x.q_2\right). \tag{10}$$

The algorithm has the next structure.

$Initial\ condition\ for\ quaternion\ values \rightarrow q_0^*=1, q_1^*=0, q_2^*=0, q_3^*=0$;
$Begin\ cycle \rightarrow i=1;$
$j=1;$
$p=0..3;$
$\omega_{(x,y,z)0}=\omega_{(x,y,z)i-1}$; $\omega_{(x,y,z)1}=\omega_{(x,y,z)i}$; ($\omega_{(x,y,z)0}\rightarrow\omega_{(x)0},\omega_{(y)0},\omega_{(z)0}$, convention is same for all used variables)
$q_p=q_p^*$; $\omega_{x,y,z}=\omega_{(x,y,z)i-1}$;
$solve \rightarrow f\left(q_p\right)_j$; (function defined after end of cycle)
$j=2, p=0..3;$
$$\omega_{x,y,z}=\omega_{(x,y,z)0}+\frac{1}{2}.\left[\omega_{(x,y,z)1}-\omega_{(x,y,z)0}\right];$$
$$q_p=q_p^*+\frac{\Delta t}{2}.f\left(q_p\right)_1;$$
$solve \rightarrow f\left(q_p\right)_j;$
$j=3, p=0..3$
$$q_p=q_p^*+\frac{\Delta t}{2}.f\left(q_p\right)_2;$$
$solve \rightarrow f\left(q_p\right)_j;$
$j=4, p=0..3;$
$\omega_{x,y,z}=\omega_{(x,y,z)1}$; $q_p=q_p^*+\Delta t.f\left(q_p\right)_3$;
$solve \rightarrow f\left(q_p\right)_j;$
$$q_p=q_p^*+\left\{\frac{\Delta t}{6}.\left[f\left(q_p\right)_1+2.f\left(q_p\right)_2+2.f\left(q_p\right)_3+f\left(q_p\right)_4\right]\right\};$$

Creating transoformation matrix $\rightarrow \left[M_q\right]_i = \prod_i^k \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T$,

where:

$$a_{11} = q_0^2 + q_1^2 - q_2^2 - q_3^2,\ a_{12} = 2.\left(q_1.q_2 + q_0.q_3\right),\ a_{13} = 2.\left(q_1.q_3 - q_0.q_2\right),$$

$$a_{21} = 2.\left(q_1.q_2 - q_0.q_3\right),\ a_{22} = q_0^2 + q_2^2 - q_3^2 - q_1^2,\ a_{23} = 2.\left(q_2.q_3 + q_0.q_1\right),$$

$$a_{31} = 2.\left(q_3.q_1 + q_0.q_2\right),\ a_{32} = 2.\left(q_2.q_3 - q_0.q_1\right),\ a_{33} = q_0^2 + q_3^2 - q_1^2 - q_2^2,$$

and $[\ ]^T$ mean the matrix transpose on each i to k.

We define the new variable array $\delta_i = \det\left[M_q\right]_i$, where δ_i is a determinant of the transformation matrix. The transformation matrix $\left[M_q\right]_i$ is strongly orthogonal. The precision of numerical integration is defined through the orthogonal matrix determinant value. We define the precision variable or error of numerical integration $E_r\left(q\right)$, where:

$i = 0$ to k
  $E_r\left(q\right)_i = 1 - \delta_i$
   increment i;
   return;

function solve $f\left(q_p\right)$
 $p = 0..3$;
$\left(q_0\right)_j = \frac{1}{2}.\left(-q_1.\omega_x - q_2.\omega_y - q_3.\omega_z\right); \left(q_1\right)_j = \frac{1}{2}.\left(q_0.\omega_x - q_3.\omega_y + q_2.\omega_z\right);$
$\left(q_2\right)_j = \frac{1}{2}.\left(q_3.\omega_x - q_0.\omega_y - q_1.\omega_z\right); \left(q_3\right)_j = \frac{1}{2}.\left(q_2.\omega_x + q_1.\omega_y + q_0.\omega_z\right);$
end function

Through the designed algorithm we were solved the transformation matrix determinant values on each step and these values are depicted in the Figure 4.

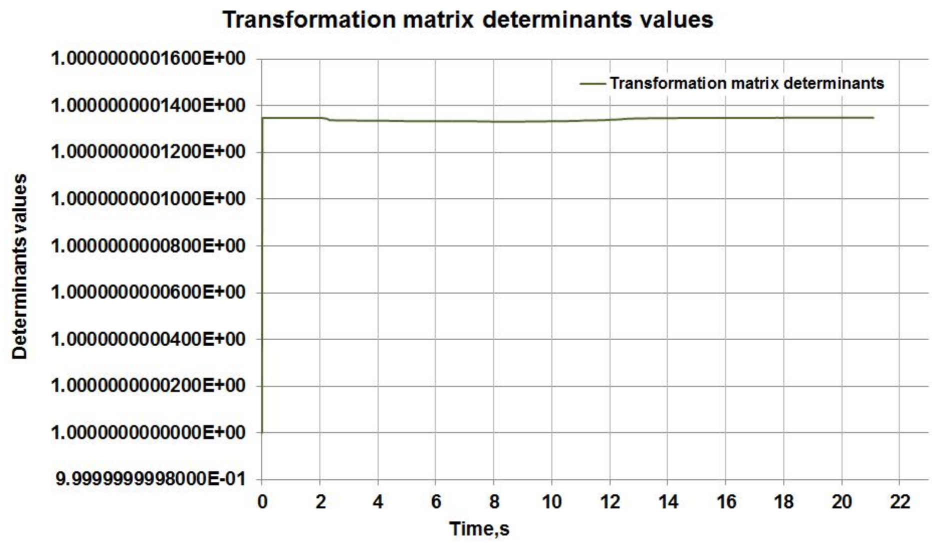

Figure 4. Transformation matrix determinants

For showing a viewable look of values of determinant we create a chart in y-axis interval $\langle 1.00000000013300, 1.00000000013500 \rangle$. The chart is depicted in Figure 5.

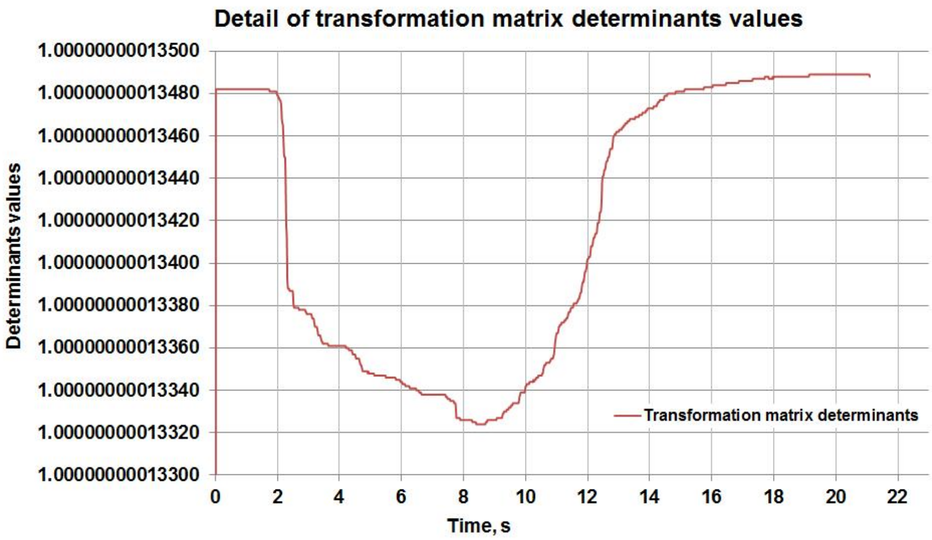

Figure 5. Transformation matrix determinants details

To control numerical integration error, we have solved the transformation matrix error on each step of counting cycles. This error is based on the orthogonality of the transformation matrix, where the determinant is equal to one. The error is depicted in Figure 6.

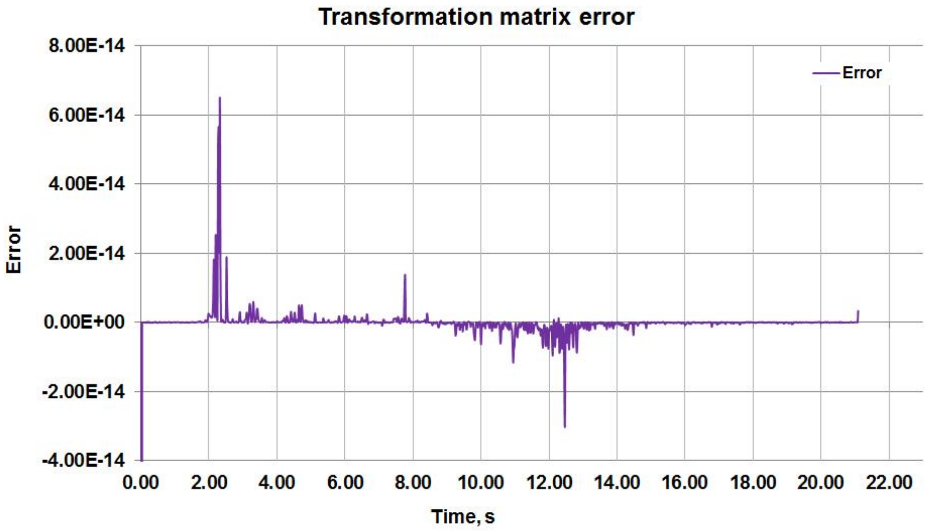

Figure 6. Numerical integration errors

CONCLUSIONS

In this paper, we are dealing with design an algorithm for numerical solving a simultaneous system of differential equations (SSDE). For the model situation, we chose the quaternion differential equation, where the angular velocities we got from the real experiment with an agricultural machine. The aim of this work was to creating and testing the optimal algorithm for solving SSDE. As a programming language, we chose the language Visual C#, where we create the function *QTSolver* and the function *SolveFQP* as a subroutine of function *QTSolver*. The published algorithm was written here in the shortest form with using the indexation of many used variables. The published algorithm has a convention like Visual C# language. The goal of this article is showing the easy way to solve any system of linear differential equations with Runge-Kutta numerical integration method with constant step size. We were chosen the quaternion system of the differential equation as a very suitable example. The benefit of these types of equations is the orthogonality as well as the control of the stability of numerical solving. In the presented example the result of solving are quaternions from which we create the transformation matrix. The determinant of transformation matrix (see Fig.4.) in respect of orthogonality is equal to one. From this assumption, we are able to solve the error of numerical integration (see Fig. 6). The process of solving is very accurate, and the errors values are in the interval $\left\langle -4.10^{-14}, 6.10^{-14} \right\rangle$.

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{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

Received: 2021-03-31
Accepted: 2021-05-31
Online published: 2021-09-30
DOI: https://doi.org/10.15414/meraa.2021.07.01.10-15

Original Paper

Evaluation of specific integrals by differentiation – part 2

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ABSTRACT

One of the most important computational techniques in higher mathematics is differentiation and its counterpart, integration (anti-differentiation). While differentiation is a routine and relatively simple procedure, integration, in general, is a much more involving task. Close (inverse) relationship between differentiation and anti-differentiation (evaluation of indefinite integrals) in some cases reveals the possibility to derive the form of the antiderivative and evaluate this antiderivative by differentiation and subsequent comparison of coefficients. This paper is a sequel to [4] and deals with some other types of elementary functions whose integrals can be evaluated by differentiation.

KEYWORDS: higher mathematics, differentiation, integration, undetermined coefficients

JEL CLASSIFICATION: I20, C20

INTRODUCTION

Integration is a widely used technique in calculus. Various standard methods of evaluation of elementary indefinite integrals, e. g. tabular, substitution, integration by parts or partial fraction decomposition can be found in [1], [2], [5]. In [4], motivated by [3], we investigated a family of simple elementary functions whose antiderivatives can be found by differentiation and comparison of coefficients. Namely, we discussed the antiderivatives of the following functions:

$P_n(x)\,\mathrm{e}^{ax}, P_n(x)\sin ax, P_n(x)\cos ax, \mathrm{e}^{ax}\sin bx, \mathrm{e}^{ax}\cos bx,$

where $P_n(x)$ denotes a polynomial of n–th degree.

Let's recall the most general case from [4].

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We showed that

$$\int P_n(x)\,\mathrm{e}^{ax}\sin bx\,\mathrm{d}x = Q_n(x)\,\mathrm{e}^{ax}\sin bx + R_n(x)\,\mathrm{e}^{ax}\cos bx$$

$$\int P_m(x)\,\mathrm{e}^{ax}\cos bx\,\mathrm{d}x = Q_m(x)\,\mathrm{e}^{ax}\sin bx + R_m(x)\,\mathrm{e}^{ax}\cos bx$$

By setting $P_n(x) = P_0$ we have

$$\int P_0\,\mathrm{e}^{ax}\sin bx\,\mathrm{d}x = Q_0\,\mathrm{e}^{ax}\sin bx + R_0\,\mathrm{e}^{ax}\cos bx$$

where P_0, Q_0, R_0 are constants. In like manner we can get all considered particular cases.

In this paper we try to extend the family of functions where this method is applicable and focus on integrals containing polynomials and various radicals of a linear function.

RESULTS AND DISCUSSION

Hereinafter, we denote polynomials of degree *n, m* as $P_n(x), Q_m(x)$ etc., respectively and their *k*-th derivatives as $P_{n-k}(x), Q_{m-k}(x)$ etc., respectively, further *a, b, c* etc. are given (real) constants and *A, B, C* etc. are unknown coefficients. All the investigated integrals are considered on intervals where they are defined.

Let us consider the simplest case first.

1. $\displaystyle\int \frac{P_n(x)}{\sqrt{ax+b}}\,\mathrm{d}x.$

Since the square root of a linear term is transformed by differentiation into its reciprocal, we will consider the function $Q_n(x)\sqrt{ax+b}$ and its derivative. In all discussions in this paper, except for the illustrative examples, we consider polynomials in general (qualitative) form, hence we deliberately neglect the constant multiples of polynomials and the factor *a*.

$$\left[Q_n(x)\sqrt{ax+b}\right]' = Q_{n-1}(x)\sqrt{ax+b} + \frac{Q_n(x)}{\sqrt{ax+b}}$$

$$Q_n(x)\sqrt{ax+b} = \int Q_{n-1}(x)\sqrt{ax+b} + \frac{Q_n(x)}{\sqrt{ax+b}}\,\mathrm{d}x$$

$$Q_n(x)\sqrt{ax+b} = \int \frac{Q_{n-1}(x)(ax+b)}{\sqrt{ax+b}} + \frac{Q_n(x)}{\sqrt{ax+b}}\,\mathrm{d}x$$

$$\int \frac{P_n(x)}{\sqrt{ax+b}}\,\mathrm{d}x = Q_n(x)\sqrt{ax+b} \quad \text{where} \quad P_n(x) = Q_{n-1}(x)(ax+b) + Q_n(x)$$

Example 1: Evaluate $\displaystyle\int \frac{3x^2-2x+4}{\sqrt{x+3}}\,\mathrm{d}x$.

Solution:

$$\int \frac{3x^2-2x+4}{\sqrt{x+3}}\,dx=\left(Ax^2+Bx+C\right)\sqrt{x+3}$$

Now we take the derivatives of both sides

$$\frac{3x^2-2x+4}{\sqrt{x+3}}=(2Ax+B)\sqrt{x+3}+\frac{Ax^2+Bx+C}{2\sqrt{x+3}}$$

$$3x^2-2x+4=(2Ax+B)(x+3)+\frac{Ax^2+Bx+C}{2}$$ from which

$$2\left(3x^2-2x+4\right)=2\left(2Ax^2+Bx+6Ax+3B\right)+Ax^2+Bx+C$$

We see that $A=\frac{6}{5}, 12A+3B=-4, 6B+C=8 \Rightarrow B=-\frac{92}{15}, C=\frac{224}{5}$, hence

$$\int \frac{3x^2-2x+4}{\sqrt{x+3}}\,dx=\left(\frac{6}{5}x^2-\frac{92}{15}x+\frac{224}{5}\right)\sqrt{x+3}+const$$

Specifically, $\int \frac{P_0}{\sqrt{x+3}}\,dx=A\sqrt{x+3}$

For example, $\int \frac{4}{\sqrt{x+3}}\,dx=A\sqrt{x+3}$

$$\frac{4}{\sqrt{x+3}}=\frac{A}{2\sqrt{x+3}} \text{ and } A=8 \Rightarrow \int \frac{4}{\sqrt{x+3}}\,dx=8\sqrt{x+3}+const$$

2. $\int \frac{P_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,dx$ and $\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,dx$

These cases represent slight generalizations of the previous case. We derive the form of the antiderivative for the first integral. Let us consider again

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]'=Q_{m-1}(x)\sqrt[n]{ax+b}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]'=\frac{Q_{m-1}(x)\sqrt[n]{ax+b}\,\sqrt[n]{(ax+b)^{n-1}}}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$\left[Q_m(x)\sqrt[n]{ax+b}\right]'=\frac{Q_{m-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}$$

$$Q_m(x)\sqrt[n]{ax+b}=\int \frac{Q_{m-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-1}}}+\frac{Q_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,dx$$

{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

$$\int \frac{P_m(x)}{\sqrt[n]{(ax+b)^{n-1}}}\,\mathrm{d}x = Q_m(x)\sqrt[n]{ax+b} \quad \text{where} \quad P_m(x) = Q_{m-1}(x)(ax+b) + Q_m(x).$$

In like manner we would derive the following equality

$$\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,\mathrm{d}x = Q_m(x)\sqrt[n]{(ax+b)^{n-1}}$$

Example 2: Evaluate $\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x$.

Solution:

$$\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x = \left(Ax^2+Bx+C\right)\sqrt[4]{(x+1)^3}$$

$$\frac{x^2+3}{\sqrt[4]{x+1}} = (2Ax+B)\sqrt[4]{(x+1)^3} + \frac{3}{4}\frac{\left(Ax^2+Bx+C\right)}{\sqrt[4]{x+1}}$$

and by doing little algebra we get

$$4\left(x^2+3\right) = 4(2Ax+B)(x+1) + 3\left(Ax^2+Bx+C\right)$$

$$4x^2+12 = 11Ax^2+8Ax+7Bx+4B+3C$$

Now we compare the coefficients and obtain $A=\frac{4}{11}, B=-\frac{32}{77}, C=\frac{1052}{231}$.

Hence the solution is

$$\int \frac{x^2+3}{\sqrt[4]{x+1}}\,\mathrm{d}x = \left(\frac{4}{11}x^2 - \frac{32}{77}x + \frac{1052}{231}\right)\sqrt[4]{(x+1)^3} + const$$

Next, we consider the most general case when the radicand is a linear function.

3. $\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}\,\mathrm{d}x$ and $\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}\,\mathrm{d}x.$

As in the previous case we derive the form of the antiderivative for the first integral.

Let us consider again

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = Q_{r-1}(x)\sqrt[n]{(ax+b)^m} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = \frac{Q_{r-1}(x)\sqrt[n]{(ax+b)^m}\ \sqrt[n]{(ax+b)^{n-m}}}{\sqrt[n]{(ax+b)^{n-m}}} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\left[Q_r(x)\sqrt[n]{(ax+b)^m}\right]' = \frac{Q_{r-1}(x)(ax+b)}{\sqrt[n]{(ax+b)^{n-m}}} + \frac{Q_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}$$

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{n-m}}}\,\mathrm{d}x = Q_r(x)\sqrt[n]{(ax+b)^m} \quad \text{where} \quad P_r(x) = Q_{r-1}(x)(ax+b) + Q_r(x)$$

In the same way we would obtain

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}\,\mathrm{d}x = Q_r(x)\sqrt[n]{(ax+b)^{n-m}}$$

Due to the nature of differentiation and the technique being used here, there is a certain kind of symmetry between these two integrals.

Of course, the last formula can also be used to evaluate products of considered functions (depending on the sign of m). Let us replace m by $-m$, then

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^{-m}}}\,\mathrm{d}x = \int P_r(x)\sqrt[n]{(ax+b)^m}\ dx = Q_r(x)\sqrt[n]{(ax+b)^{n+m}}$$

Example 3: Evaluate $\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x$.

Solution:

$$\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x = \frac{Ax^3 + Bx^2 + Cx + D}{\sqrt[3]{(2x+1)^2}}$$

$$\frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}} = \frac{3\left(3Ax^2 + 2Bx + C\right)\sqrt[3]{(2x+1)^2} - 4\left(Ax^3 + Bx^2 + Cx + D\right)\sqrt[3]{(2x-1)^{-1}}}{3\sqrt[3]{(2x+1)^4}} \quad ,$$

and after clearing the radicals it is the same as

$$3\left(4x^3 - x + 2\right) = 3\left(3Ax^2 + 2Bx + C\right)(2x+1) - 4\left(Ax^3 + Bx^2 + Cx + D\right)$$

$$12x^3 - 3x + 6 = 14Ax^3 + (8B + 9A)x^2 + (6B + 2C)x + 3C - 4D$$

$$\Rightarrow A = \frac{12}{14},\ 8B + 9A = 0,\ 6B + 2C = -3,\ 3C - 4D = 6$$

The solution of this system is $A = \frac{6}{7},\ B = -\frac{27}{28},\ C = \frac{39}{28},\ D = -\frac{51}{112}$

and

$$\int \frac{4x^3 - x + 2}{\sqrt[3]{(2x+1)^5}}\,\mathrm{d}x = \frac{\frac{6}{7}x^3 - \frac{27}{28}x^2 + \frac{39}{28}x - \frac{51}{112}}{\sqrt[3]{(2x+1)^2}} = \frac{1}{112}\left(\frac{96x^3 - 108x^2 + 156x - 51}{\sqrt[3]{(2x+1)^2}}\right) + const$$

Example 4: Evaluate $\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x$.

Solution:

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = (Ax^2 + Bx + C)\sqrt[4]{(x-1)^5}$$

$$(x^2+1)\sqrt[4]{x-1} = (2Ax+B)\sqrt[4]{(x-1)^5} + \frac{5}{4}(Ax^2+Bx+C)\sqrt[4]{x-1}$$

$$4(x^2+1) = 4(2Ax+B)(x-1) + 5(Ax^2+Bx+C)$$

$$4x^2 + 4 = 13Ax^2 + (-8A+9B)x - 4B + 5C$$

$$A = \frac{4}{13},\ -8A + 9B = 0,\ -4B + 5C = 4 \Rightarrow B = \frac{32}{117},\ C = \frac{596}{585}$$

and

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = \left(\frac{4}{13}x^2 + \frac{32}{117}x + \frac{596}{585}\right)\sqrt[4]{(x-1)^5} + const$$

CONCLUSIONS

All integrals considered in the paper can also be evaluated by means of the “substitution” method. But using substitution to solve Example 3 by hand would become a tedious work, especially in the case of a higher order polynomial, so the method introduced in this paper can facilitate the process of integration of the herein given families of functions. There are other types of functions whose antiderivatives can be found without the “necessity” of integration. We will explore such functions in the upcoming paper. The use of the presented method is left to the reader in every particular case.

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{MERAA} Mathematics in Education, Research and Applications

ISSN 2453-6881

Math Educ Res Appl, 2021(7), 1

Received: 2021-05-24
Accepted: 2021-06-23
Online published: 2021-09-30
DOI: https://doi.org/10.15414/meraa.2021.07.01.16-24

Original Paper

Comparison of full-time and distance education of Mathematics at the Slovak University of Agriculture in Nitra

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ABSTRACT

The dangerous coronavirus SARS-CoV-2, which causes COVID-19, has been spreading around the world from the Chinese city of Wuhan for more than 15 months. More than 3.4 million people in the world have already contracted this disease, more than 12,290 in our country. As the virus spreads mainly through physical contact, to protect the health and lives of students and teachers, it was necessary to close universities and transfer teaching to the online space. The Slovak University of Agriculture in Nitra has also switched to the distance form of study since March last year. In this paper, we compare the methods and forms of teaching mathematics in full-time study before and after the appearance of COVID. We compared teaching in the school year 2019/2020, when we taught in-person, and 2020/2021, when the teaching was carried out in a distance form. In both cases, we took into account the study at the Faculty of Economics and Management in the subject Mathematics IA, which is taught in the winter semester of the given school year. In this paper, we evaluated the partial score results those students achieved during the semester and their results of final exams. Both years differed in the number of partial semester works. We used the methods of mathematical descriptive statistics for this evaluation. We created databases in Excel, from which we calculated the average point evaluation of individual components and the total number of points. We also calculated the correlation coefficients between the individual parts of this course, for which students could get points. In this way we recorded the data obtained in tables and graphs. We tested hypothesis that students of full-time study would achieve better results than distance-learning students. After comparing the obtained results, we found that this hypothesis was not approved.

KEYWORDS: full-time education, distance mathematics education, coronavirus pandemic, mathematical-statistical analysis, correlation coefficient

JEL CLASSIFICATION: C02, C11, I210

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INTRODUCTION

Since the coronavirus was discovered in Wuhan, China, in December 2019, a COVID-19 pandemic has spread around the world. As of 24 May 2021, more than 167 million positive cases and more than 3.48 million deaths from COVID-19 had been confirmed in more than 190 countries and regions of the world, and these numbers are expected to be several times higher [3]. The USA, India, Brazil, France, and Turkey are most affected.

The first cases appeared in Slovakia at the beginning of March 2020, based on which the government took tough measures. As of 24 May 2021, 388,854 people had tested positive and 12,296 had died because of the disease [4].

This pandemic has affected all areas of life, not only healthcare, but also industry, catering, tourism, and the like. Of course, she did not bypass education at all levels, not excluding higher education. The virus is most often spread by close physical contact, so it was necessary to transfer full-time teaching to the online space. From the beginning of March 2020, students of the Slovak University of Agriculture switched to distance learning. This applies to lectures, exercises, but also verification of knowledge, either continuously, but also by the final exam.

As stated in [11]: "Mathematics is more than science, it is the language of all sciences." According to Álvarez et al. [1] Mathematics plays a key role in every engineer's curriculum because it provides a theoretical basis for various theories of natural sciences. Pechočiak and Kecskés [9] say that the use of mathematical and statistical methods not only allows the detection of the occurrence of certain phenomena in the new global environment, but indirectly requires special attention. Országhová and Horváthová also dealt with statistical analysis of mathematical and linguistic competencies [7]. Also Cígler [2] deals in details with mathematical abilities and mathematical skills, stating the main differences, ways in which they arise and could be measured. Results of didactic research in different countries have confirmed that students' interest in science subjects is decreasing and students come across difficulties in STEM subjects (i.e., science, technology, engineering, and mathematics) [10].

In the distance form of study, teachers as well as students had to reorient themselves to work with computer technology. Teachers became programmers, creators of online teaching materials. In many cases, they used their own computing resources for this work, working at home, where they met other members of the household, and had to harmonize the requirements of all.

MATERIAL AND METHODS

In our paper, we decided to compare the teaching of mathematics before and after Covide. We compared the teaching of the subject Mathematics IA, which is taught at the Faculty of Economics and Management in the winter semester. We researched teaching in the school years 2019/2020 and 2020/2021. In the first of the mentioned years, the teaching took place classically, in person, in the second we switched to distance education online. Matušek and Hornyák Gregáňová [6] also dealt with the comparison of exam results in Mathematics.

The scoring system according to the ECTS scale has been operating at our school for several years. Therefore, both continuous evaluations and exams are awarded points.

The winter semester of the school year 2019/2020 took place in person, so the students personally participated in lectures and exercises led by teachers from the Department of Mathematics FEM SUA in Nitra. During this semester, students were able to earn 35 points for the preliminary test they wrote in about the sixth week. They wrote the test at school in practice and had 60 minutes to complete it. We present an example of such a preliminary test:

1. If the function $f: y = x^2 - 3x + 4$ is called, which is its graph, sketch it.
2. Calculate the definition area $D(g)$ of the function $g: y = 2 - 5\ln(3 - 2x)$ and find the inverse function g^{-1} to it.
3. Find the asymptote without the direction of the function graph and sketch it, when $h: y = \frac{2-x}{3+x}$.
4. Write a definition of the bounded function from above and give an example.

In about the eighth month, after taking over the issue, students were given a homework seminar, for which they could get another 15 points. It concerned the examination of the course of the function, i.e., finding out the properties of the function and then drawing its graph. The function was in the form of a fraction, for example $f: y = \frac{2-x}{3x^2}$. They had a week to work it out.

During the semester, students could get an additional 5 bonus points for completing short homework or activity in exercises. Thus, during the semester, students were able to gain a total of 55 points. They needed at least 30 points to be awarded credit. 92 students got to this stage of the semester. Students with credit could take the exam, which lasted 90 minutes and could get 50 points for it. Here is an example of such an exam written work:

1. If the function is called $f: y = \log_6(x - 4)$, which is its graph, sketch it.
2. Find the monotonicity intervals of the function $g: y = 2x^3 + 6x^2 - 18x + 7$.
3. Find the inflection points of the function $h: y = 2x^4 - 4x^3 - 24x^2 + 10x + 7$.
4. Find the local extrema of the function $k: z = x^2 + 6y^3 + 18y^2 - 6xy + 18y - 18x$.
5. Calculate the second partial derivatives of the function $z = \cos(3x - 4y)$.
6. Write a definition of a descending function and give an example.

During the winter semester in the school year 2020/2021, which was already taking place online, students wrote 6 individual assignments, in which they found 3 tasks. All teachers of our department participated in the creation of these assignments. We evaluated each correctly calculated assignment with 9 points. In the first assignment, students had to display graphs of linear and quadratic functions and write their properties and write the definition of some given property. In the second assignment they had a given exponential or logarithmic function (for example $f: y = 4^{x+1} - 32$), they had to express its inverse function and display graphs of both functions. In the third assignment, they had to find out the equations of asymptotes without a directive and with the directive of the graph of the function *f* (for example $f: y = \frac{3x^2}{2x-6}$) and write one of the theorems or definitions concerning the limits of functions. In the fourth assignment, students counted 2 examples of derivatives of functions and had to write a definition or sentence concerning the derivation of a function of one variable. The sixth assignments included examples from economic applications of derivatives, the calculation of the first and second partial derivatives of the function of two real variables, and a theoretical question in this area. Here is an example of such an assignment:

1. The function of total costs has the expression $TC(x) = x^4 - 8x^2 + 6000$, where x is the level of production in thousands. Find out for which level of production the total costs are minimal and state the size of the costs.
2. Calculate the partial derivatives of the 1st and 2nd order of the function $f : z = \frac{x^2}{4y}$.
3. Define the partial derivative of the function of two real variables according to the variable *x*.

Students from these six assignments could get 54 points. Of this total number of points, in order for a student to obtain a credit and be able to register for the exam, he also needed at least 30 points as in the previous year. Together we evaluated the results of 114 students. After obtaining the credit, students wrote an exam written work for 50 points, which contained 5 examples for calculation and 1 theoretical question from the curriculum of the whole semester. We also give an example of such an exam written work:

1. Find the equation of the asymptote with the direction of the graph of the function $f : y = \frac{3-2x}{3x-2}$ and sketch it.
2. Find the local extrema of the function $g : y = 4x^3 - 12x^2 - 36x - 10$.
3. Find the convexity and concavity intervals of the function $h : y = x^4 + 2x^3 - 12x^2 + 20x + 17$.
4. Find the local extrema of the function $k : z = 4x - y - x^2 - y^2 - xy + 5$.
5. Calculate the second partial derivatives of the function $z = 5x^2y^3 - 4xy^2 + 3xy + 2\cos y - 8$
6. Write the definition of the descending function and give its example (by the equation).

The points obtained during the semester and the points from the exam were counted and we evaluated the overall success of the student using the ECTS scale. At least 64 points were needed to successfully complete the study of this subject.

The results we obtained during these two years were evaluated in the next part of the work. In our work we were also inspired by the textbook of colleagues from CPU Nitra [5]. We used the methods of mathematical descriptive statistics, while we created databases in Excel by arranging the results of individual parts of this subject for each student under each other. From the data arranged in this way, we calculated the average point evaluation of individual components and the total number of points. We also calculated the correlation coefficients between the individual parts of this course, for which students could get points, i.e., whether it applies if students master one part, they manage others, that is, whether there is any dependence between them. A weak dependence is when the correlation coefficient is from an interval $\left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle$, a medium dependence for values from intervals $\left\langle -\frac{2}{3}, -\frac{1}{3} \right) \cup \left(\frac{1}{3}, \frac{2}{3} \right\rangle$ and a strong dependence for values from intervals $\left\langle -1,-\frac{2}{3}\right) \cup \left(\frac{2}{3},1\right\rangle$. Similarly, we addressed such an issue in the paper [8]. We also set a hypothesis: Students who attend full-time will achieve better results compared to students who have studied remotely.

RESULTS AND DISCUSSION

As mentioned above, students in the school year 2019/2020 could get 35 points for preliminary test, 15 points for home seminar work, 5 bonus points and 50 points for exam written work. The average number of points obtained in this way, even with a percentage expression for all 92 students, is given in Table 1 and we can also see them in Figure 1.

Table 1 Points obtained during the winter semester in the school year 2019/2020

	PT	HSW	BP	∑ P+H+B	EWW	TP
AVERAGE NUMBER OF POINTS	23.13	13.67	3.4	40.20	35.36	75.56
%	66.09	91.11	68.00	73.09	64.28	68.69

Source: own

Explanations of the abbreviations in the Table 1:
PT – the points obtained of preliminary test, HSW – the points obtained of home seminar work, BP – bonus points
∑ P+H+B – the sum of points obtained during the semester
EWW – the points obtained of the exam written work, TP – total points

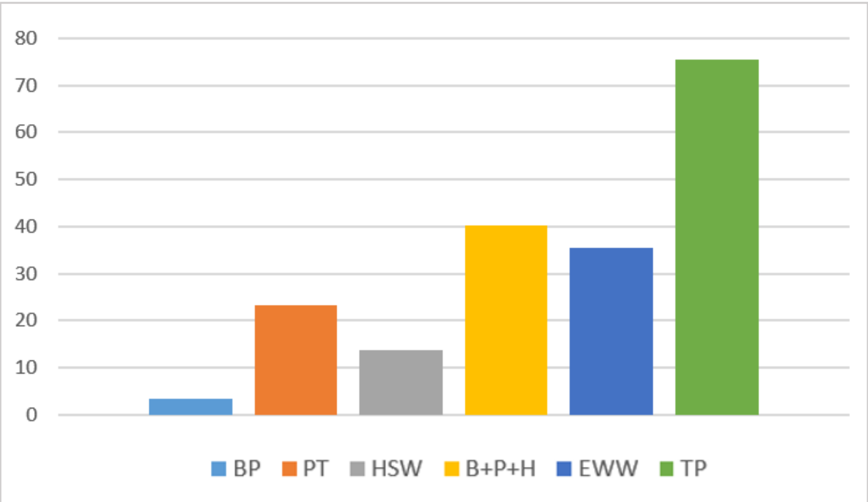

Figure 1 Points earned during the winter semester 2019/2020
Source: Calculation of author

They received the most points for the home seminar work, on average 13.67 points, which represents 91.11% of the total number of points. For the semester, they gained an average of 40.2 points, which is 73.09% of the total number of 55 points. They wrote the exam written work on average at 64.28%, which is 35.36 points out of 50 points. The sum of all points obtained averaged 75.56 points out of 105.

We also calculated the correlation relations between the points obtained for the preliminary test, the sum of the points obtained for the credit (preliminary test + seminar work + bonus), the points for the exam written work and the total sum of points (Table 2).

Table 2 Correlation coefficients between individual items for the year 2019/2020

	PT	**∑ P+H+B**	**EWW**	**TP**
PT		0.961	0.257	0.706
∑ P+H+B	0.961		0.304	0.759
EWW	0.257	0.304		0.851
TP	0.706	0.759	0.851	

Source: Calculation of author

As can be seen from this table, there was a weak relationship between the preliminary test and exam written work and the sum of points for the semester and the exam written work. We assume that the student underestimated the exam when he obtained a relatively large number of points per semester, or vice versa, when he gained few points during the semester, he was better able to prepare for the exam. There was a strong dependence between the other categories.

In the school year 2020/2021, students were able to obtain 54 points for 6 individual assignments and 50 points for the written exam. These obtained average numbers of points, even with a percentage expression for all students, are shown in Table 3 and Figure 2.

Table 3 Points obtained during the winter semester in the school year 2020/2021

	$\sum A_i$	**EWW**	**TP**
AVERAGE NUMBER OF POINTS	43.11	38.03	81.14
%	79.84	76.05	78.02

Source: Calculation of author

Explanations of the abbreviations in the Table 3:
$\sum A_i$ – the sum of points obtained during the semester
EWW – the points obtained of the exam written work
TP – total points

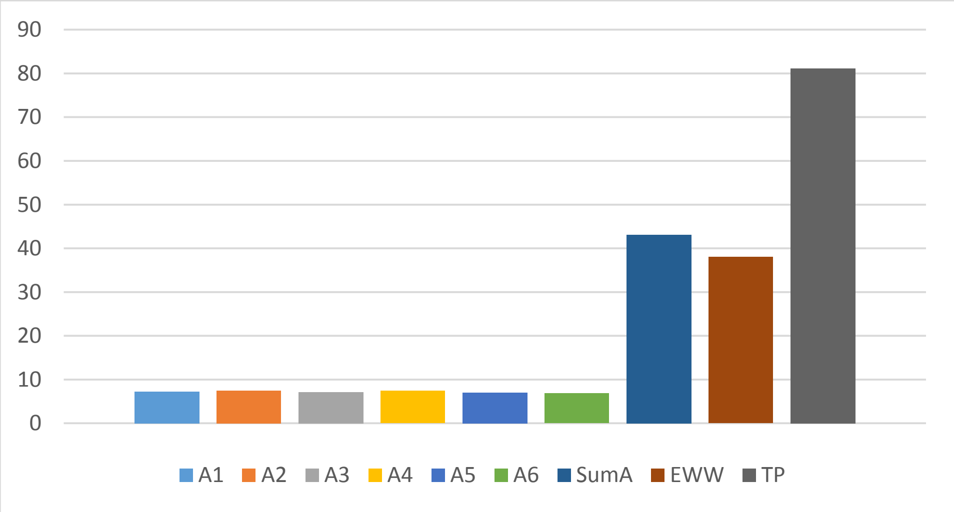

Figure 2 Points earned during the winter semester 2020/2021
Source: Calculation of author

The average number of points for individual assignments was from 6.85 to 7.46 out of 9, which is in percentages from 76.07% to 82.9%. The average total of points was 43.11, which represents 79.84% of 54 points. They wrote the exam written work on average at 76.05 percent, or 38.03 points out of 50. The average sum of points for the semester and the exam was 81.14 points, which is 78.02% out of 104 points.

Also in this case, we calculated the correlation coefficients between the individual points obtained during the semester. We entered them in Table 4.

Table 4 Correlation coefficients between individual items for the year 2020/2021

	$\sum A_i$	EWW	TP
$\sum A_i$		-0.119	0.204
EWW	-0.119		0.315
TP	0.204	0.315	

Source: own

As we can see from Table 4, all dependencies are weak, so they are out of range $\left\langle -\frac{1}{3}, \frac{1}{3} \right\rangle$.

This means that if a student had enough points from the assignment, he did not have to get a large number of points from the exam, or if he got a larger number of points from the exam, he did not have to get a large number of points from the assignment. It was similar in the previous school year, when there were cases where students had relatively many points during the semester and few points from the exam work, or vice versa.

In the introduction, we set the hypothesis: Students who attend full-time will achieve better results compared to students who study in distance form. We assumed that during contact teaching the student would be better prepared for the written work, as the students counted the examples on the board in the exercises, we commented on the whole procedure and students could react immediately if they had a problem solving. However, this assumption was not met. Students studying in distance form achieved on average 9.33% better results than full-time students. We explain this by saying that students in online teaching had a lot of time to do partial written work better (they earned 6.75% more points on average) and similarly in the exam, although they had almost the same time to do it, these students could use aids, or even cheat (they gained up to 11.77% more points here than students who took the full-time exam).

CONCLUSIONS

The coronavirus pandemic, which has been spreading around the world for almost a year and a half, has also affected higher education. From face-to-face teaching, from face-to-face teaching, we had to switch to distance learning, which we did mainly online. Because we did not want to reduce the level of mathematics education at our university, we, teachers, and students had to switch to education through online means. Many of us used our own computer technology to do this. In our work, we showed how we had to change the way of teaching and especially the control of acquired knowledge by students. We compared the results of mathematics studies obtained in person and online. We found that students studying online achieved better results than full-time students.

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{MERAA} Mathematics in Education, Research and Applications

ISSN 2453-6881

Math Educ Res Appl, 2021(7), 1

Received: 2021-06-04
Accepted: 2021-06-30
Online published: 2021-09-30
DOI: https://doi.org/10.15414/meraa.2021.07.01.25-32

Original Paper

Outcomes of distance education in Mathematics at secondary technical school: a case study

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ABSTRACT

Education at secondary vocational schools has an important role in preparing graduates to perform professional activities in practice. These schools provide a complete secondary education, and Mathematics is also a part of the study plan. In this paper, we focused on the success of students in solving selected types of mathematical problems during the school year 2020/2021, when the study process at secondary schools was carried out in a distance form. The main goal was to evaluate the mathematical knowledge of the secondary school students through a set of selected tasks. The research sample consisted of 21 students of the 3rd year of the secondary technical school with a focus on civil engineering study program, for which we analyzed tasks of online tests in the first and second half of the school year. Using methods of mathematical statistics, we compared and evaluated the success of students in solving problems that were part of the tests from the mathematics curriculum. The results show that students have mastered the necessary electronic tools and methods of distance learning in acquiring mathematical knowledge.

KEY WORDS: mathematics education, secondary technical school, sequences, combinatorics, probability, Wilcoxon signed-rank test

JEL CLASSIFICATION: D40, C50, M10

INTRODUCTION

Education is one of the important factors for the development and quality of human resources, that are key for the knowledge society and the economy of society. The interest in vocations of children (or pupils) in primary schools is gradually evolving and changing, among other

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things because new professions are emerging depending on changes in society. After completing primary education, students decide at which secondary school to continue education, and this choice is conditioned by the interest in the profession they want to do in adulthood. In the Slovak Republic the system of secondary education is governed by the *Act on Education* (School Act) [1], which lists in § 32 the following types of secondary schools in the Slovak Republic: a) Grammar school, b) Secondary vocational school, c) Secondary sports school, d) School of the Arts Industry, e) Conservatory.

Educational institutions, and thus secondary schools, are an important factor in the development of each region and provide conditions for linking education and the needs of the labor market in each region. As stated in the *National Program for Learning Regions* [5]: "strong links need to be established between educational institutions, educational establishments, higher territorial units and municipalities, employers and professional associations".

The aim of the educational process at a secondary vocational school is the quality preparation of pupils for practice and other forms of study. Secondary vocational schools are focused on the development of practical skills and the creation of conditions to support the activity form of teaching. In addition to professional subjects, students also have general education subjects in their study plan. Complete secondary vocational education is finished by successfully passing the school-leaving examination. Graduates of a secondary vocational schools can continue their study at universities.

During the school years 2019/2020 and 2020/2021, the course of education at all levels was influenced by the coronavirus pandemic. In 2020, schools were closed in the Slovak Republic to protect students and teachers from COVID-19, and teaching was moved to online space with the support of information technology (IT) tools. Instructions for secondary schools were issued at the Ministry and the „Decision“ of the Minister of Education, Science, Research and Sport of the Slovak Republic [3] showed “with effect from October 12, 2020 until the appeal, the decision as follows: a) exceptionally interrupts school teaching in secondary schools pursuant to § Section 32 of the Education Act, with the exception of school teaching in the first to fourth years of the eight-year secondary school curriculum; secondary school directors will provide distance education for secondary school students". Teaching at secondary schools took place from mid-October in the school year 2020/2021 in distance form. After the improvement of epidemic conditions, the entry of secondary school students in 2021 was determined in April, first for the graduation years, and then in May also for students from the 1st to the 3rd school year.

The transition from full-time contact study to distance learning has brought many changes for teachers in organizing the educational process, as well as for students who have had to start using IT tools to join on-line teaching and perform the required educational tasks. As Turek [11] states “during the distance learning, the following principles of programmed teaching are appropriate for the effectiveness of mathematics teaching: 1) Principle of small steps, 2) Principle of active response, 3) Principle of immediate fixation, 4) Principle of individualization, and 5) Principle of evaluation and improving the teaching process”.

The distance education takes place in a virtual space, where the current young generation gains new experiences, skills, and knowledge. In a research study [2], authors analyzed five main topics in the secondary education during the pandemic, including the structure of mathematics content, teachers' readiness for distance learning, psychological and emotional aspects arising during a coronavirus pandemic.

The course of distance education at Slovak primary and secondary schools in 2019/2020 was assessed via a questionnaire survey in which an estimate of the pupils' involvement in this form of education was made. The main findings of this survey include an estimate that 52,000 primary and secondary school pupils (7.5% of the student population) were not involved in distance learning. Approximately 128,000 pupils (i.e. 18.5% of the student population) did not learn online; most of them were probably educated by other forms of distance learning, e.g. through printed worksheets prepared and delivered to them by teachers [4].

The complete secondary vocational education is finished after successfully passing the school-leaving examination in compulsory and optional subjects. Many graduates of secondary vocational schools continue their studies at universities, choosing faculties that correspond to the focus of the completed secondary school. Knowledge from secondary school is the basis for the study of mathematical subjects at the university and the level of knowledge of newly admitted students is examined through an entrance math test. Students motivation is still one of the important determinants of success in university study [6].

In this paper, we focused on mathematical education at secondary vocational schools and its outputs during the period of distance education in the school year 2020/2021. The current network of secondary technical schools (as a kind of vocational schools) in the Slovak Republic was created gradually and has base in industrial enterprises that need qualified workers for special areas of practice. Table 1 contains data on the number of secondary technical schools in the Slovak Republic by region in year 2020.

Table 1. Number of secondary technical schools in the Slovak Republic by region

Region	**Number**	**Region**	**Number**
Bratislavský	6	Žilinský	3
Trnavský	4	Banskobystrický	4
Trenčiansky	4	Prešovský	6
Nitriansky	5	Košický	5
Sum: 37 schools			

Source: [9], author's processing

In the period 2012 - 2017 data of the Statistical Yearbook showed a declining trend in the number of high school graduates in the Slovak Republic; subsequently, in years 2019 and 2020, there was a slight increase [10]. As reported Pechočiak and Drábeková [7], such trend was also reflected in the decline in the number of students enrolled in universities, until a slight increase was recorded in 2018. A part of secondary school graduates continues in study at foreign universities, thus the number of students at Slovak universities deepens every year.

MATERIAL AND METHODS

The motivation to write the paper followed from the distance form of education at secondary vocational schools. We were interested in how students mastered selected topics from the curriculum of mathematical subjects. The logical structure of the curriculum, symbolic notations, abstract considerations and generalizations are used in mathematics. Mathematical relations, graphs and diagrams are basis for acquiring of mathematical knowledge and computational skills. During the distance learning, students needed to be motivated to take an active part in online teaching and to perform assigned individual tasks.

The main goal of this paper was the analysis of the results of mathematical education at a secondary technical school with a focus on the subject matter of the 3rd year of study. The selected research sample was formed of students of the 3rd study year at the Secondary Technical School of Civil Engineering in Žilina. Students of the 3rd study year started full-time teaching again from May 11, 2021.

In the evaluation of students' mathematical knowledge, we focused on the following thematic areas in the 1st and 2nd parts of the school year 2020/2021:
1st half of year: Sequences and their properties, Arithmetic sequence, Geometric sequence.
2nd half of year: Combinatorics and Probability.

We present selected examples to individual topics in the analyzed tests.

Test 1. Sequences
Task 1: Write the first five members of the sequence $\left\{\frac{n-1}{n+1}\right\}_{n=1}^{\infty}$ and find out if it is bounded.

Task 2: Display the first five members of a sequence $\left\{\frac{n+2}{n}\right\}_{n=1}^{\infty}$.

Test 2. Arithmetic sequence
Task 3: In the arithmetic sequence $a_1 = 23\,, s_7 = 35$. What is the difference of this sequence?

Task 4: Find the first term and the difference of the arithmetic sequence to which it applies

$$a_2 + a_3 = 2$$
$$a_2 + a_7 = -8$$

Test 3. Geometric sequence
Task 5: Write the first five terms of a geometric sequence if $a_2 = 27,\ a_3 = 81$.
Task 6: Calculate the quotient of the geometric sequence if $a_3 = \frac{1}{8},\ a_5 = 2$.

Test 4. Combinatorics
Task 7: How many different natural five-digit numbers with different digits can be formed using digits 0,1, 2, 3, 4, 5?

Task 8: There are 30 students in the class, 20% of them boys. How many ways can we choose 4 girls for a trip to Paris?

Test 5. Probability

Task 9: What is the probability of having the sum of 9 after rolling 3 dice?

Task 10: There are 38 students in the class. Just 7 students do not have homework. The teacher randomly checks 8 students. Calculate the probability that no more than three (out of the checked students) have a homework assignment.

In the evaluation of mathematical tasks and tests, the current evaluation scale was used, which is a part of the approved evaluation criteria of study subjects at secondary school (Table 2).

Table 2. Evaluation criteria for Mathematics

Percentage success	Final grade
90 % to 100 %	Excellent (1)
75% to 89.99 %	Very good (2)
50% to 74.99%	Good (3)
33.33% to 49.99%	Sufficient (4)
Less than 33.33%	Insufficient (5)

Source: [8], author's processing

Due to the size of the research sample (21 students), we applied the Wilcoxon signed-rank test (which belongs to the non-parametric statistical tests) to determine the significance of the differences in the mean value between individual tests.

RESULTS

We evaluated 3 tests in Mathematics from the 1st half and 2 tests from the 2nd half in the 3rd year of study in academic year 2020/2021. Each test contained 5 tasks from a given topic and for the correct solution of all tasks the student could get 6 points together. Results of the analysis of tests in the 1st and 2nd part of school year are in Figure 1.

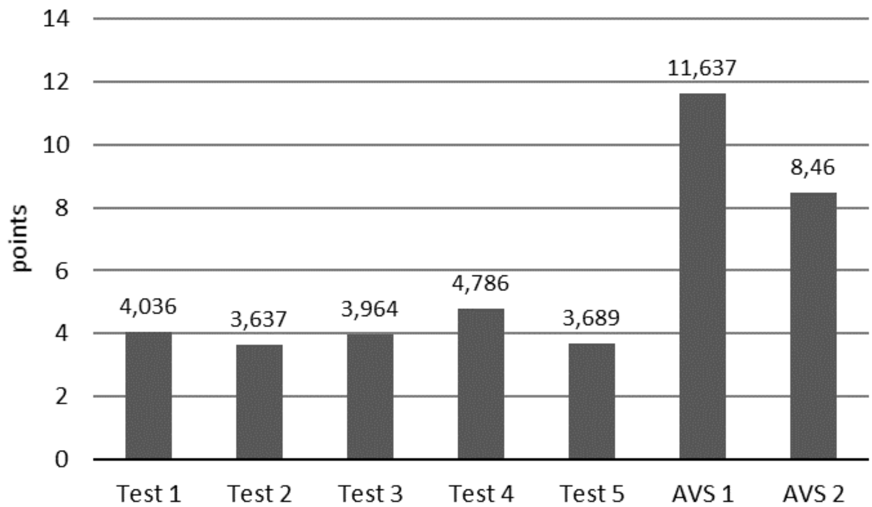

Figure 1. Evaluation of the average point score in the solution of tests
Source: authors

The best average point score was achieved by students in the Test 4 in the 2nd part of school year and the worst average evaluation was obtained by students in Test 2 in the 1st half of the year. The value of AVS 1 expresses the average score for Tests 1 to 3 together. AVS 2 expresses the average number of points for Tests 4 and 5. To compare these data, the average percentage success evaluated for tasks in the first half was 64.7% (AVS 1 recalculated) and for tasks in the second half of the year, the average percentage success rate was 70.5% (recalculated AVS 2).

According to the evaluation scale, we conclude that students managed the analyzed tasks on average grade Good (3). The evaluated tasks are the part of the curriculum in the 3rd year and other thematic tests and students' activities are included in the final semi-annual and end-year evaluation. The resulting average grade from Mathematics in this research sample was 2.77 in the first half of the school year 2020/2021and 2.33 in the second half of the school year. Students solved tasks better in the 2nd half of the year and they already knew from the 1st half of the study means and methods of distance education.

If we rank all evaluated tasks together according to the assigned grade from 1 to 5 using the evaluation scale, then we obtain the following histogram for all tasks evaluation (Figure 2). In the analyzed sample of tasks, the highest number was achieved by the grade 3 (Good).

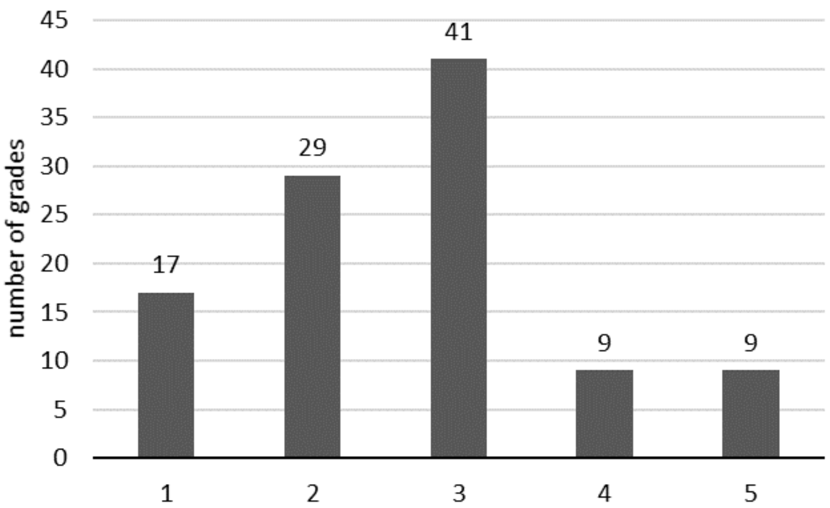

Figure 2. Histogram of grades for all tasks together
Source: authors

If we display the evaluation of success in solving problems according to grades obtained in individual tests, then we see that the problems from Combinatorics in the Test 4 were solved by most students with a grade 1 (Excellent) (Figure 3). This also corresponds to the highest average score of the Test 4. The grade 2 (Very good) was awarded to the most students in assignments on Sequences in the Test 1, which also corresponds to the highest average score of Test 1 in the first half of the year.

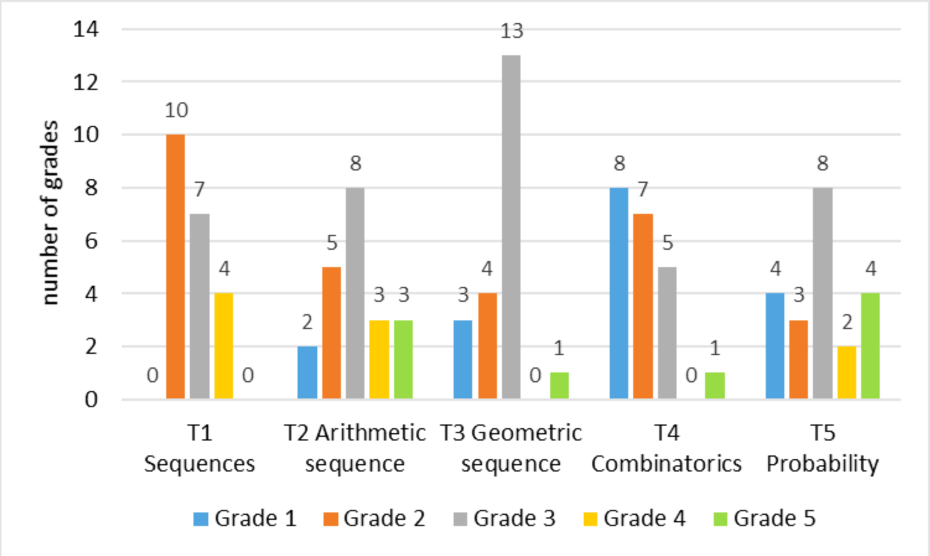

Figure 3. Assessment of tasks according to particular tests
Source: authors

Using the Wilcoxon signed-rank test, we found that there are significant differences in the obtained score of Test 4 and Test 5 at the selected level of significance $\alpha = 0.05$. When testing the average score for the 1st and 2nd half of the school year (AVS 1, AVS 2) there was also confirmed the significance of the differences in the mean value. Using another statistical method - the correlation coefficient - a strong correlation was confirmed between the average score of AVS 1 and AVS 2 (Table 3). This means that students who had good results in the first half of the year tried to gain knowledge and good evaluation in Mathematics in the second half of the year too. There is a moderate relationship between the Test 1 and Test 3, and between Test 4 and Test 5. The other evaluated correlations are weak.

Table 3. Results of Wilcoxon signed-rank test and correlations ($\alpha = 0.05$)

Topic 1	**Topic 2**	***p*-value**	**Correlation**
T1 Sequences	T2 Arithmetic sequence	0.29	0.314
T1 Sequences	T3 Geometric sequence	0.92	0.367
T2 Arithmetic sequence	T3 Geometric sequence	0.26	0.266
T4 Combinatorics	T5 Probability	0.008*	0.392
AVS 1	AVS 2	0.000*	0.675

Source: author's calculations

CONCLUSIONS

Secondary vocational schools prepare graduates for professional work and creative activity in their future profession. Some graduates of secondary vocational schools will start their internships and some graduates will continue in study at technical faculties. These faculties also have mathematical subjects included in their study programs, where students apply and expand the acquired knowledge of high school mathematics.

By analyzing the data of the empirical research, we found the average grade in Mathematics at the level of 2.77 in the first half of the school year 2020/2021; and in the second half of the year, it was 2.33. Using a non-parametric statistical test, it was confirmed the significance in differences between the evaluated tests T4 and T5 in the second half of the school year. In the research sample it was also approved the significance in differences between sum of points together for individual half-years.

During the distance education in 2020/2021, secondary school teachers used modern didactic means, which required their own creative contribution to achieve teaching goals. The realization of the process of mathematical education with the use of IT tools also affects the learning outcomes. In any change in the process of the teaching, the teacher must consider many of direct and indirect factors affecting the course of education.

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{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

Received: 2021-08-16
Accepted: 2021-10-08
Online published: 2021-10-15
DOI: https://doi.org/10.15414/meraa.2021.07.01.33-49

Original Paper

Differences in the use of electronic and printed versions of a university mathematics workbook

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ABSTRACT

The recent months have shifted contact teaching to the online environment and distance learning and students are dealing more and more with digital materials in various e-learning systems. The question is whether the online electronic materials are as effective as their printed versions for the students using them for self-study purposes. This paper presents research focusing on university students' work with an electronic and printed version of a mathematics workbook. The main research focuses on differences regarding error rate, the number of used hints, and the time they need to spend to solve 111 mathematical problems covering four topics of their introductory course of Mathematics such as limits, graphs, differentiation, and applications of derivatives. One hundred fifty-seven university students participated in the research working with sets of mathematical problems with multi-choice answers taken from the Khan Academy, including step-by-step hints. At the same time, the students were recording their errors, time, and the number of used hints using a questionnaire. The electronic sets were transformed into an electronic workbook and afterward into a printed version of this workbook. Obtained data were analysed using the Random Mixed Model as it enables to mix the used mathematical problems with different variance. The most exciting finding of this research was that the students working with the electronic version of the workbook work significantly faster but at the expense of errors. Students working with the interactive version of the workbook used significantly fewer hints.

KEYWORDS: Khan Academy, mathematics, e-workbook/textbook, printed workbook/textbook, random-mixed model, technology of education, university student learning

JEL CLASSIFICATION: I20, I21, C12

INTRODUCTION

With today's massive spread of electronic media into schools, we question whether new technologies will not suppress classic textbooks, blackboards, and chalk. It has been many

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years since new electronic materials appeared for the first time. However, traditional printed textbooks still hold their solid position in the teaching and learning process. Therefore, the question is not if printed textbooks belong to the educational market or not, but rather, the critical issue is the differences between printed and electronic textbooks differences into possibilities in education content dissemination.

Generally speaking, we may understand textbooks as a necessary part of the education process, in which teachers and pupils are involved at school and parents outside school. However, priority users are pupils. It is undeniable that a good textbook should be an essential and irreplaceable means in the educational process, easing the teachers' workload at school and serve the pupils for their self-study.

We are fully aware that the concept of the electronic textbook is very general and broad. On the one hand, it may only be an electronic version of the paper text. On the other hand, it may be a very profoundly structured and multimedia construct with hypertext links and interactive elements. In our research, we worked with both printed textbooks and electronic teaching materials with multimedia elements, hypertext links, and interactive tools used for the evaluation of pupils, which are understood as interactive educational objects with didactically justified sets of elements (figures, graphs, videos, and texts) forming one profound whole enabling the participants to interact. We use, in this paper, the terms 'textbook' and 'workbook' in this sense, more specifically, 'an electronic version of a textbook/workbook' and 'a printed version of a textbook/workbook'.

The offer of digital teaching and learning materials spreads worldwide in two ways. On the one hand, more and more publishers issue besides printed textbooks also their digital versions (as the most common format – PDF). On the other hand, an increasing number of teaching portals offer various interactive materials, including Computer-Aided Assessment (CAA). Students praise their immediate feedback and possibilities of repeated practicing with different variables or parameters, teachers, them, the easy variability when preparing tests for their students.

Our research focuses on students' practice at home during their introductory mathematics course using digital interactive tests and different aspects between the printed and electronic versions of a mathematics workbook. First, let us start to deal with mathematical texts generally. Nebeský [23] focused on the specifics of the Czech mathematical text. He examined the functioning of natural language in the mathematics environment and tried to set borders between the natural and artificial aspects. Units of the mathematical text, where their linguistic function is precisely given, he calls rigid, while the other units he calls live. He claims that thanks to the richness of the Czech language, the same expressions may function in mathematical texts as a rigid unit and a life unit, always depending on its particular meaning. In this case, it is necessary to replace life expressions with a synonym or to reformulate the given sentence.

On the contrary, rigid units have to be left. Authors of mathematical texts face a choice as to which content they should use brief expressions based on life units or if they should use a ponderous formulation based on rigid units. Moreover, they have to consider carefully if the consequence of thoughts is helpful sufficiently for the readers to understand the text's content correctly. Writing mathematical texts means mastering a complex apparatus in which content, methodological, pedagogical, and psychological components intermingle. It is crucial to find their mutual harmony.

The specificity of mathematical texts considering their comprehension was investigated by Fang & Schleppegrell [11]. They claim that it is essential to understand mathematical terminology and comprehend everyday words and their meanings for solving mathematical problems. On the other hand, the understanding of everyday vocabulary is not sufficient for language comprehension in mathematics. Morgan, Craig & Schuette [21] concluded the same together with Abedi & Lord [1], who conducted an exhaustive study focusing on testing word problems in which 1,174 students, for whom English is no their mother tongue, participated. The students were given the origins of word problems and then edited versions with more straightforward vocabulary and shorter sentences. They recorded an average improvement in performance in mathematics in the case of more than 1,000 students. The improvement was statistically significant with students with a lower command of English, a lower social-economic status, and more unsatisfactory performance in mathematics.

The structure of the mathematical text should not be underestimated when assigning test problems. Gueudet & Trouche [14] claim that it is essential to combine suitably three important components when testing students in mathematics. They mention the material component (paper, computer), the mathematical content component (terminology, tasks, and techniques), and finally, the didactical component (organizational elements and effective planning of the mathematical subject matter). Also, the organization committee of the international Kangaroo of Mathematics Competition investigates formulations of mathematics competition problems every year in detail as these problems are then translated into many other languages. The translations have to be made very carefully, knowing both languages well, but it is also necessary to understand the language of mathematics well.

The fact that mathematicians think very carefully about every word they use in their texts was verified by research conducted by Shanahan, Shanahan & Misischia [29], who made a profound inter-subject study of reading as a tool for the development of suitable teaching strategies. On top of all that, they proved that the successful performance in solving mathematical problems goes hand in hand with the language mastery of the students and the complexity of the text.

Dostal & Robinson [8] define in their paper, called Doing Mathematics with Purpose: Mathematical Text Types, four types of mathematical texts (proof text, algorithmic text, algebraic/symbolic text, and visual text), their purpose and their key functions. Dostal & Robinson [8] investigated mathematical literacy, too. They claim that mathematics learning includes reading and writing various types of mathematical texts that may be control, algorithmic, algebraic (symbolic), and visual. Shanahan, Shanahan & Misischia [29] investigated the differences in how chemists, historians, and mathematicians read text specific to their disciplines. Unlike the chemists and historians, the mathematicians in their study did not consider sources when reading and evaluating a text or another visual element. They believe these elements as unified and identically necessary.

The sets of problems in our research consisted of rather algebraic and visual texts. The control text was partially present in particular problems. In our study, we consider all types of the mathematical text of the test problems as unified. We do not distinguish whether the problems' assignment was algebraic or visual in the consequent statistical analysis.

Computers provide a range of opportunities for developing more interactive, authentic, and engaging tests [33], they are also increasingly used in the workplace and in everyday life to deal with problems involving numbers, quantities, two or three-dimensional figures, and data. It is also important to point out that some computer-based tasks cannot exist in a paper test because of their response format (e.g., “drag and drop”), or they require students to use the computer as a mathematical tool by interacting with the stimulus to solve a mathematics problem. However, in our study, we consider the used mathematical problems assigned in the form of paper and on the computer as identical with no unique response format in the electronic version. Therefore, there was no need to take the transferability into account, as mentioned, for example, by Lenhard, Schroeders & Lenhard [17] or Noyes & Garland [24].

Jahodová Berková [15] deals with the contribution of CAA in the teaching and learning of mathematics from university students’ point of view and concludes the potential of CAA may be seen, above all, in its formative assessment and online basis. The students think that CAA systems should be used mainly to revise newly acquired subject matter but certainly not for the summative testing. On the other hand, teachers see the most important benefit in testing their students. Formative tests (in the form of homework) may monitor improvements during the learning process. In contrast, the summative tests (as final examination tests) function as an assessment tool at the end of the teaching and learning process.

Besides the mentioned factors, there are many other advantages and disadvantages of CAA. Repeated testing problems should improve students’ performance, but only if their attitude to the tests is not based on trial and error. If there is no supervision of the students, for example, at home, it is hard to prevent cheating. Therefore, formative testing should help the students consolidate their knowledge and prepare them for a final examination. For this reason, they do not tend to cheat. However, it is crucial to mention that Axtell & Curran [3] claim that if students do not make notes and comments while solving CAA systems problems, they cannot use homework to help them acquire better the subject matter.

Our research is based on voluntary testing of the knowledge our students learned during weekly lectures and seminars. We were interested in the level of acquiring the theory and the ability to solve various mathematical problems during the pandemic period. To see more deeply into the acquisition level, we also recorded the number of hints the students used during the solving process. Unfortunately, we have not found any relevant research dealing with step-by-step hints in solving mathematics problems, representing the interactive structural component of the mathematical apparatus (Zujev [39]; Průcha [25]; Krotký & Mach [16]).

To conclude the introduction part, we would like to say that in the case of any testing in mathematics (formative, summative, paper-based, or digital), it is necessary to provide students with sufficient technical conditions, to formulate unambiguously and precisely mathematical problems and to pay always attention to the mathematical syntax. In the case of summative testing, it is crucial to set the time limit appropriately and select such testing problems so that they cover the required subject matter and may reveal possible defects in students’ understanding. The question is if the time limit should be the same as students are taking the same test at school. Another question is if students make mistakes to the same extent when dealing with the paper-based and digital tests. These are the reasons we deal with these two factors (time and error rate) factors in comparing the paper-based and digital formative testing in our study, as we consider that the findings may be beneficial to the teaching practice.

MATERIAL AND METHODS

Our study was based on voluntary formative testing of our students' knowledge during weekly lectures and seminars. We were interested in the level of acquiring the theory and the ability to solve various mathematical problems during the pandemic period. To see more deeply into the acquisition level, we also recorded the number of hints the students used during the solving process. We selected the testing problems used in our study according to the following requirements: digital materials are available in the English and Czech languages, they contain step-by-step hints, they enable student’s automatic assessment, and they cover the selected topics of our course of Mathematics. Having explored several educational portals, we chose the open educational resource Khan Academy which meets all the mentioned criteria. Schwartz [30] summarized five key observations about authentic understanding: thanks to the pedagogical experience of the author, Khan Academy is a suitable basis for authentic understanding stabilized with practical examples and problems to solve, offers relevant feedback, it is context-sensitive, and the particular pieces of knowledge are ordered hierarchically.

In the presented study were formulated the following research questions:

- Does the workbook version influence the students' error rate when solving the assigned mathematical problems?
- Does the workbook version influence the number of the used hints the students need to solve the assigned mathematical problems?
- Does the workbook version influence the time the students need to solve the assigned mathematical problems?

The research was conducted during the online teaching period starting in mid of March to the end of May 2020, and 157 students of the Faculty of Economics of the University of South Bohemia in České Budějovice were involved in it; specifically, there were 67 men and 90 women. The students were dealing with 111 mathematics problems in total, forming 27 sets covering four mathematics topics. The sets of problems were taken from the Khan Academy1, including all step-by-step hints, and offered multiple-choice questions with one or more correct answers. The advantages and limitations of MCQ used in mathematics tests are widely discussed, for example, in Torres et al. [36], Sangwin [27] or Sangwin & Köcher [28]. We take advantage that formative tests with MCQ can be assessed automatically using computer systems, such as LMS, and that the consequent statistical analysis can be easily performed. However, we have to admit that our findings may be biased because of the limitations of MCQ, mainly by the fact that MCQ cannot measure some types of learning objectives, and students can guess correct answers.

Students could choose between a printed or an electronic version of the assignment in each of the sets. Those, who decided on the printed version, obtained three sets of materials: assignments of the mathematical problems with multiple-choice questions, corresponding

[1] https://cs.khanacademy.org/math/differential-calculus/dc-limits/dc-limits-intro/e/limits-intro?modal=1

step-by-step hints, and the correct answers. Those, who decided on the digital version, were given a link to a website with all the assignments, hints, and where their answers were automatically assessed. Most of the sets consisted of 4 problems; only one set contained seven problems. The electronic sets were transformed into an electronic workbook and afterward into a printed version of this workbook. The following images illustrate the electronic version of the workbook. The conducted research and subsequent analysis focused on error rate, the score of used hints, and the time necessary for solving particular mathematical problems.

Stoop, Kreutzer & Kircz [34] conducted similar research to authors of this study when they researched the difference in reading and learning from paper-based versus electronic media in a professional and educational setting. The paper-based set consisted of several paragraphs of a book, a dictionary, and a list of questions; the digital version was in the form of consecutive web pages, including test questions and a possibility of translation using a mouse. The group dealing with the digital version gained better results. The authors claim that better performance and quicker work with the text lie in better orientation in electronic texts.

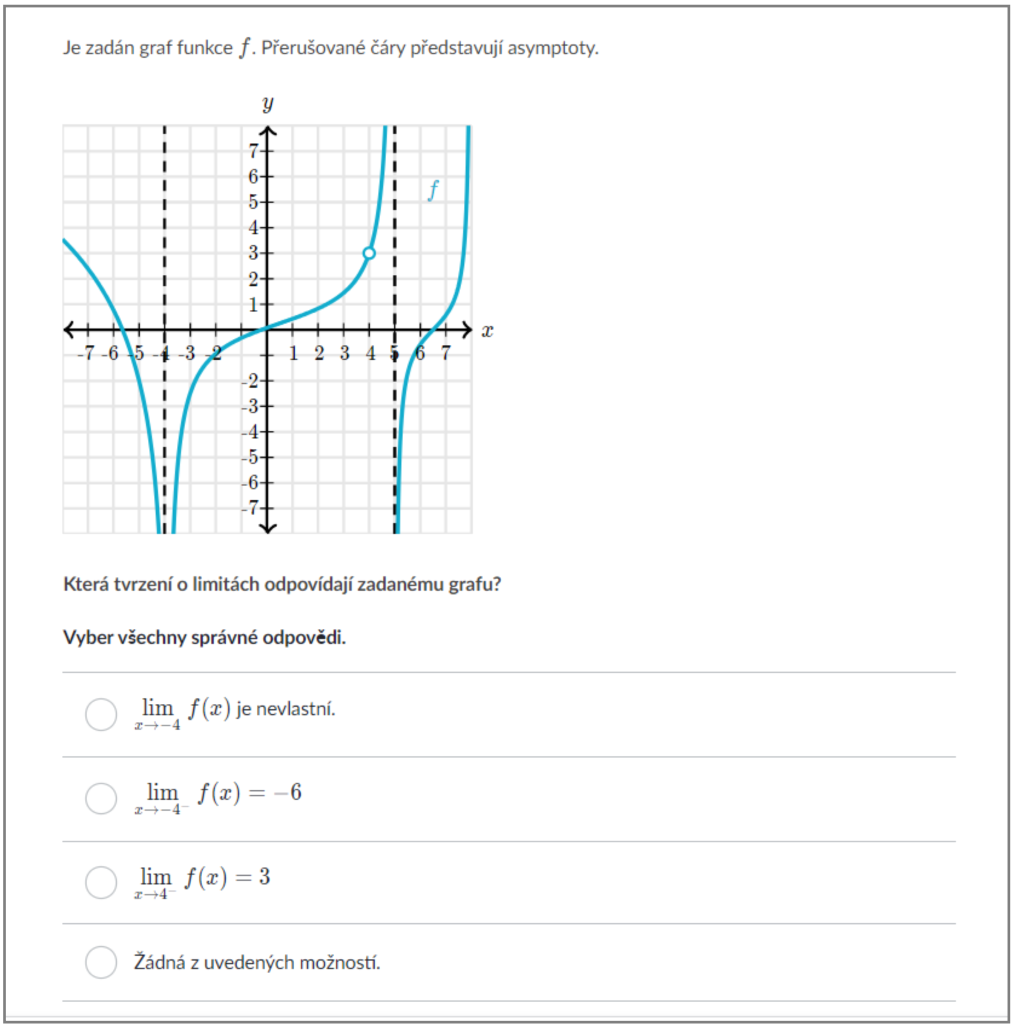

Figure 1. Assignment sample from testing
Source: Khan Academy

{MERAA} Mathematics in Education, Research and Applications ISSN 2453-6881
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However, Mangen, Walgermo & Brønnick [18] claim that scrolling down in digital texts, which should help the readers understand them and orientate in texts, may thwart it, especially when texts are longer than a page. In our study, all problems were assigned within one page, but the students had to use scrolling when they used step-by-step hints.

In Figure 1, you can see a sample of one of the problems. How did the students proceed when trying to solve it? If students knew how to solve it and did not need a hint, they choose one of the offered answers and, in case of the correct answer, they could proceed to another problem. If they worked with the printed version, they had to check their answer with the workbook key. In the case of the electronic workbook, their answers were assessed automatically. If their answer was correct, they were rewarded with a winning notice 'Nice work!'

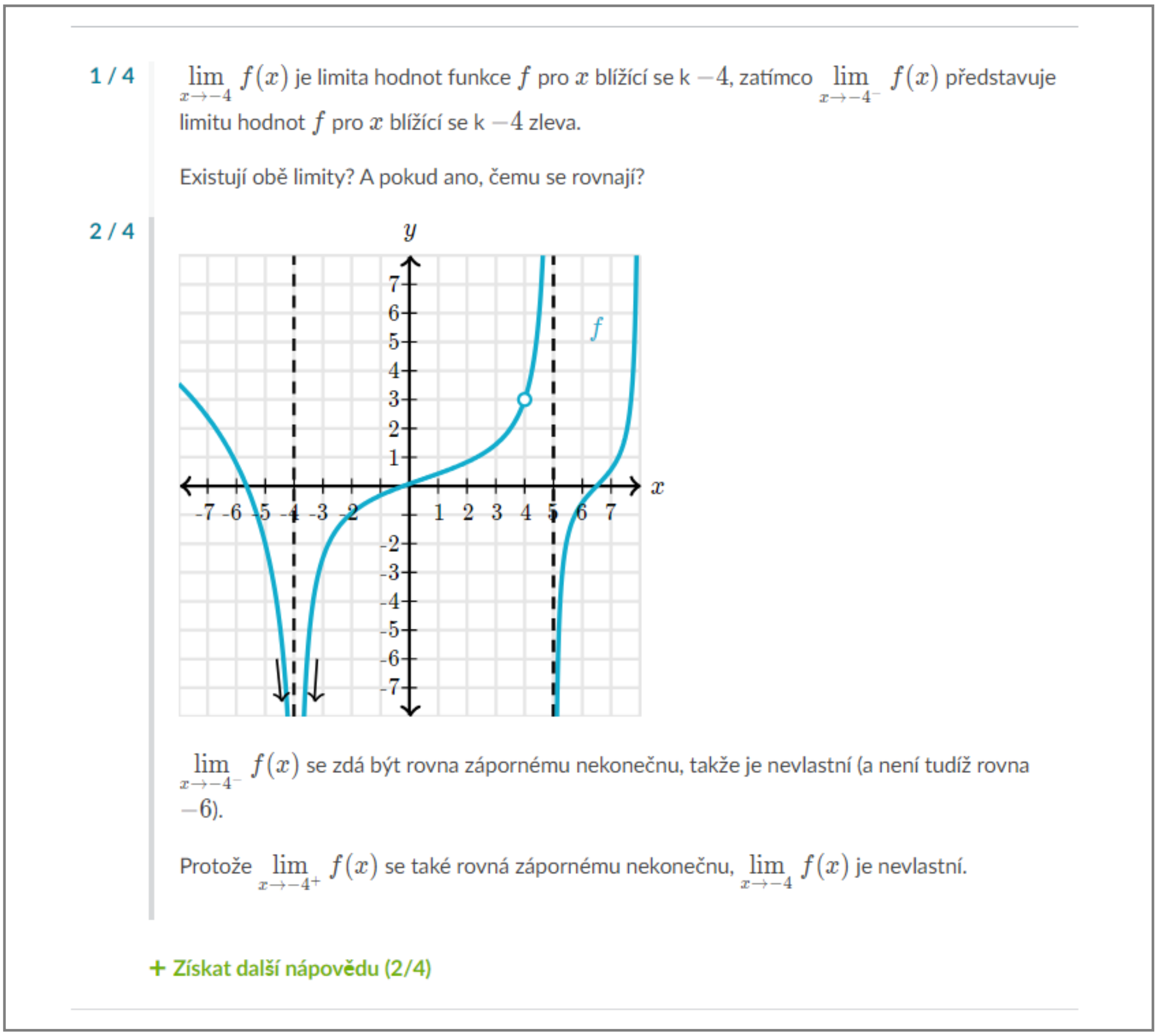

Figure 2. Step-by-step hint sample
Source: Khan Academy

If the students knew how to solve the problem, but their answer was not correct, they calculated the given problem again. Usually, the reason for the wrong answer was the students' inadvertence. In both the printed and the electronic version, the students made records of their errors. In the case of a wrong answer in the electronic version, there was an automatic notice 'Not quite yet ... Try again, Get help, or move on.' If the students did not know how to solve some of the problems, they could use hints. A sample of step-by-step hints is presented in Figure 2.

This particular problem offered four hints in total. The students working with the printed workbook could see the printed hints. In the case of the digital version, the students could ask for a hint by clicking on 'Stuck? Watch a video or use a hint'. In both versions, the students made records of the number of hints for each of the problems. Therefore, we could analyze the 'number of the used hints' for both versions of the workbook afterward.

As each of the problems has a different number of hints, the number of used hints was divided by the total number of the hints to get the indicator called 'score of the used hints', which we used in the further analysis.

All sets of mathematical problems served the students to revise and consolidate their knowledge and check their understanding and skills according to the consolidation and control functions of textbooks used by Zuyev [39] that they had learned from lectures and seminars.

The students knew that their work with the workbook was voluntary and would have no impact on their assessment and final marks. Those working with the printed versions of the workbook were asked to record their progress (the number of used hints, their time necessary to solve the problems) into standardized forms and upload the forms in the LMS Moodle.

All the obtained data were then processed using the random-mixed model (McLean, Sanders & Stroup [20]), a statistical model considering random and fixed effects that are useful when measurements are made repeatedly using the same data and the same date may be missing. The random-mixed model was used to analyze the data obtained from 119 students who worked consistently with only one version of the workbook (50 students worked with the electronic and 69 students with the printed version).

The version of the workbook, denoted in our model as 'VERSION', is a fixed effect. The set of problems, denoted as 'SET', and the students, denoted as 'STUDENT', are random effects. The research question was if the version of the workbook (electronic or printed) had any impact on the error rate, the score of the used hints, and the time necessary for solving particular mathematical problems denoted in the model as 'ERROR', 'PROHINTS', and 'TIME'.

RESULTS AND DISCUSSION

The first of the observed aspects was the error rate indicating how many times the students chose the wrong answer within one particular mathematical problem. Table 1 presents 'Total error rate' indicating the mean of the total error rate made by all students in the given problem. Columns 'Error rate E' and 'Error rate P' indicate the means of the total error rates regarding the electronic and printed version. Columns 'Difference E' and 'Difference P' present differences between the total error rate and the error rates in the electronic and printed versions. If the value in these columns is positive, it means that the error rate in that particular version is higher than the total one and vice versa. Observing Table 1, it is evident that the error rate of the students using the electronic version is higher than in the printed version. In total, this is true for 26 problems out of 27.

Table 1. Mean values of error rates and differences between versions E and P

	Total error rate	Error rate E	Error rate P	Difference E	Difference P
1	0.190	0.314	0.107	0.124	-0.083
2	0.226	0.265	0.200	0.039	-0.026
3	0.509	0.708	0.359	0.199	-0.150
4	0.444	0.563	0.362	0.119	-0.082
5	0.462	0.612	0.353	0.150	-0.109
⋮			⋮		
23	0.521	0.773	0.370	0.252	-0.151
24	0.425	0.465	0.403	0.040	-0.022
25	0.729	0.952	0.605	0.223	-0.124
26	0.600	0.857	0.452	0.257	-0.148
27	0.400	0.558	0.312	0.158	-0.088

Source: Results processed by the authors

The identified differences were also verified statistically. To verify that the version of the workbook influences the error rate, we used the Poisson distribution in the random-mixed model. The random effects were particular sets of problems (‘SET’) and particular students (‘STUDENT’). The fixed effect is the used version of the workbook (‘VERSION’). All this means that each of the 27 sets may have higher and lower error rates, and it also applies to the students. Using the random effect, we give to the shift of the error rate, within the sets of problems and within the students, a random influence, as some sets may be more difficult than the others and some of the students being weaker and some stronger. The random-mixed model enables us to prove the significance via the random shift within the sets of problems and the students. The created model ‘model.glmer1’ is described below. Table 2 presents the variance and the standard deviation concerning a particular student or a particular set of problems. Table 3 shows calculated p-values revealing if the differences in the error rates concerning the version of the workbook are statistically significant, even on a low significance level. The statistical investigation verified that the students working using the electronic version of the workbook had had a statistically higher error rate than the students working with the printed version.

$$> model.glmer1 < -glmer\big(ERROR \sim 1 + VERSION + (1|STUDENT) + (1|SET), +data = M, family = poisson(link = \text{‘log’})\big)$$

$$> summary(model.glmer1)$$

Table 2. Random-mixed model – random effects, error rate

	variance	standard deviation
STUDENT	2.08526	1.444
SET	0.07236	0.269

Source: Results processed by the authors

Table 3. Random-mixed model – fixed effect, error rate

	p-value
VERSION	$< 2 \cdot 10^{-16}$ ***

Source: Results processed by the authors

Another observed aspect was the number of used hints for each of the set of problems, which indicates the percentage of the used hints to the total of all available hints in each of the sets. Therefore, the values are from 0 (no hints used) to 1 (all available hints were used). The column 'ProHints Total' presents the mean values of the total percentage of the hints used in all 27 sets of problems. Columns 'ProHints E' and 'ProHints P' present the mean values of the percentage of the used hints concerning the electronic and the printed versions of the workbook. Again, columns 'Difference E' and 'Difference P' present the differences between the mean values of the total percentage of the used hints and the percentage for each of the workbooks' versions. The positive values represent a higher percentage of the used hints to the total, and the negative values represent the opposite. Regarding all 27 sets of problems, the mean value of the used hints is lower for the electronic version. It means that the students working with the electronic version of the workbook do not use hints as often as their counterparts, and they tried to solve the problems without the available hints.

Table 4. Mean values of percentage for used hints and differences between versions E and P

	ProHints Total	ProHints E	ProHints P	Difference E	Difference P
1	0.050	0.024	0.068	-0.026	0.018
2	0.048	0.023	0.064	-0.025	0.016
3	0.188	0.164	0.204	-0.024	0.016
4	0.221	0.158	0.263	-0.063	0.042
5	0.223	0.126	0.284	-0.097	0.061
⋮			⋮		
23	0.173	0.146	0.187	-0.027	0.014
24	0.133	0.118	0.142	-0.015	0.009
25	0.223	0.134	0.268	-0.089	0.045
26	0.150	0.127	0.162	-0.023	0.012
27	0.126	0.079	0.151	-0.047	0.025

Source: Results processed by the authors

The identified differences were also verified statistically. To verify that the version of the workbook influences the use of the available hints, we used the normal distribution to model the available hints. The random effect was again one of the sets of problems ('SET') and a student ('STUDENT'), the fixed effect, on the other hand, was again the version of the workbook ('VERSION'). The created model 'model.glmer2' is described below. Table 5 presents the variance and the standard deviation regarding the students or the set of problems. As the model 'model.glmer2' does not give us a *p*-value, we created another model 'model.glmer21' (see below) and using F-test, we compared these two models. The calculated *p*-value, presented in Table 6, revealed that the difference between the number of the used hints regarding the electronic and the printed version is statistically significant. The statistical investigation verified that the students working with the electronic version of the workbook had used the available hints less than the students working with the printed version (Table 7).

$$> model.glmer2 < -lmer(PROHINTS \sim 1 + VERSION + (1|STUDENT) + (1|SET), +data = M)$$

$$> model.glmer21 < -lmer(PROHINTS \sim 1 + (1|STUDENT) + (1|SET), +data = M)$$

$$> anova(model.glmer2, model.glmer21)$$

Table 5. Random-mixed model – random effects, used hints

	variance	standard deviation
STUDENT	0.028002	0.16734
SET	0.001962	0.04429

Source: Results processed by the authors

Table 6. Random-mixed model – fixed effect, used hints

	p-value
VERSION	0.0238*

Source: Results processed by the authors

The third observed aspect was the time necessary for solving the mathematical problems measured in seconds. Table 6 presents the mean values of the total time necessary for solving the problems ('Total Time') and the mean values of each version's total time ('Time E' and 'Time P'). Similar to the previous aspect, columns 'Difference E' and 'Difference P' present the differences in times regarding the electronic and printed versions and the total time. The investigation identified the fact that for 23 sets of problems, the students working with the electronic version needed less time than their counterparts.

Table **1.** Mean values of time (in seconds) and differences between versions E and P

	Total Time	Time E	Time P	Difference E	Difference P
1	179	162	190	-16	11
2	173	119	208	-54	35
3	508	491	521	-17	13
4	599	534	645	-66	46
5	519	458	563	-61	44
⋮			⋮		

23	389	392	388	3	-1
24	400	362	422	-38	22
25	397	395	397	-2	0
26	516	485	534	-31	18
27	555	475	600	-80	45

Source: Results processed by the authors

The identified differences were also verified statistically. To verify that the version of the workbook influences the time necessary for solving the set of problems, we used the normal distribution. In the random-mixed model, we again compared a random effect ‘SET’ and ‘STUDENT’ with the fixed effect ‘VERSION’. The models were constructed in the same way as the previous models ‘model.glmer2’ and ‘model.glmer21’, only instead of ‘PROHINTS’ we tested the time ‘TIME’. Table 8 presents the corresponding variance, and the standard deviation and Table 9 presents the calculated *p*-value. The calculated *p*-value shows that the differences are not statistically significant.

Table 2. Random-mixed model – random effects, time

	variance	**standard deviation**
STUDENT	73171	270.5
SET	19802	140.7

Source: Results processed by the authors

Table 3. Random-mixed model – fixed effect, time

	***p*-value**
VERSION	0.1367

Source: Results processed by the authors

It is necessary to mention that the input parameters of particular pieces of research differ. In some studies, students could choose the reading format. In others, students were divided randomly into two groups, so some of them might have worked with the format they disliked. It is also essential to mention the length of the text they worked with. Most of the studies considered narrative texts or longer study texts. This makes a significant difference considering our research when our students worked with short texts, formulae, and also with graphs. Moreover, mathematical texts have, because of their strict logical structure, used symbols, and other specifics, more complicated structure than other texts of scientific character.

The conducted research revealed that the students working with the electronic version of the workbook have a statistically higher error rate than those working with the printed version. This conclusion has also been verified by the random-mixed model and is in accordance with the conclusions made by Lenhard, Schroeders & Lenhard [17]. They revealed that students function more quickly when their knowledge is tested digitally but at the expense of accuracy. This evokes several questions. Do electronic teaching and learning materials distract students’ concentration and attention? Do students read assignments in electronic materials less carefully, which leads to a higher error rate? Unfortunately, we have not found any relevant research comparing electronic and printed mathematics textbooks concerning students' performance. However, there are pieces of research on e-textbooks focused primarily on comprehension and reading speed of individuals accessing text content through a stand-alone computer, even though they are not consistent.

Studies by Green et al. [12] focused on differences in comprehension between numerical data presented in illustrated diagrams and tables and written text on paper or a screen. Green et al. [12] suggest that the presentation of numerical information in graphs and tables shortens students' response time compared to data described in plain text. Sidi et al. [31] minimized the burden of reading in their study, and they tested short demanding logical problems. Their outcomes confirmed a significantly lower success rate of students taking tests on a computer.

Some studies concluded that reading traditional (printed) textbooks comes with better reading comprehension (Mayer, Heiser & Lonn [19]; Dillon [7]). On the other hand, Rockinson-Szapkiw et al. [26] claim that students using the electronic version of a core textbook were more active than their counterparts using the paper textbook. However, their final academic performance did not differ significantly. Some researchers, such as Daniel & Woody [5], Sun, Shien & Huang [35]; Young [37] or Grzeschik et al. [13] revealed in their studies that reading comprehension between paper-based and electronic documents differs only negligibly.

A long-term (2000 – 2017) comparison of a paper-based and electronic reading was conducted by Delgado et al. [6], who found that the advantage of paper-based reading was gradually increasing over the years, mainly regarding informational texts or a combination of informational and narrative texts. The benefit of paper-based reading was also confirmed when the reading time was constrained. Ackerman & Goldsmith [2] claimed that self-paced paper-based and screen-based reading performance differed. The lower performance of the screen-based reading was caused by excessive self-assurance, as the common perception of presentations lowers mobilization of cognitive sources. Also, Singer & Alexander [32] revealed a higher paper-based reading performance regarding questions on particular pieces of information. Regarding questions on the main and key points, the reading medium was not essential.

Another aspect of the presented research was the number of used hints. The mean value of the used hints is statistically lower in the case of the electronic version, which could be caused by the fact that the students cannot reach 100% if they asked for any of the hints. To achieve 100%, the students had to solve the sets repeatedly. This may lead to the conclusion that the students think carefully about solving problems before using any of the hints. Unfortunately, as already mentioned above, we have not found any relevant research dealing with the use of step-by-step hints in solving mathematics.

The last observed aspect was the total time necessary for solving the presented mathematical problems. Although the mean values of time required for solving the sets were higher regarding the printed version in 23 sets out of all 27, the difference was not statistically significant. Findings of some studies dealing with the time used by pupils and students when reading printed and electronic texts are not unequivocal. Initial experimental studies suggested that reading long passages of information took longer when using an electronic format than reading a paper text (Dillon [7]; Mayer, Heiser & Lonn [19]). Dillon [7] found reading from a screen increased the length of time it took to read a text by 20–30%. Mayer, Heiser & Lonn [19] confirmed that readers had faster reading rates for paper text when compared to screen text during 25-minute reading sessions. Daniel & Woody [5] detected higher reading time of students reading newspaper texts at home.

On the other hand, Najjar [22] investigated the efficiency of the teaching process when multimedia is implemented. He lists many pieces of research that confirm that teaching with multimedia shortens the learning process significantly, and he also declares that interactivity has a strong influence on learning; students learn faster and gain better attitudes to learning. However, recent studies indicate that the differences in time between various media are not statistically significant (Eden & Eshet-Alkalai [10]; Young [37]).

Many researchers focus on screen-based reading and comprehension, but their outcomes are not consistent, not reading time and understanding. The authors mostly agree that digital reading is characterized as non-linear when readers skip from one place to another, search for keywords, and select the content. Due to this style, readers do not read in a concentrated and profound way (Durant & Horava [9]; Wolf [38]; Cull [4]).

We may pose an obvious question whether the less time the students spent working with the electronic version does not go hand in hand with the proven higher error rate. Lenhard, Schroeders & Lenhard [17] made the same conclusions as they revealed that students function more quickly when their knowledge is tested digitally but at the expense of accuracy. They also worked with the error rate, which they defined as the ratio of errors to the number of completed items in tests focusing on general comprehension of elementary school pupils. They conclude that the higher error rate is closely connected with the age of pupils when younger pupils make errors more often than their older counterparts. They also mention reasons for the higher error rate in the case of digital materials. The first reason may be playing computer games where speed is very often more important than accuracy. Another reason may be the manipulation with a computer mouse. While marking a correct answer with a pen requires a movement of the whole hand or even an arm, clicking on a mouse requires only a small movement of a finger. This may cause the marking of a wrong answer instead of the intended.

CONCLUSIONS

The presented research seems to be one of the first research pieces focusing on the difference in student’s work with an electronic and printed version of a mathematics workbook regarding the error rate, the number of used hints, and the time needed to solve several sets of mathematical problems. We are fully aware that some aspects may influence our study's outcomes and that it is not possible to generalize the conclusions. As the students worked with the workbook at home without any supervision, we cannot guarantee that they recorded all the requested items. Our findings are based purely on the evidence provided by our students and by the computer system in which the students were working. Therefore, more research in this field will be necessary to conduct to do so. There is a trend to digitize course content to be more accessible to students, so it seems unavoidable to conduct more research on digital resources and differences between the printed and digital textbooks and workbooks in various elementary, secondary, and university subjects and courses.

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{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

Received: 2021-09-07
Accepted: 2021-10-08
Online published: 2021-10-15
DOI: https://doi.org/10.15414/meraa.2021.07.01.50-57

Original Paper

Analysis of consumer behavior zero waste consumers

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ABSTRACT

The article analyzes the consumer behavior of zero waste consumers based on a questionnaire survey, which was attended by 205 respondents of the Slovak Republic. The chi-square test of independence was used for these data, where strength of relationship between qualitative variables was examined, specifically between the place of purchase and the frequency of purchase, the place of purchase and the percentage of cash income spent on the food purchases. In both cases, a dependence was found between the given traits. Subsequently, the differences in the behavior of men and women were examined, while the price and psychological factor was analyzed, namely the discount factor and the environmental impact factor. In the first case, based on calculations in the SAS system using the Mann Whitney U test, we can state the difference in behavior between the sexes, with women being more influenced by the price factor and on the other hand with psychological factor, men being more influenced by the environmental impact than women.

KEYWORDS: consumer behavior, zero waste, questionnaire survey, Chi-square independence test, Mann-Whitney U test

JEL CLASSIFICATION: C14, E20, Q56

INTRODUCTION

Knowing consumer behavior is an essential part of planning a company's marketing activities, which should then contribute, among other things, to profit. The use of a questionnaire survey is generally preferred to examine consumer behavior, and sophisticated methods should be used in the comprehensive analysis and interpretation of the results so that we can consider the conclusions of the survey relevant. It is a fact that the zero waste consumer is a new category among consumers for the trader, despite the fact that the zero waste philosophy is nothing new. That is the mean reason why is very important to know the dependencies

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between the factors that affect the buyer and also correctly identify these factors. Despite the fact that the zero waste philosophy has recently been addressed by more and more authors, the zero waste consumer as such examines only a negligible amount. For example, authors from Spain Calderon-Monge, Pastor-Sanz and Garcia [1] analyzed how consumers incorporate sustainability issues into their buying behavior or authors from Romania analyzed if it is sustainable consumption translated into ethical consumer behavior while Tomsa, Maniu and Scridon [6] found out that the decision-making process of environmentally conscious customers is significantly connected to the intention to engage in politically ethical conduct.

Mauch states that [2] Zero waste is a semi-philosophy of a set of procedures aimed at minimizing waste at the lowest level. The separate concept of zero waste represents the impact of the way we live and manage our resources on the level of environmental pollution. Based on this philosophy, resources should be reused, on the example of how it happens in nature. At the moment, zero waste is considered to be the most cost-effective method by which we as a society can fight climate change or contribute to sustainability. It is for this reason that traders should not only be able to meet the needs of such consumers, but also directly encourage consumers who do not know this philosophy or are not following it to do so. According to Zaman [7] waste is a representation of misallocated resources and a symbol of inefficiency in every modern civilization. As ZWIA states [8]: Everybody has a part to play in leading civilization to a zero-waste future. As everyone produces trash, and if everyone begins to take modest moves in the Zero Waste direction, the process of achieving zero waste will be accelerated. Zero waste should indeed be promoted by all entities (government agencies, non-governmental organizations, companies, and local governments), as well as people. Cities should implement long-term policies and initiatives. Current service providers should adopt zero waste and make every effort to decrease trash, as well as offer a deposit refund service. The research made by Romano G. et al [5] in 2019 shows that municipal policies have an impact on trash strategy implementation. The findings reveal that when municipal trash services are administered by privately held firms, so when the average taxable income of persons per capita is smaller, municipal garbage output is bigger. Ripple et al. states [4] that in last year (2020) cattle numbers have surpassed 4 billion for the first time ever, that is more live weight than the human race and wildlife together. Brazil's deforestation rate went up to a 12-year peak, with the country vowing to end deforestation completely by 2030. The pricing of CO_2 produced by burning fossil fuels is far too low, and it should be numerous times higher if usage need to be reduced. Scientists have hailed the sudden increase of renewables usage, despite the fact that it is still 19 times lower than fossil fuel use. Furthermore, all co2 levels have reached new highs, along with historic highs ocean heat content and, opposite, historically low ocean pH, and also rapid melting of mountain glaciers.

Nowadays we can also calculate the zero waste index which is based on the worth of materials that might be used to replace virgin materials. The zero waste index is a technique for calculating how much virgin material may be offset by zero waste management systems. Zero depletion of natural resources is one of the most essential aims of the zero waste philosophy [5].

MATERIAL AND METHODS

The article used the calculations of the chi-square test of independence, the Mann-Whitney U test, the Zero waste consumer questionnaire survey using the SAS software and Microsoft Excel.

Characteristics of survey respondents
205 zero waste respondents participated in the questionnaire survey, with 80 men and 125 women. 10% of zero waste respondents was up to 18 years old, 22% was 19-25 years old, 39% was 26-35 years old, 21% was 36-45 years old, 5% was 46-65 years old and 2% was upon 65 years old. Most respondents came from the Nitra region and the least respondents came from abroad. The most numerous group of respondents were women aged 26-35 from the Bratislava region.

Chi-square test of independence
When testing associations, we find out whether between the given attributes, there is a dependence, i.e., we examine whether a given occurrence of a certain attribute *X* is likely to assume the occurrence of another attribute *Y*.

The article examines the dependence between attribute *X* - Place of purchase and attribute *Y* - % share of income spent on food purchase, attribute *X* - Place of purchase and attribute *Z* - frequency of purchase [3]. This dependence was examined on the basis of a questionnaire survey among zero waste consumers.

When examining the dependence between the attributes we verify the following hypotheses:
Null hypothesis: Attributes are independent.
Alternative hypothesis: Attributes are dependent.
The test criterion is expressed by formula:

$$X^2 = \sum_{i=1}^{m} \sum_{j=1}^{r} \frac{(E - T)^2}{T}$$

We measured the intensity of the dependence using:
Pearson's coefficient

$$C = \sqrt{\frac{\chi^2}{n + \chi^2}}$$

where:
$\chi 2$ – symbolizes the calculated test criterion,
n – symbolizes total number of respondents.

Cramer V coefficient

$$V = \sqrt{\frac{\chi^2}{n(\min((m, r) - 1)}}$$

where:
m – number of rows, *r* – number of columns,

n – total number of respondents, $\chi 2$ – symbolizes the calculated test criterion.

Mann-Whitney U test
The Mann-Whitney U test is a nonparametric test. It is used for ordinal data when two samples are independent. When analyzing consumer behavior, it can be used, for example, to examine whether a given product is evaluated by respondents (e.g. by gender or age range) in the same way or whether there is a difference in evaluation.

Calculation of test characteristics according to formulas:

For the first sample : $$U = n_1 * n_2 + \frac{n_1 * (n_1 + 1)}{2} - R_1$$

For the second sample: $$U = n_1 * n_2 + \frac{n_2 * (n_2 + 1)}{2} - R_2$$

where:
n_1 – total number of the first sample,
n_2 – total number of the second sample,
R_1 – sum of the order of the first sample,
R_2 – sum of the order of the second sample,
Analyzes were performed at the selected level of significance alpha = 0.05.

RESULTS AND DISCUSSION

In the first part of the presented paper, we paid attention to determining the dependence, respectively independence between the frequency of purchase of zero waste consumers and the place of purchase of zero waste foods. Table 1 presents the input data from the questionnaire survey conducted in 2021.

Table 1. Theoretical abundance

	Place of the purchase					
Frequency of purchase		**Local stores**	**The market/ local farms**	**Zero waste stores**	**Chain stores**	**Total**
	Once per week	11	14	8	29	**62**
	2 - 3 per month	3	11	33	5	**52**
	Daily	2	2	0	5	**9**
	More than once a week	18	10	6	48	**82**
	Total	**34**	**37**	**47**	**87**	**205**

Source: author's processing

The test characteristic reached a value of 77.55, critical value = 16.92. Based on the results, we can state that the frequency of zero waste consumers' food purchases is statistically significantly dependent on the place of zero waste consumers' food purchases. Based on the results of the coefficients measuring the intensity of the dependence, we can confirm a medium-strong dependence (Pearson's coefficient = 0.524, Cramer's V coefficient = 0.435).

In the following part of the article, we again focused on determining the dependence or independence between the place of purchase of food and the percentage of money spent on food by zero waste by consumers. Table 2 presents data from a questionnaire survey in 2021.

Table 2. Theoretical abundance

		Place of purchase				
		Local stores	**The market / local farms**	**Zero waste store**	**Chain store**	**Total**
% share of income spent on food purchase	**10 - 25%**	14	21	36	35	**106**
	25 - 45%	11	12	7	38	**68**
	About 50%	6	3	1	7	**17**
	< 10%	3	0	3	6	**12**
	> 50%	0	1	0	1	**2**
	Total	**34**	**37**	**47**	**87**	**205**

Source: author's processing

The critical value is 21.03 and the calculated test characteristic is 29.42. Based on the results, we can observe that the percentage of money income spent on food purchases is statistically significantly dependent on the place of purchase of food zero waste consumers. Based on the results of comparison criteria, which the intensity of dependence can confirm a weak dependence (Pearson's coefficient = 0.354, Cramer's V coefficient = 0.219). SAS software using Mann-Whitney U test, was used to detect differences in the behavior of zero waste consumers by gender, and we assumed that there was no significant difference in the assessment of the discount factor between men and women.

Based on the results (Figure 1), we can say that our assumption was not correct (test characteristic = -3.32, *P*-value = j 0.0005), which means that men and women behave significantly differently when buying stock, respectively discounted products purchased according to the zero waste philosophy.

From Figure 2, it can be seen that women were more influenced by the price factor when buying food than men.

Subsequently, the impact factor on ecology was examined in the same way, while our assumption was determined similarly, while according to the following outputs we can state that our assumption was again incorrect.

According to the results (Figure 3) (test characteristic = 5.22, and *P*-Value < j 0.0001), we find out that men and women behave significantly differently when buying foods that have a negative / positive impact on the environment.

As can be seen in Figure 4, there are gender differences between the sexes, with men being more affected by environment factors than women.

{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

Wilcoxon Scores (Rank Sums) for Variable Value Classified by Variable Gender					
Gender	N	Sum of Scores	Expected Under H0	Std Dev Under H0	Mean Score
Female	126	14300.0	12978.0	398.209152	113.492063
Male	79	6815.0	8137.0	398.209152	86.265823
Average scores were used for ties.					

Wilcoxon Two-Sample Test	
Statistic	6815.0000
Normal Approximation	
Z	-3.3186
One-Sided Pr < Z	0.0005
Two-Sided Pr > |Z|	0.0009
t Approximation	
One-Sided Pr < Z	0.0005
Two-Sided Pr > |Z|	0.0011
Z includes a continuity correction of 0.5.	

Figure 1. Output from SAS
Source: author's processing

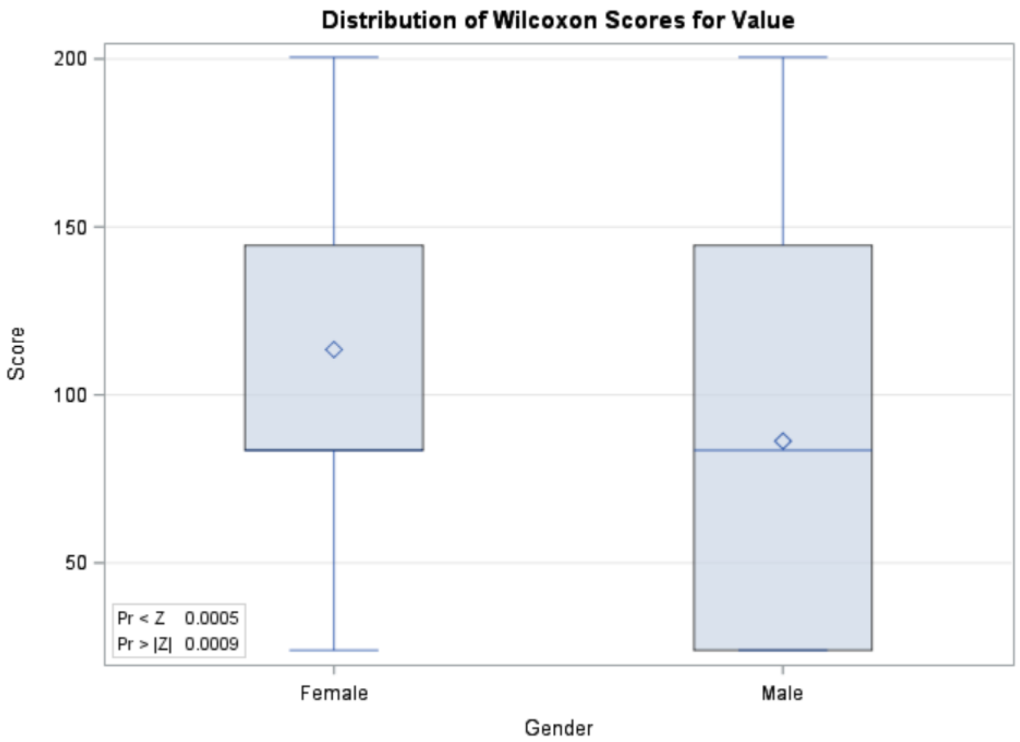

Figure 2. Boxplot by gender (discount factor)
Source: author's processing

{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

The NPAR1WAY Procedure

Wilcoxon Scores (Rank Sums) for Variable Value Classified by Variable Gender					
Gender	N	Sum of Scores	Expected Under H0	Std Dev Under H0	Mean Score
Male	79	10188.50	8137.0	392.752328	128.968354
Female	126	10926.50	12978.0	392.752328	86.718254
Average scores were used for ties.					

Wilcoxon Two-Sample Test	
Statistic	10188.5000
Normal Approximation	
Z	5.2221
One-Sided Pr > Z	<.0001
Two-Sided Pr > |Z|	<.0001
t Approximation	
One-Sided Pr > Z	<.0001
Two-Sided Pr > |Z|	<.0001
Z includes a continuity correction of 0.5.	

Figure 3. Output from SAS
Source: author's processing

Figure 4. Boxplot by gender (impact on environment factor)
Source: author's processing

CONCLUSIONS

Our intention was based on a questionnaire survey to analyze the relationship between the place of purchase and the frequency of buying food zero waste consumers, in the next part we analyzed the relationship between the place of purchase and the percentage of money spent on food purchase. We can argue that there are differences in the assessment of factors that influence respondents when buying food, so that women are more affected by the price factor than men and men are more affected by the impact on the environment than women. The article presented the use of statistical methods in the analysis of consumer behavior of zero waste respondents. Based on the results, the trader can then guide the planning of his marketing activities. Such results could also help, for example, in targeting the advertising of non-packaging products or in the use of emotional appeals to the customer.

ACKNOWLEDGEMENTS

The research presented in the article was supported by the VEGA project: Challenges for ensuring food security in Europe in the 21st century - key factors, socio-economic and environmental contexts, No. VEGA 1/0755/21.

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{MERAA} Mathematics in Education, Research and Applications
ISSN 2453-6881
Math Educ Res Appl, 2021(7), 1

Received: 2021-06-01
Accepted: 2021-10-11
Online published: 2021-10-15
DOI: https://doi.org/10.15414/meraa.2021.07.01.58-62

Original Paper

Logical reasoning in life situations and in mathematics

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ABSTRACT

Having a certain amount of knowledge from the area of Propositional Logic can refine logical reasoning in life situations and in mathematics. Judgment is an important activity where we obtain logical consequences arising from the assumptions made. The article provides examples of correct and incorrect judgments. It draws attention to the mistakes that students make in determining the correctness of their judgments. By using appropriate methods, the mentioned inaccuracies can be eliminated and thus the teaching process will be improved, and the students' level of knowledge increased. We analyzed the knowledge of students at the Slovak University of Agriculture in Nitra in the years 2018 to 2019 from the area of propositional logic. The tasks in the tests were aimed at on the basic knowledge of propositional logic. The main hypothesis was that by introducing statements of logic into teaching we can improve logical reasoning not only in mathematics, but also in everyday life situations. In determining the main hypothesis of our research, we relied on mathematical knowledge related to definitions and sentences and teaching experience. We performed a pedagogical experiment in two different groups. After evaluating the test results, differences in knowledge were found between the two tested groups.

KEYWORDS: logical reasoning, propositional logic, animate situations, tests, teaching experience

JEL CLASSIFICATION: C02, C11, I210

INTRODUCTION

Being familiar with mathematical logic clarifies correct reasoning in both, different life situations and in mathematics. Raclavský [10] says, that when reasoning from the stated assumptions, we obtain the resulting logical consequences. To be able to do that we must know the truth values of the statements. In the end, we assign truth values to other statements. In mathematics, correct judgment is important in determining the properties of a function, sequence, and in solving problems. Gahér [3] says, that students need to be taught to think logically and correctly, the aim is they are able to recognize judgments based on valid

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implications. Mathematical logic has its history. In his work, Vrhovski [11] dealt with the history of mathematical logic. The first examples of mathematical logic began to appear when Bertrand Russell, a famous English philosopher, visited China in 1925. Chapters of books written by Russell penetrated into the surrounding states of China and dealt with the basics of mathematics and mathematical logic. Zhang Shenfu and Zhang Dongsun were prominent philosophers of Chinese liberalism and extended the general concepts of Russell's foundations of mathematical logic. Later, two Chinese mathematicians, Fu Zhongsun and Zhang Bangming, translated Russell's English book into Chinese. According to Hornyák Gregáňová and Országhová [4] a skilled teacher can motivate students to study all educational subjects. Math teachers are a special category because Maths is one of the less popular subjects taught at school and is generally referred to as "a difficult subject". Hornyák Gregáňová, R. et al. [6] deal with the status and importance of mathematics in university education. Dawson [2] also motivates students to think and look for relationships between the mathematical formulas they use in mathematical subjects. Bronkhorst et al. [1] say that more and more topics related to logical reasoning are gradually penetrating the mathematical curriculum. According to the results of our research, we can say that students are able to solve mathematical problems focused on mathematical logic and the use of mathematical symbols.

Gregáňová and Országhová [5] focused on the fact that successful results of mathematical examples solved by students are related to understanding of definitions, sentences, logical arrangement of individual steps in solving problems and their correct use in solving applied tasks. The results of pedagogical research will bring the improvement of mathematical education in selected topics of mathematical analysis. We will improve students' level of knowledge by focusing on topics that have been least mastered. Another way is online education using online materials. Jiang et al. [7] designed a logical framework for thinking in imperfect information games. The possibility of using the language to express rules of an imperfect information game and to formalize the common features of the logical game was shown in the works of Ndungo and Majuma [9].

The aim of our study was to point out abilities of university students to form basic mathematical proofs using mathematical logic and make effective teaching and learning maps. Students' responses were analysed. In teaching, we were exploring the understanding of mathematical proofs using the symbols of mathematical logic, and we also paid attention to the correct procedures. Materials for teaching mathematical logic have been developed by using methods for proving mathematical statements. The two different groups of students have been compared.

MATERIAL AND METHODS

Logical reasoning is not a separate topic in the teaching of mathematics and is not stated in its curriculum in the direct way. However, it is closed linked with propositional logic. Therefore, it is necessary to include tasks for logical reasoning in the teaching of mathematical logic, where we can practice and review principles and rules of logical conjunctions. However, logical reasoning can also be included in other parts of mathematics, e.g., teaching sequences, functions, etc. Our paper presents methodological procedures of solving problems related to logical reasoning. There are two types of examples:

- Examples 1 determines the correctness of a judgment from everyday life situations,

- Example 2 verifies the veracity of a statement about the construction of an angle using a ruler and a compass.

Methodical procedure for solving tasks:

Example 1

Emil's father said, "If Emil scores 100% in all the exams, he will receive a brand-new tablet as a reward." Consider the two situations:
a) When talking about Emil in summer we found that he had received a brand-new tablet.
b) When meeting Emil, his father said that he had scored 100% in all the exams
From which situation (a) and (b) can we conclude that Emil has received a brand-new tablet?

Solution

Let’s write the truth table No.1 with the implication.

a) The logical implication should be true, and we mark the values of 1 are shown in bold. Let's have a look at row number one and three of the table (rows, where the implication and the statement q are true - the assumptions are met) and we will find out that the truth value of the statement p is 1 (first row) or 0 (third row). So, Emil did not necessarily have to score 100% in all the exams to receive a tablet. In fact, he could have received a tablet as a reward for something else (e.g., for working in the garden, excellent behaviour at school, etc.).

b) The implication should be true, and we mark the values of 1 are shown in bold. Let's have a look at the row number one of the tables (the row where the implication and the statement p are true - the assumptions are met) and we will find out that the truth value of the statement q is 1 (first row). So, Emil has received a tablet.

Conclusions. Real situation says that we cannot say whether Emil scored 100% in all the exams.

Table 1 Mathematical logic for solution in example 1

Statement p: „Emil scored 100% in all the exams. “	Statement q: „Emil received a computer. “	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Source: author

From examples a) and b) we can see distinct ways of logical reasoning:
- Example 1 - no conclusion can be drawn,
- Example 2 - it is possible to draw a clear conclusion.

Example 2

Consider the statement: “If we cannot construct an angle $\alpha = 1^\circ$ with a compass and a ruler, then we cannot construct an angle $\beta = 19^\circ$. What can we say about its truthfulness?

Solution

Let's use the property of implication and write an equivalent statement to the given statement: "If we can construct an angle $\beta = 19^\circ$, then we can also construct an angle $\alpha = 1^\circ$.
Let's suppose that we can construct an angle $\beta = 19^\circ$. Then we can also construct an angle $19 \cdot 19^\circ = 361^\circ$, and from that we can easily construct an angle $\alpha = 1^\circ$, because we can construct an angle of 360°.

If we cannot construct an angle $\beta = 19^\circ$, it means that the assumptions are not met and it does not matter whether the conclusion is true or false, the given statement is true.

RESULTS AND DISCUSSION

Logical reasoning was used in the experimental group. In the control group the logical reason was not used.

Before carrying out the research, we set the goal of the research: to compare results in both groups.
Null hypothesis
H_0: The level of knowledge of students in experimental as well as control group is the same.
Alternative hypothesis
H_1: The level of knowledge of students in experimental group has significantly improved owing to the control one.

The choice of methods was subject to the aim and hypothesis of the research. The main methods used in research were directly related to the educational activities. The research included methods that could be used to monitor the set goals. This gave more accurate and objective data: pedagogical experiment, observation, interview, tests (making up, distribution, statistical evaluation), study of professional literature (textbooks, methodological manuals, etc.), study and analysis of students works. Out research was carried out in the summer semester 2018/2019 in two groups: an experimental group and a control group. There were 5 tasks in the test, including the examples given in our paper.

The experimental research was realized out in two different groups during the winter term of the academic year: the experimental group (75 students) logical reasoning method was used in the 1st week of the term and the control group (71 students) in the 5th week of the term. The students of the control group had been studying logical reasoning and were solving problems from propositional logic for 4 weeks. The changes were followed in both groups: experimental and control group.

We assumed that the students of the control and experimental groups formed a random sample from the basic set with a normal distribution. The level of knowledge in each group was determined by using a Two-sample t-test, but before that we had to perform the *F*-test to determine the equality of variances. *T*-test belongs to statistical tests, it is the most commonly used parametric test for testing two mean values [8]. According to the statistical significance, we focus on monitoring the statistical significance of the tested difference of mean values.

Using the *F*-test, we calculated the value of the test $F = 3.125625$, the critical value with a significance level of $\alpha = 0.05$ is equal to 1.7, so $F > 1.7$, so we reject the equality of variances. Since we rejected the equality of variances, for testing we would use a Two-sample

T-Test with inequality of variances. We would test H_0 versus H_1, where H_0 and H_1 were mentioned above. We calculated the t-test value 4.265. The critical value with a significance level α = 0.05 is 2.0. Since the value of the t-test is greater than the critical value, we reject the hypothesis H_0. This means that we accept the hypothesis H_1 and claim that the average level of knowledge is significantly improved in experimental group.

CONCLUSION

The research has shown that using logical reasoning has significantly improved the results of students in the subject of Mathematics.

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