Generalized Lindley-Quasi Xgamma distribution

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Abstract

We obtained a new generalization of Lindley-Quasi Xgamma distribution by adding weight parameter to it through weighting technique and have shown the flexibility of proposed model. Expression for reliability measures, order statistics, Bonferroni curves & indices, Renyi entropy along with some other important properties are derived. Maximum likelihood estimation method is put to use for estimation of unknown parameters of proposed model. Simulation study for checking the performance of maximum likelihood estimates and for model comparison is carried out. Proposed model and its related models are fitted to real life data sets and goodness of fit measure Kolmogorov statistic & p-value, loss of information criteria's AIC, BIC, AICC & HQIC are computed through R software to check the applicability of proposed model in real life. The significance of weight parameter is also tested by using likelihood ratio test for both randomly generated data as well as real life data.

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1. INTRODUCTION

Probability models are and have been generalized for providing more flexibility in terms of hazard rate, reliability, prediction and moments. Because of presence of extra parameter generalized probability models find greater applicability in real life. Probability models can be generalized by using various methods. One of the method employed for generalizing probability models by adding weight parameter is weighting technique. Weighted models are mostly used in those situations where observations are recorded with unequal probabilities which generally happens in encountered sampling. Weighted models find applications in many fields of real life like forestry, medical sciences, socioeconomic surveys etc. Warren [1] applied the size biased distributions in connection with sampling wood cells. Hassan, Dar and Para [2] introduced a new generalization of Ishita distribution and obtained vital mathematical properties of the distribution along with applications of the proposed model. Hassan, Wani and Para [3] formulated three parameter Quasi Lindley

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distribution by using weighting technique and obtained various properties of that model. Das and Roy [4] studied the length biased weighted generalized Rayleigh distribution with properties and applications. Patil and Rao [5] introduced weighted distributions and size biased sampling with applications to wild life populations and human families and obtained its properties. Hassan, Wani and Shafi [6] introduced Poisson Pranav distribution and obtained its various mathematical properties along with obtaining applications of the proposed model. Rezaeia, Nadarajah and Tahghighnia [7] worked on a new three parameter life time distribution and studied its properties & applications. Hassan, Wani, Bilal and Akhtar [8] introduced weighted Quasi Xgamma distribution and studied its properties and applications.

Hassan, Wani, Shafi and Sheikh [9] introduced Lindley-Quasi Xgamma Distribution (LQXD). With p.d.f, c.d.f and c^{th} moment about origin $E(x^c)$ given below in (1.1), (1.2) and (1.3) respectively

$$f(x,\alpha,\theta)=\frac{\theta e^{-\theta x}}{(\alpha+\theta)^2}\left\{(\alpha+\theta)(\alpha+\frac{x^2\theta^2}{2})+\theta(\theta-1)(1+\alpha x)\right\} \qquad x>0,\theta>0,\alpha\ge 0 \tag{1.1}$$

$$F(X)=\frac{1}{2(\theta+\alpha)^2}\left[\begin{array}{l}(\alpha+\theta)\left\{2\alpha+2-\left(2\alpha+x^2\theta^2+2\theta x+2\right)e^{-\theta x}\right\}\\ +2(\theta-1)\left\{\theta+\alpha-\left(\theta+\alpha\theta x+\alpha\right)e^{-\theta x}\right\}\end{array}\right] \qquad x>0,\theta>0,\alpha\ge 0 \tag{1.2}$$

$$E(x^c)=\frac{c!\left\{(\theta+\alpha)\left(\alpha+\frac{(c+1)(c+2)}{2}\right)+(\theta-1)(\theta+\alpha(c+1))\right\}}{\theta^c(\theta+\alpha)^2} \tag{1.3}$$

The important statistical properties along with application in real life were studied for the model given in (1.1).

In this paper we have obtained weighted version of Lindley-Quasi Xgamma (LQXD) distribution with p.d.f given in (1.1).

2. WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Consider X to be a non-negative random variable following Lindley Quasi Xgamma distribution with p.d.f $f(x)$. Suppose $W(x)$ is the non-negative weight function, then the probability density function of the weighted Lindley Quasi Xgamma distribution (WLQXD) is

$$f_w(x)=\frac{W(x)f(x)}{E(w(x))}\ , \qquad x>0$$

Where $w(x)=x^c$ is a non-negative weight function and $f(x), E(x^c)$ are given in (1.1), (1.3) respectively.

$$f_w(x)=\frac{x^c f(x)}{E\left[x^c\right]} \tag{2.1}$$

$$f_w(x)=\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)} \qquad x>0\,,\theta>0,\alpha\geq 0\,,c\geq 0$$

(2.2)

Proof of (2.2) being a probability density function is given below

PROOF: $\displaystyle\int_0^\infty f_w(x)=\int_0^\infty \frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$

$$=\frac{\theta^{c+1}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\int_0^\infty x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}$$

$$=\frac{\theta^{c+1}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\times\frac{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}{\theta^{c+1}}$$

$$=1$$

$$\int_0^\infty f_w(x)=1$$

So (2.2) is a probability density function

where $\int_0^\infty x^{\beta-1} e^{-\alpha x} dx = \frac{\Gamma\beta}{\alpha^\beta}$ is a gamma function

The plots of p.d.f and c.d.f for different values of parameters are given in graphs 1 & 2 below. Graph 1 indicating that proposed model is positively skewed & platykurtic as well as leptokurtic. Graph 2 Indicating curve starts with zero and ends at one and showing c.d.f is non-decreasing function.

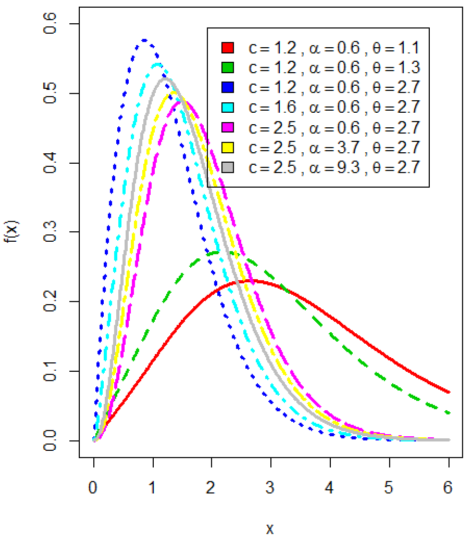

Figure 1 Graph of density function

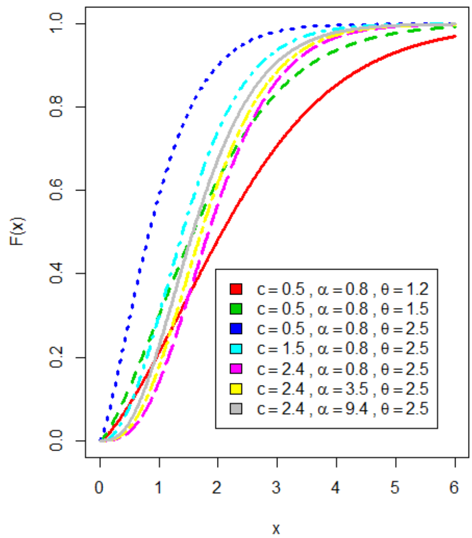

Figure 2 Graph of distribution function

The corresponding c.d.f of Weighted Lindley-Quasi Xgamma distribution is given in (2.3) & obtained as

$$F_w(x) = \int_0^x \frac{\theta^{c+1} x^c \left((\alpha+\theta)\left(2\alpha + x^2\theta^2\right) + 2\theta(\theta-1)(1+\alpha x) \right) e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1))\right)} dx \qquad (2.3)$$

Put $\theta x = t$ in (2.3)

$\theta \, dx = dt$

$as\, x \to 0, t \to 0 \;\; and \;\; x \to x, t \to \theta x$

$$F_w(x)=\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{2.4}$$

where $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}\,dt$ is a lower incomplete gamma function

3. NEED OF PROPOSED MODEL AND MOTIVATION FOR DEVELOPING PROPOSED MODEL

There are many situations in real life where models with less parameters don't perform better. Here in this paper we added an extra parameter known as weight parameter to two parameter Lindley-Quasi Xgamma distribution to obtain a generalized model which will find greater applicability in dealing with complex data. It can be observed from Table 1 that proposed model can be used for over dispersed as well as under dispersed data. Addition of weight parameter also brings the flexibility in terms of moments as can be seen from graphs of probability density function and hazard rate. Complex data applicability and increased flexibility were two important points which motivated us to work on this model.

4. RELIABILITY MEASURES

This division of paper presents survival function, hazard rate, reverse hazard rate of the proposed Weighted Lindley-Quasi Xgamma distribution for random variable X , where X represents the lifetime of a system.

4.1. Reliability Function R(x)

The reliability function or survival function $R(x)$ gives the numerical value of odds of surviving of a system beyond a specified time (t) .

Mathematically

$$R_w(x)=P(X>t)=1-F_w(x)$$

The reliability function or the survival function of Weighted Lindley-Quasi Xgamma distribution is obtained as:

$$R_w(x)=1-\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]$$

4.2. Hazard Rate

The hazard rate which is defined as chance that a system which is surviving up to time *"t"* will fail in the small time interval after *"t"* is obtained for WLQXD as:

$$H.R = h_w(x) = \frac{f_w(x)}{R_w(x)}$$

$$= \left(\frac{\theta^{c+1} x^c \left((\alpha+\theta)\left(2\alpha + x^2\theta^2\right) + 2\theta(\theta-1)(1+\alpha x) \right) e^{-\theta x}}{\left(\begin{array}{l} c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1))\right) - \\ (\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x) + \gamma(c+3,\theta x)\right) + 2(\theta-1)\left(\theta\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right) \end{array} \right)} \right)$$

The graphs of reliability function and hazard rate of WLQXD for different values of parameter are given below

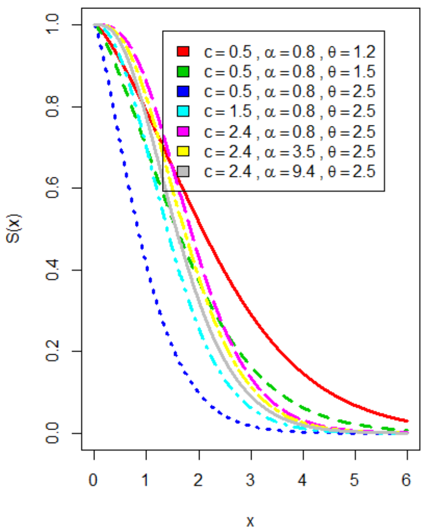

Figure 3 Graph of Reliability function

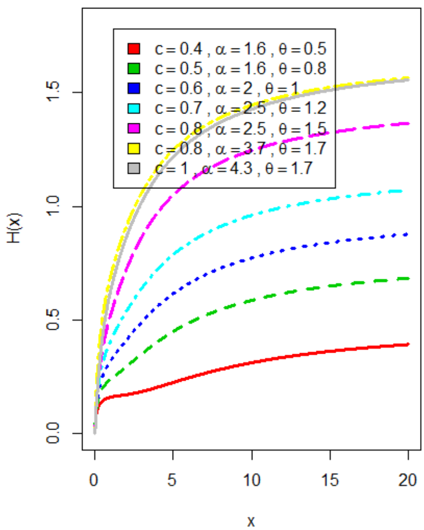

Figure 4 Graph of Hazard rate

From the above graph of hazard rate it is revealed that our proposed model possesses non-decreasing hazard rate and it can be also seen that hazard rate becomes constant as value of x increases. There are many situations in real life where hazard rate is non-decreasing, like lifetime of human beings, animals etc.

4.3. Reverse Hazard Rate

The reverse hazard rate of the Weighted Lindley-Quasi Xgamma distribution is given as:

$$R.H.R = h_{r\,w}(x) = \frac{f_w(x)}{F_w(x)}$$

$$= \frac{\theta^{c+1} x^c \left((\alpha+\theta)\left(2\alpha + x^2\theta^2\right) + 2\theta(\theta-1)(1+\alpha x)\right) e^{-\theta x}}{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x) + \gamma(c+3,\theta x)\right) + 2(\theta-1)\left(\theta\gamma(c+1,\theta x) + \alpha\gamma(c+2,\theta x)\right)}$$

5. STATISTICAL PROPERTIES

Moments, characteristic function, mean deviation, harmonic mean, coefficient of variation characterize probability models. Here we have obtained these statistical properties for proposed Weighted Lindley-Quasi Xgamma distribution.

5.1. Moments and Related Measures

Assuming X to be a random variable having Weighted Lindley-Quasi Xgamma distribution with parameters θ, c and α .Then the r^{th} moment about origin for a given probability distribution is given by

$$\mu_r{}' = E\left(X^r\right) = \int_0^\infty x^r f_w(x) dx \qquad r=1, 2, 3\ldots$$

$$= \int_0^\infty x^r \frac{\theta^{c+1} x^c \left((\alpha+\theta)\left(2\alpha + x^2\theta^2\right) + 2\theta(\theta-1)(1+\alpha x)\right) e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1))\right)} dx$$

$$\mu_r{}' = \left[\frac{(r+c)!\left((\alpha+\theta)(2\alpha+(r+c+1)(r+c+2)) + 2(\theta-1)(\theta+\alpha(r+c+1))\right)}{\theta^r\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1))\right)\right)}\right] \tag{5.1.1}$$

Put r=1 in equation (5.1.1) we get

$$\mu_1{}' = \left[\frac{(c+1)\left((\alpha+\theta)(2\alpha+(c+2)(c+3)) + 2(\theta-1)(\theta+\alpha(c+2))\right)}{\theta\left(((\theta+\alpha)(2\alpha+(c+1)(c+2)) + 2(\theta-1)(\theta+\alpha(c+1)))\right)}\right]$$

Which is mean of the WLQXD.

Put r=2 in equation (5.1.1) we get

$$\mu_2{}' = \left[\frac{(c+1)(c+2)\big((\alpha+\theta)(2\alpha+(c+4)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+3)\big)\big)}{\theta^2\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]$$

The variance μ_2 of WLQXD is

$$\mu_2 = \left(\left[\frac{(c+1)(c+2)\big((\alpha+\theta)(2\alpha+(c+4)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+3)\big)\big)}{\theta^2\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right] - \left[\frac{(c+1)\big((\alpha+\theta)(2\alpha+(c+2)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+2)\big)\big)}{\theta\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]^2\right)$$

The coefficient of variation (C.V) of WLQXD is

$$C.V = \frac{(\mu_2)^{\frac{1}{2}}}{\mu_1{}'}$$

$$= \frac{\left(\left[\frac{(c+1)(c+2)\big((\alpha+\theta)(2\alpha+(c+4)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+3)\big)\big)}{\theta^2\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right] - \left[\frac{(c+1)\big((\alpha+\theta)(2\alpha+(c+2)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+2)\big)\big)}{\theta\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]^2\right)^{\frac{1}{2}}}{\left[\frac{(c+1)\big((\alpha+\theta)(2\alpha+(c+2)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+2)\big)\big)}{\theta\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]}$$

The index of dispersion (γ) of WLQXD is

$$\gamma = \frac{(\mu_2)}{\mu_1{}'}$$

$$= \frac{\left(\left[\frac{(c+1)(c+2)\big((\alpha+\theta)(2\alpha+(c+4)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+3)\big)\big)}{\theta^2\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right] - \left[\frac{(c+1)\big((\alpha+\theta)(2\alpha+(c+2)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+2)\big)\big)}{\theta\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]^2\right)}{\left[\frac{(c+1)\big((\alpha+\theta)(2\alpha+(c+2)(c+3)\big)+2(\theta-1)\big(\theta+\alpha(c+2)\big)\big)}{\theta\big(((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))\big)}\right]}$$

Table 1: Description of WLQXD for different parameter values.

α	θ	c	Mean	Variance	Index of dispersion	Coefficient of variation
0.5	0.5	0.3	6.353521	14.5477	2.289707	0.6003198
2.5	0.5	0.5	4.704545	14.00362	2.976614	0.7954308
2.5	5.5	1.5	0.6199305	0.1368836	0.2208048	0.5968055
0.5	5.5	2.0	0.7485493	0.1655399	0.2211476	0.5435393

It can be observed from Table 1 that WLQXD is over dispersed as well as under dispersed for different parameter values.

5.2. Harmonic Mean of Weighted Lindley-Quasi Xgamma Distribution

The harmonic mean H of the WLQXD is determined as

$$\frac{1}{H}=E\left[\frac{1}{X}\right]=\int_0^\infty \frac{1}{x} f_w(x)dx$$

$$=\int_0^\infty \frac{1}{x}\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}dx$$

$$H=\frac{c\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}{\theta\left((\alpha+\theta)(2\alpha+c(c+1))+2(\theta-1)(\theta+\alpha c)\right)}$$

5.3. Mean Deviation about Mean and Median of WLQXD

We have derived the expressions for mean deviation about mean and median of WLQXD in this section.

THEOREM 1. If X has the WLQXD (θ,α,c), then the mean deviation about mean $(\delta_1(X))$ and mean deviation about median $(\delta_2(X))$ are given as:

$$\delta_1(X)=\left[\begin{array}{l} 2\mu\left\{\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta\mu)+\gamma(c+3,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta\mu)+\alpha\gamma(c+2,\theta\mu)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\} \\ -2\left\{\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\}\end{array}\right]$$

And

$$\delta_2(X)=\left[\mu-2\left\{\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\}\right]$$

respectively.

PROOF. Mean deviation about mean and mean deviation about median are defined as

$$\delta_1(X)=\int_0^\infty |x-\mu|\, f_w(x)dx$$

And $\delta_2(X)=\int_0^\infty |x-M|\, f_w(x)dx$

respectively.

Where μ and *M* are mean and median respectively of random variable *X~WLQXD.* The measures $\delta_1(X)$ and $\delta_2(X)$ can be obtained by using the simplified relationships.

$$\delta_1(X)=\int_0^\mu (\mu-x)\, f_w(x)dx+\int_\mu^\infty (x-\mu)\, f_w(x)dx$$

$$\delta_1(X)=2\mu F_w(\mu)-2\int_0^\mu x f_w(x)dx \tag{5.3.1}$$

And

$$\delta_2(X)=\int_0^M (M-x)\, f_w(x)dx+\int_M^\infty (x-M)\, f_w(x)dx$$

$$\delta_2(X)=\mu-2\int_0^M x f_w(x)dx \tag{5.3.2}$$

Where $f_w(x)=\dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$

Now

$$\int_0^{\mu} x f_w(x)dx = \left[\frac{(\alpha+\theta)\big(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\big)+2(\theta-1)\big(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\big)}{\theta c!\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)}\right] \tag{5.3.3}$$

And

$$\int_0^{M} x f_w(x)dx = \left[\frac{(\alpha+\theta)\big(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\big)+2(\theta-1)\big(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\big)}{\theta c!\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)}\right] \tag{5.3.4}$$

Using expressions (5.3.1), (5.3.2), (5.3.3) and (5.3.4) and expression for c.d.f (2.4) we obtain mean deviation about mean $(\delta_1(X))$ and mean deviation about median $(\delta_2(X))$

$$\delta_1(X) = \left[\begin{array}{l} 2\mu\left\{\dfrac{(\alpha+\theta)\big(2\alpha\gamma(c+1,\theta\mu)+\gamma(c+3,\theta\mu)\big)+2(\theta-1)\big(\theta\gamma(c+1,\theta\mu)+\alpha\gamma(c+2,\theta\mu)\big)}{c!\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)}\right\} \\ -2\left\{\dfrac{(\alpha+\theta)\big(2\alpha\gamma(c+2,\theta\mu)+\gamma(c+4,\theta\mu)\big)+2(\theta-1)\big(\theta\gamma(c+2,\theta\mu)+\alpha\gamma(c+3,\theta\mu)\big)}{\theta c!\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)}\right\}\end{array}\right]$$

&

$$\delta_2(X) = \left[\mu - 2\left\{\frac{(\alpha+\theta)\big(2\alpha\gamma(c+2,\theta M)+\gamma(c+4,\theta M)\big)+2(\theta-1)\big(\theta\gamma(c+2,\theta M)+\alpha\gamma(c+3,\theta M)\big)}{\theta c!\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)}\right\}\right]$$

5.4. Generating functions of Weighted Lindley-Quasi Xgamma Distribution

We will derive moment generating function and characteristic function of WLQXD in this segment of paper.

THEOREM 2. If $X \sim WLQXD(\theta,\alpha,c)$ then the moment generating function $M_X(t)$ and characteristic generating function $\phi_X(t)$ are

$$M_X(t) = \left(\begin{array}{l}\dfrac{\theta^{c+1}}{\big((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\big)} \\ \left((\alpha+\theta)\left(\dfrac{2\alpha}{(\theta-t)^{c+1}}+\dfrac{\theta^2(c+1)(c+2)}{(\theta-t)^{c+3}}\right)+2\theta(\theta-1)\left(\dfrac{1}{(\theta-t)^{c+1}}+\dfrac{(c+1)\alpha}{(\theta-t)^{c+2}}\right)\right)\end{array}\right)$$

And

$$\phi_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-it)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-it)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-it)^{c+1}}+\frac{(c+1)\alpha}{(\theta-it)^{c+2}}\right)\right)\right)$$

respectively.

PROOF. We begin with the well-known definition of the moment generating function given by

$$M_X(t)=E\left(e^{tX}\right)=\int_0^\infty e^{tx}\, f_w(x)\,dx$$

$$=\int_0^\infty e^{tx}\,\frac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\,dx$$

$$M_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-t)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-t)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-t)^{c+1}}+\frac{(c+1)\alpha}{(\theta-t)^{c+2}}\right)\right)\right) \quad (5.4.1)$$

Which is the m.g.f of Weighted Lindley-Quasi Xgamma distribution.

Also we know that $\phi_X(t)=M_X(it)$

Therefore,

$$\phi_X(t)=\left(\frac{\theta^{c+1}}{((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1)))}\left((\alpha+\theta)\left(\frac{2\alpha}{(\theta-it)^{c+1}}+\frac{\theta^2(c+1)(c+2)}{(\theta-it)^{c+3}}\right)+2\theta(\theta-1)\left(\frac{1}{(\theta-it)^{c+1}}+\frac{(c+1)\alpha}{(\theta-it)^{c+2}}\right)\right)\right) \quad (5.4.2)$$

Which is the characteristic function of WLQXD distribution.

6. ORDER STATISTICS OF WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Consider $X_{(1)}, X_{(2)}, X_{(3)}...., X_{(n)}$ to be the ordered statistics of the random sample $x_1, x_2, x_3,....x_n$ obtained from the Weighted Lindley-Quasi Xgamma distribution with cumulative distribution function $F_w(x)$ and probability density function $f_w(x)$, then the probability density function of v^{th} order statistics $X_{(v)}$ is given by:

$$f_{vw}(x) = \frac{n!}{(v-1)!(n-v)!} f_w(x)[F_w(x)]^{v-1}[1 - F_w(x)]^{n-v} \qquad v=1, 2, 3\ldots n$$

Using the equations (2.2) and (2.3), the probability density function of v^{th} order statistics of Weighted Lindley-Quasi Xgamma distribution is given by:

$$f_{(v)w}(x) = \left[\begin{array}{l} \dfrac{n!}{(v-1)!(n-v)!}\left\{\dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\} \\ \left[\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^{v-1} \\ \left[1-\left(\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)\right]^{n-v}. \end{array}\right]$$

Then, the p.d.f of first order statistic $X_{(1)}$ of Weighted Lindley-Quasi Xgamma distribution is given by:

$$f_{(1)w}(x) = \left[\begin{array}{l} n\left\{\dfrac{\theta^{c+1}x^c\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\} \\ \left[1-\left(\dfrac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)\right]^{n-1}. \end{array}\right]$$

and the pdf of n^{th} order statistic $X_{(n)}$ of Weighted Lindley-Quasi Xgamma distribution is given as:

$$f_{(n)w}(x)=\left[n\left\{\frac{\theta^{c+1}x^{c}\left((\alpha+\theta)\left(2\alpha+x^{2}\theta^{2}\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right\}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+1,\theta x)+\gamma(c+3,\theta x)\right)+2(\theta-1)\left(\theta\gamma(c+1,\theta x)+\alpha\gamma(c+2,\theta x)\right)}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right]^{n-1}\right].$$

7. BONFERRONI AND LORENZ CURVES AND INDICES OF WLQXD

The Bonferroni curve $(B(p))$, Lorenz curve $(L(p))$, Bonferroni index (B) and Gini index (G) have find applicability in fields of economics, demography, reliability, life testing and medical sciences. The Bonferroni and Lorenz curves are defined as

$$B(p)=\frac{1}{p\mu}\int_{0}^{q}xf_{w}(x)dx \tag{7.1}$$

$$L(p)=\frac{1}{\mu}\int_{0}^{q}xf_{w}(x)dx \tag{7.2}$$

Where $\mu=E(x)$ is the mean of WLQXD and $q=F^{-1}(p)$.

The Bonferroni and Gini indices are defined as

$$B=1-\int_{0}^{1}B(p)dp \tag{7.3}$$

$$G=1-2\int_{0}^{1}L(p)dp \tag{7.4}$$

Using the p.d.f (2.2) of WLQXD we get

$$\int_{0}^{q}xf_{w}(x)dx=\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.5}$$

Using equation (7.5) in (7.1) & (7.2) we get

$$B(p)=\frac{1}{p\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.6}$$

And

$$L(p)=\frac{1}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.7}$$

Using (7.6) & (7.7) in (7.3) & (7.4) we get

$$B=1-\frac{1}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.8}$$

$$L=1-\frac{2}{\mu}\left[\frac{(\alpha+\theta)\left(2\alpha\gamma(c+2,\theta q)+\gamma(c+4,\theta q)\right)+2(\theta-1)\left(\theta\gamma(c+2,\theta q)+\alpha\gamma(c+3,\theta q)\right)}{\theta c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right] \tag{7.9}$$

8. RENYI ENTROPY

The information associated with various values of random variable following a particular distribution is called entropy. Renyi entropy $T_R(\delta)$ of random variable X following Weighted Lindley-Quasi Xgamma distribution is obtained as

$$T_R(\gamma)=\frac{1}{1-\delta}\log\left(\int_0^\infty f_w{}^{\delta}(x)dx\right)$$

Where $\delta>0$ and $\delta\neq1$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^\infty\left(\frac{\theta^{c+1}x^c\left(\left(\alpha+\theta\right)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)^{\delta}dx\right)$$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^\infty\frac{\theta^{\delta(c+1)}x^{\delta c}\left(\left(\alpha+\theta\right)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)^{\delta}e^{-\delta\theta x}}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}}dx\right) \tag{8.1}$$

Using the binomial expansion

$$\left((\alpha+\theta)\left(2\alpha+x^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x)\right)^{\delta}$$

$$=\sum_{k=0}^{\delta}\binom{\delta}{k}2^k\theta^k(\theta-1)^k(1+\alpha x)^k(\alpha+\theta)^{\delta-k}\left(2\alpha+x^2\theta^2\right)^{\delta-k}$$

$$=\sum_{k=0}^{\delta}\binom{\delta}{k}2^k\theta^k(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\alpha^p x^p\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}x^{2l}\theta^{2l}2^{\delta-k-l}\alpha^{\delta-k-l} \tag{8.2}$$

Using (8.2) in (8.1) we get

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\left(\sum_{k=0}^{\delta}\binom{\delta}{k}(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}\theta^{c\delta+\delta+k+2l}2^{\delta-l}\alpha^{p+\delta-k-l}\int_0^{\infty}x^{c\delta+p+2l}e^{-\delta\theta x}dx\right)}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}}\right\}$$

$$T_R(\delta)=\frac{1}{1-\delta}\log\left\{\frac{\left(\sum_{k=0}^{\delta}\binom{\delta}{k}(\theta-1)^k(\alpha+\theta)^{\delta-k}\sum_{p=0}^{k}\binom{k}{p}\sum_{l=0}^{\delta-k}\binom{\delta-k}{l}\theta^{\delta+k-p-1}2^{\delta-l}\alpha^{p+\delta-k-l}(c\delta+p+2l)!\right)}{\left(c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\right)^{\delta}\delta^{c\delta+p+2l+1}}\right\}$$

9. ESTIMATION OF PARAMETERS OF WEIGHTED LINDLEY-QUASI-XGAMMA DISTRIBUTION

Parameters are estimated by using method of maximum likelihood estimation. Considering $x_1, x_2,...x_n$ to be the random sample of size n drawn from Weighted Lindley-Quasi Xgamma distribution having density function given by (2.2), then the likelihood function of WLQXD is given as:

$$L(x\mid\theta,\alpha,c)=\prod_{i=1}^{n}\left(\frac{\theta^{c+1}x_i^{\,c}\left((\alpha+\theta)\left(2\alpha+x_i^{\,2}\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)e^{-\theta x_i}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}\right)$$

Taking log on both sides of likelihood function we get log likelihood function as:

$$\log L=\begin{pmatrix} n(c+1)\log\theta+c\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left((\alpha+\theta)\left(2\alpha+x_i{}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)\\ n\log c!-n\log\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)\end{pmatrix} \tag{9.1}$$

Now partially differentiate (9.1) with respect to θ,α,c and equating the result to zero, we obtain the following normal equations,

$$\frac{\partial\log L}{\partial\theta}=\begin{bmatrix}\frac{(c+1)n}{\theta}-\frac{n(2\alpha+(c+1)(c+2)+4\theta+2\alpha(c+1)-2}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}+\\ \sum_{i=1}^{n}\left(\frac{2\alpha\theta x_i{}^2+2\alpha+3x_i{}^2\theta^2+2(1+\alpha x_i)(2\theta-1)}{\left((\alpha+\theta)\left(2\alpha+x_i{}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)}\right)-\sum_{i=1}^{n}x_i\end{bmatrix}=0 \tag{9.2}$$

$$\frac{\partial\log L}{\partial\alpha}=\begin{bmatrix}-\frac{n(2\theta+(c+1)(c+2)+4\alpha+2\theta(c+1)-2(c+1)}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}+\\ \sum_{i=1}^{n}\left(\frac{4\alpha+x_i{}^2\theta^2+2\theta+2\theta(\theta-1)x_i}{\left((\alpha+\theta)\left(2\alpha+x_i{}^2\theta^2\right)+2\theta(\theta-1)(1+\alpha x_i)\right)}\right)\end{bmatrix}=0 \tag{9.3}$$

$$\frac{\partial\log L}{\partial c}=\left[n\log\theta-\frac{n\left((\theta+\alpha)(2c+3)+(2\alpha\theta-2\alpha)\right)}{\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}-n\left(\log(c+1)-\frac{1}{2(c+1)}\right)+\sum_{i=1}^{n}\log x_i\right]=0 \tag{9.4}$$

MLEs of θ,α,c cannot be obtained by solving above complex equations as these equations are not in closed form. So we solve above equations by using iteration method through R software.

10. SIMULATION STUDY

In this part of paper we have carried out the simulation study for checking the performance of maximum likelihood (ML) estimates by taking different sample sizes (n=30, 50, 70, 110). We have used the inverse CDF technique for data simulation for WLQXD using R software. The process was repeated 1000 times for calculation of bias, variance & mean square error (MSE) as are given values in Table 2. For two parameter combinations of WLQXD, decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence, the performance of ML estimators is quite good and consistent in case of Weighted Lindley-Quasi Xgamma Distribution.

Table 2: Simulation Study of ML estimators for WLQXD.

Parameter	n (sample size)	$\alpha = 0.8, \theta = 0.7, c = 0.8$			$\alpha = 0.5, \theta = 1.7, c = 0.6$		
		Bias	Variance	MSE	Bias	Variance	MSE
α	30	0.4862598	0.4406534	0.677102	0.40865	0.540489	0.7074845
θ		1.625651	2.056811	4.699553	0.6685304	1.925161	2.372094
c		0.4383506	0.5523539	0.7445051	0.5508202	0.5480579	0.8514608
α	50	0.478908	0.438007	0.6673606	0.36540	0.454492	0.5880101
θ		1.434994	0.9519631	3.011171	0.4003302	1.001894	1.162159
c		0.3290241	0.2367339	0.3449908	0.4239412	0.2483344	0.4280606
α	70	0.467299	0.4364103	0.6547787	0.30907	0.447506	0.5430311
θ		1.261992	0.6351194	2.227743	0.338809	0.6839647	0.7987562
c		0.2534664	0.1573207	0.2215659	0.3918621	0.1738424	0.3273983
α	110	0.459871	0.4333168	0.6447982	0.26780	0.398899	0.4706166
θ		1.17978	0.3789148	1.770795	0.2331814	0.4179511	0.4723247
c		0.2156018	0.09346848	0.1399526	0.3397029	0.1026403	0.2180383

11. TESTING SIGNIFICANCE OF WEIGHT PARAMETER ON BASIS OF SIMULATED DATA FROM WLQXD

For comparing proposed model with base model and for testing the significance of weighted parameter we generated random samples of sizes (50, 100, 200, 500) from WLQXD using inverse CDF technique. It is evident from the Table 3 that weighted parameter plays a highly significant role for large samples. Even though in small samples, the AIC, AICC, BIC and Negative Log likelihood values are also minimum in case of Weighted model as compared to base model. LR statistic for testing H_0 versus H_1 is $\psi = 2(L(\hat{\Theta}) - L(\hat{\Theta}_0))$, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are the MLEs under H_1 and H_0. The statistic ψ is asymptotically $(as\, n \to \infty)$ distributed as χ_k^2, with k degrees of freedom which is equal to the difference in dimensionality of $\hat{\Theta}$ and $\hat{\Theta}_0$. H_0 will be rejected if the LR-test p-value is <0.01 (or LR Statistic value >3.841) at 95% confidence level

Table 3: Model Comparison Based On Simulated Data from WLQXD.

$\hat{c} = 0.6, \hat{\alpha} = 0.9, \hat{\theta} = 1.5$				Parameter Estimates		Likelihood Ratio Statistic
Criterion	WLQXD	LQXD	Sample Size (n)	WLQXD	LQXD	
-logL	13.02580	17.33376	50	$\hat{c} = 0.7756921$ (0.3682644) $\hat{\alpha} = 0.7541315$ (3.0450987) $\hat{\theta} = 4.8614277$ (0.9328238)	$\hat{\alpha} = 1.317147$ (1.466524) $\hat{\theta} = 3.107664$ (0.352400)	8.615
AIC	32.05161	38.66752				
AICC	24.05161	32.66752				
BIC	37.78767	42.49157				
-logL	19.48443	31.00223	100	$\hat{c} = 0.9303426$ (0.2523067) $\hat{\alpha} = 1.9131216$ (3.0526314) $\hat{\theta} = 5.3480262$ (0.6873024)	$\hat{\alpha} = 1.6114539$ (1.1262244) $\hat{\theta} = 3.1965086$ (0.2635093)	23.035
AIC	44.96885	66.00447				
AICC	36.96885	60.00447				
BIC	52.78436	71.21481				

-logL	33.61176	61.80406	200	$\hat{c}=1.1748909$ (0.2734827). $\hat{\alpha}=0.4499443$ (1.9152323) $\hat{\theta}=5.9318359$ (0.6560791)	$\hat{\alpha}=1.5711807$ (0.7844924) $\hat{\theta}=3.2011676$ (0.1854941)	56.384
AIC	73.22353	127.60812				
AICC	65.22353	121.6081				
BIC	83.11848	134.2048				
-logL	106.5939	134.1149	500	$\hat{c}=0.54764349$ (0.08917304) $\hat{\alpha}=1.50125869$ (1.05332253) $\hat{\theta}=4.64187400$ (0.26435388)	$\hat{\alpha}=1.5740119$ (0.5568915) $\hat{\theta}=3.2970302$ (0.1221850)	55.042
AIC	219.1878	272.2297				
AICC	211.1878	266.2297				
BIC	231.8316	280.6589				

12. APPLICATIONS OF WEIGHTED LINDLEY-QUASI XGAMMA DISTRIBUTION

Proposed model, its base model and some other lifetime models are fitted to two real life data sets for testing how fine proposed model fits to real life data sets as compared to other mentioned models.

DATA SET 1. The data set given in Table 4 represents the tensile strength measures in GPa of 69 carbon fibres tested under tension at gauge lengths of 20mm and has been taken from Bader and Priest [10].

Table 4: Tensile strength measures in GPa of 69 carbon fibres.

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865
1.944	1.958	1.966	1.997	2.006	2.021	2.027	2.055
2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426
2.434	2.435	2.478	2.490	2.511	2.514	2.535	2.554
2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821
2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096
3.128	3.233	3.433	3.585	3.585			

DATA SET 2. This data set given in Table 5 is due to Smith and Naylor [11] consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory.

Table 5: The strengths of 1.5 cm glass fibres of 63 observations.

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11
1.13	1.24	1.25	1.27	1.28	1.29	1.30	1.36
1.39	1.42	1.48	1.48	1.49	1.49	1.50	1.50
1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59
1.60	1.61	1.61	1.61	1.61	1.62	1.62	1.63
1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69
1.70	1.70	1.73	1.76	1.76	1.77	1.78	1.81
1.82	1.84	1.84	1.89	2.00	2.01	2.24	

For the analysis of both the data sets we used R Software version 3.5.3.

Table 6: Summary Statistic of data set 1 and 2.

Data set	Number of observations	Min.	Mean	First Quartile	Median	Third Quartile	Max.
1	69	1.312	2.451	2.098	2.478	2.773	3.585
2	63	0.550	1.507	1.375	1.590	1.685	2.240

Table 7: Model function of fitted models.

Distribution	Model function
Weighted Lindley Quasi Xgamma Distribution (WLQXD)	$\dfrac{\theta^{c+1}x^{c}\left((\alpha+\theta)\left(2\alpha+x^{2}\theta^{2}\right)+2\theta(\theta-1)(1+\alpha x)\right)e^{-\theta x}}{c!\left((\theta+\alpha)(2\alpha+(c+1)(c+2))+2(\theta-1)(\theta+\alpha(c+1))\right)}$
Lindley Quasi Xgamma Distribution (LQXD)	$\dfrac{\theta e^{-\theta x}}{(\alpha+\theta)^{2}}\left\{(\alpha+\theta)(\alpha+\dfrac{x^{2}\theta^{2}}{2})+\theta(\theta-1)(1+\alpha x)\right\}$
Quasi Lindley Distribution (QLD)	$\dfrac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x}$
Exponential Distribution (ED)	$\dfrac{1}{\theta}e^{-\frac{x}{\theta}}$
Quasi Akash Distribution (QAD)	$\dfrac{\theta^{2}}{(\alpha\theta+2)}\left(\alpha+\theta x^{2}\right)e^{-\theta x}$

Table 8: ML estimates, -logL, AIC, AICC, BIC, HQIC, KS-distance, for fitted WLQXD and other mentioned models for data set 1.

Distribution	WLQXD	LQXD	QLD	ED	QAD
-logL	49.9985	96.20767	105.7322	130.8676	92.28544
AIC	105.997	196.4153	215.4644	263.7352	188.5709
AICC	106.3662	196.5972	215.6462	263.7949	188.7527
BIC	112.6993	200.8836	219.9326	265.9693	193.0391
HQIC	108.656	198.188	217.2371	264.6216	190.3436
KS-Distance (D)	0.057772	0.355	0.36154	0.44828	0.31119
Likelihood ratio statistic	92.418				
ML Estimates	$\hat{c} = 21.760292$ (5.169830) $\hat{\alpha} = 125.672241$ (958.215785) $\hat{\theta} = 9.793707$ (1.716879)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 0.98255426$ (0.05831841)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 0.8156695$ (0.06359975)	$\hat{\theta} = 2.4513330$ (0.2951056)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.22351031$ (0.06779732)

Table 9: ML estimates, -logL, AIC, AICC, BIC, HQIC, KS-distance, for fitted WLQXD and other mentioned models for data set 2.

Distribution	WLQXD	LQXD	QLD	ED	QAD
-logL	23.75059	73.63075	66.34654	88.83032	54.55285
AIC	53.50119	151.2615	136.6931	179.6606	113.1057
AICC	53.90797	151.4615	136.8931	179.7262	113.3057
BIC	59.93059	155.5478	140.9793	181.8038	117.392
HQIC	56.0299	152.9473	138.3789	180.5035	114.7915
KS-Distance (D)	0.2156	0.30991	0.34707	0.418	0.30549
Likelihood ratio statistic	99.760				
ML Estimates	$\hat{c} = 14.829858$ $\hat{\alpha} = 0.001000$ $\hat{\theta} = 11.732504$ (1.464477)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.4647349$ (0.1004913)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.3269391$ (0.1075059)	$\hat{\theta} = 1.506826$ (0.189842)	$\hat{\alpha} = 0.001000$ $\hat{\theta} = 1.9900949$ (0.1132172)

Different criteria's of goodness of fit like AIC, BIC, AICC, HQIC and K-S distance have been computed by using R software for both the data sets and it has been observed from Tables 8 and 9 that proposed model possesses lesser AIC, BIC, AICC, HQIC & K-S distance values as compared to Lindley Quasi Xgamma distribution, Quasi Lindley distribution, Exponential distribution and Quasi Akash distribution for both the data sets. Hence proposed model provides a good fit to both the mentioned data sets.

AIC = 2k-2log$L$ AICC = AIC+ $\frac{2k(k+1)}{n-k-1}$

BIC = k log$n$-2log$L$ HQIC=2k log(log(n))-2 log L

Where k= number of parameters in model

n= size of the sample (number of observations in data set)

logL= value of likelihood function of model

Likelihood ratio statistic is used for testing the significance of weight parameter in both the data sets. For testing $H_0 : c = 0$ versus $H_1 : c \neq 0$ the LR statistic for testing H_0 is $\omega_1 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 92.418$ for data set 1 as in Table 8, $\omega_2 = 2\{L(\hat{\Theta}) - L(\hat{\Theta}_0)\} = 99.760$ for data set 2 as in Table 9, where $\hat{\Theta}$ and $\hat{\Theta}_0$ are MLEs under H_1 and H_0 .LR statistic $\omega \sim (\chi_{(1)}{}^2(\alpha = 0.05) = 3.841)$ as $n \to \infty$, where 1= degrees of freedom is the difference in dimensionality. From Table 8 $\omega_1 = 92.418 > 3.841$ & from Table 9 $\omega_2 = 99.760 > 3.841$ at 5% level of significance for both the data sets, so we reject H_0 and conclude that weight parameter c plays statistically a significant role.

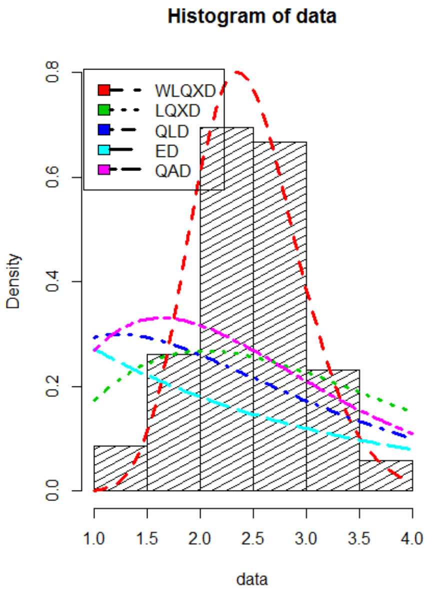

Figure 5 curve fitting of data set 1

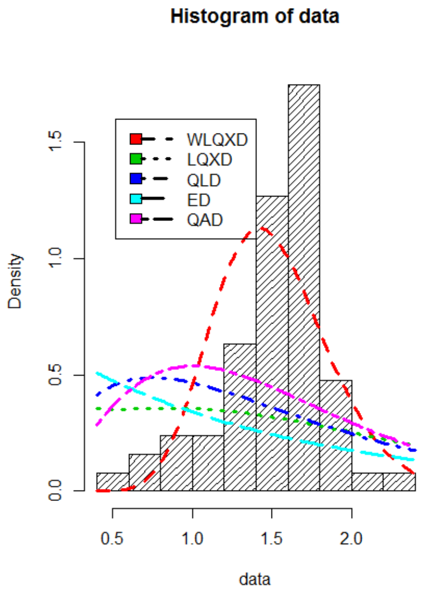

Figure 6 curve fitting of data set 2

13. CONCLUSION

Generalized version of Lindley-Quasi Xgamma distribution by using weighting technique has been proposed in this paper. We obtained the important statistical properties like moments, reliability, moment generating function, order statistics, Bonferroni and Gini indices of formulated model. Expression for Renyi entropy has been derived. For obtaining the estimates of unknown parameters maximum likelihood estimation method is used. For testing the suitability of ML estimates simulation study has been carried which showed that ML estimation method performs well for proposed model. For testing the goodness of fit of proposed model and for investigating the application of proposed model in real life we fitted our proposed model and its related models to two real life data sets and computed log-likelihood values, AIC, AICC, HQIC, BIC and Kolmogorov statistic (D). We observed that our model possesses lesser values of AIC, BIC, AICC, HQIC and D values. Hence our model finds greater applicability in modeling survival times. From generated data as well as from real life data significant role of weight parameter has been observed.

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Some fractional derivatives of *A*-function of multivariable

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Abstract

In the present paper, we study and develop Fractional derivatives of multivariable *A* – function. We derive two theorems which will act as the key formulas from which can obtain their special cases.

Mathematics Subject Classification 2010: 26A33, 30C45, 11B65
Keywords: Generalized multivariable A – function , Hypergeometric function, Mellin – Barnes contour integral and Horn's function.

1. INTRODUCTION

A number of earlier works on the subject of fractional calculus give interesting account of the theory and application of fractional calculus operators in many different areas of mathematical analysis. In this paper, we define the Fractional Derivatives involving A – function of multivariable and derive two main theorems involving Fractional Derivative of the product of A – function of multivariable and the Horn's function. Some new and known results are also established as special cases of our main results. The Fractional Derivative of the product of the multivariable A – function and Horn's function has not been established so far, and some new Fractional Derivative formulae for the product of the multivariable A – function and Horn's function are derived by making use of generalized Leibnitz rule. Recently, Berndt and Bowman [1], Chaurasia and Godika [2], Saxena [3], Tripathi et al [4] gives some integrals and series.

Gautam and Asgar [5, 6], Ram and Kumar [7], Srivastava and Panda [8] and several other authors have evaluated some definite and indefinite integrals involving the *A* – function of one, two and multivariables.

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2. DEFINITION OF FRACTIONAL DERIVATIVE

Following Oldham and Spainer [9], we define the (Riemann Liouville) fractional derivatives of a function f(x) of complex order ϑ or alternatively $(a-\vartheta)^{th}$ by the following

$$\alpha D_x^{\vartheta}\{f(x)\} = \begin{cases} \frac{1}{\Gamma(-\vartheta)}\int_{\alpha}^{x}(x-t)^{-\vartheta-1}f(y)dy\, , Re\,(\vartheta) < 0, \\ \frac{d^n}{dx^n}\alpha D_x^{\vartheta-k}\{f(x)\}, 0 \le Re\,(\vartheta) < n, \end{cases} \tag{2.1}$$

where n is a positive integer .

For simplicity, the special case of the Fractional Derivative Operator αD_x^{ϑ} when $\alpha = 0$ will be written as αD_x^{ϑ}. Thus, we have

$$0D_x^{\vartheta} = D_x^{\vartheta}. \tag{2.2}$$

3. MAIN RESULTS

THEOREM 1. If $\min\{\rho_{r,}\sigma_r\} > 0, |\arg(x/\xi)| < \pi$, Re (m) $+\rho_r$ min {Re $(b_{j,}\beta_j)$}$>$ $-1\ (j = 1, \ldots, r)$,$|z_r x^{\rho_r}| < r_1, |(x+\xi)^{\sigma_r} z_r| < r_2, r_1 + r_2 = 1$; then

$$D_x^{\vartheta}\{x^m(x+\xi)^{\lambda} A_{p_r,q_{r:Y}}^{0.n_r;X}\begin{bmatrix} z_1 x^{\rho_1}(x+\xi)^{\sigma_1} \\ \vdots \\ z_r x^{\rho_r}(x+\xi)^{\sigma_r}\end{bmatrix} \left| \begin{matrix} ,\ldots,\ldots \\ ,\ldots,\ldots, \end{matrix}\right.\Bigg] G_1(\gamma,\delta,\delta' : Z_2\, x^{\rho_2},(x+\xi)^{\sigma_2} z_3, \ldots, \ldots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!}(z_2 x^{\rho_2})^r (z_3\xi^{\sigma_2})^s z_r \xi^{\lambda} x^{m-\vartheta} \sum_{R=0}^{\infty}\frac{(x/\xi)^R}{(R)!}$$

$$A_{p_{r+2,}q_{r+2:Y}}^{0.n_r+2;X}\begin{bmatrix} z_1\xi^{\sigma_1}x^{\rho_1} \\ \vdots \\ z_r\xi^{\sigma_r}x^{\rho_r}\end{bmatrix}\left| \begin{matrix} (-\lambda-\sigma_2 s,\sigma_1,\ldots\ldots,\sigma_r),(-R-m-\rho_2 r,\rho_1,\ldots,\ldots\rho_r),\ldots,\ldots\ldots \\ \ldots,\ldots(R-\lambda-\sigma_2 s,\sigma_1,\ldots,\ldots\sigma_r)(\vartheta-m-R-\sigma_2 s,\rho_1,\ldots,\ldots,\rho_r)\end{matrix}\right.\Bigg]. \tag{3.1}$$

PROOF. We first replace the A – function of multivariable occurring on the left –hand – side by its Mellin –Barnes type contour integral and Horn’s function G_1, and changing the order of integration and differentiation, which is readily justified in view of conditions stated above and collecting the powers of x and (x+ξ), we get

$$\sum_{r,s=0}^{\infty}\frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!}\, z_2{}^r z_3{}^s \frac{1}{(2\pi\omega)^r}\int_{L_1},\ldots,\int_{L_r}\emptyset(s_1,\ldots\ldots,s_r)\prod_{i=1}^{r}\emptyset_i\,(s_i) z_i{}^{s_i}$$

$$\{D_x^{v} x^{m+\rho_1 s+\rho_2 r}\alpha(x+\xi)^{\lambda+\sigma_1 s+\sigma_2 r})\}\ ds_1,\ldots,ds_r \tag{3.2}$$

Now, applying well known binomial expansion , we have

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1}, \ldots, \int_{L_r} \emptyset(s_1, \ldots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} \xi^{\lambda+\sigma_1 s+\sigma_2 r}$$

$$D_x^{\nu} x^{m+\rho_1 s+\rho_2 r} \sum_{R=0}^{\infty} \binom{\lambda+\sigma_1 s+\sigma_2 r}{R} \left(\frac{x}{\xi}\right)^R \mathrm{d}s_1, \ldots, ds_r \,. \tag{3.3}$$

Making use of the formula [the result Oldham and Spanier [9]], we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!} \frac{1}{(2\pi\omega)^r} \int_{L_1}, \ldots, \int_{L_r} \emptyset(s_1, \ldots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} \xi^{\lambda+\sigma_1 s+\sigma_2 r-R}$$

$$\times \frac{\Gamma[1-(-\lambda-\sigma_2 s)+\sigma_1 s, \ldots, \sigma_r s]\,\Gamma[1-(-m-R-\rho_2 r)+\rho_1 s, \ldots, \rho_r s]}{(R)!\;\Gamma[1-(R-\lambda-\sigma_2 S)+\sigma_1 s, \ldots, \sigma_r s]\,\Gamma[1-(\vartheta-m-R-\rho_2 r)+\rho_1 s, \ldots, \rho_r s]}$$

$$\times (z_2 x^{\rho_2})^r (z_3 \xi^{\sigma_2})^s x^{m+\rho_1 s+\rho_2 r+R-\vartheta} {z_1}^s, \ldots, {z_r}^s \mathrm{d}s_1, \ldots, ds_r \tag{3.4}$$

If we interpret the resulting Mellin–Barnes contour integral as an A- function of multivariable, we shall arrive (3.1).

THEOREM 2. If $\min\{\rho_{r,}\sigma_r\} > 0, |\arg(-x/\xi)| < \pi, \mathrm{Re}\ (\mathrm{m}) + \rho_r \min\ \{\mathrm{Re}\ (\delta_{j,}\gamma_j)\} > -1\ (j = 1,2, \ldots, r), |z_2(x-\xi)^{\rho_r}| < r_1, |(\eta - x)^{\sigma_r} z_r| < r_2, r_1 + r_2 + r_n = 1$; then

$$D_x^{\vartheta}\{(x-\xi)^{\lambda}(\eta-x)^{\lambda} A_{p_r, q_{r:Y}}^{0.n_r;X} \left[\begin{matrix} z_1(x-\xi)^{\rho_1}(\eta-\xi)^{\sigma_1} \\ \vdots \\ z_r x-\xi)^{\rho_r}(\eta-\xi)^{\sigma_r} \end{matrix}\middle|\, \begin{matrix} \ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$G_1(\gamma, \delta, \delta' \colon Z_2\, x^{\rho_2}, (\text{x-}\xi)^{\sigma_2}, (\eta-x)^{\sigma_r} z_3, \ldots, \ldots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} \left[(z_2(-\xi)^{\rho_2}\right]^r \left[z_3 \eta^{\sigma_2}\right]^s (\text{-}\xi)^m (\eta)^{\lambda}$$

$$\sum_{R_1=0}^{\infty} \sum_{R_2=0}^{\infty} \frac{x^{-\vartheta}\,(x/\xi)^{R_1}(x/\eta)^{R_2}}{(R_1)!(R_2)!} \frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)}$$

$$A_{p_{r+2,} q_{r+2:Y}}^{0.n_r+2;X} \left[\begin{matrix} z_1(-\xi)^{\rho_1}\eta^{\sigma_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\eta^{\sigma_r} \end{matrix}\middle|\, \begin{matrix} (-\lambda-\rho_2 s, \rho_1, \ldots\ldots, \rho_r), (-\rho-\sigma_2 s, \sigma_1, \ldots, \ldots \sigma_r), \ldots, \ldots\ldots \\ \ldots, \ldots (R_1-m-\rho_2 r, \rho_1, \ldots, \ldots \rho_r)(R_2-\sigma_2 s, \sigma_1, \ldots, \ldots, \sigma_r) \end{matrix}\right]. \tag{3.5}$$

PROOF. we first replace the A-function of several variable occurring on the left – hand side by its Mellin –Barnes type contour integral and Horn's function G_1 by its definition and changing the order of integration and differentiation, which is readily justified in view of conditions stated above and collecting the powers of (x-ξ) and (η-x), we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!}\, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i}$$

$$\{D_x^v (x-\xi)^{m+\rho_1 s+\rho_2 r} \times (\eta - x)^{\lambda+\sigma_1 s+\sigma_2 r})\}\, \mathrm{d}s_1, \dots, ds_r \qquad (3.6)$$

Now, applying well known Binomial expansion, we have

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!}\, {z_2}^r {z_3}^s \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_i) {z_i}^{s_i} (-\xi)^{m+\rho_1 s+\rho_2 r}$$

$$\times (\eta)^{\lambda+\sigma_1 s+\sigma_2 r} D_x^v \sum_{R_1=0}^{\infty} \binom{m+\rho_1 s+\rho_2 r}{R_1} \left(\frac{-x}{\xi}\right)^{R_1} \sum_{R_2=0}^{\infty} \binom{\lambda+\sigma_1 s+\sigma_2 r}{R_2} \left(\frac{-x}{\eta}\right)^{R_2} \}$$

$${z_1}^s, \dots, {z_r}^s\, \mathrm{d}s_1, \dots, ds_r \qquad (3.7)$$

Making the use of the formula [the result Oldham and Spanier [9]], we get

$$\sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!\,(s)!}\, {z_2}^r {z_3}^s (-\xi)^{m+\rho_1 s} (\eta)^{\lambda+\sigma_1 s} \sum_{R_1=0}^{\infty} \sum_{R_2=0}^{\infty} \frac{(-1)^{R_1+R_2} \left(\frac{x}{\xi}\right)^{R_1} \left(\frac{x}{\eta}\right)^{R_2} x^{-\vartheta}}{(R_1)!\,(R_2)!}$$

$$\frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)} \frac{1}{(2\pi\omega)^r} \int_{L_1}, \dots, \int_{L_r} \emptyset(s_1, \dots\dots, s_r) \prod_{i=1}^{r} \emptyset_i\,(s_r) {z_i}^{s_r} (-\xi)^{\rho_1 s} (-\eta)^{\sigma_1 s}$$

$$\times \frac{\Gamma[1-(-m-\rho_2 r)+\rho_1 s, \dots, \rho_r s]\, \Gamma[1-(-\lambda-\sigma_2 s)+\sigma_1 s, \dots, \sigma_r s]}{\Gamma[1-(R_1-m-\rho_2 r)+\rho_1 s, \dots, \rho_r s]\, \Gamma[1-(R_2-\sigma_2 S)+\sigma_1 s, \dots, \sigma_r s]}$$

$${z_1}^s, \dots, {z_r}^s\, \mathrm{d}s_1, \dots, ds_r\ . \qquad (3.8)$$

If we interpret the resulting Mellin –Barnes contour integral as an A- function of multivariable, we shall arrive (3.5).

4. SPECIAL CASES OF (3.1) AND (3.5)

(1) Putting $\sigma_r \to 0$ another four Fractional Derivative formulae corresponding to (3.1) and (3.5):

$$D_x^{\vartheta} x^m (x\xi)^{\lambda} A^{0.n_r;X}_{p_r,q_{r:Y}} \left[\begin{matrix} z_1(-\xi)^{\rho_1}\eta^{\sigma_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\eta^{\sigma_r} \end{matrix} \middle| \begin{matrix} \dots,\dots \\ \dots,\dots \end{matrix} \right] G_1(\gamma, \delta, \delta' : Z_2\, x^{\rho_2}, (\mathrm{x}+\xi)^{\sigma_2} z_3, \dots, \dots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} (z_2 x^{\rho_2})^r (z_3 \xi^{\sigma_2})^s z_r \xi^{\lambda} x^{m-\vartheta} \sum_{R=0}^{\infty} \frac{(x/\xi)^R}{(R)!}$$

$$\times \frac{\Gamma(1+\lambda+\sigma_2 s)}{\Gamma(1+\lambda+\sigma_2 s-R)} A^{0.n_r+1;X}_{p_{r+1},q_{r+1:Y}} \left[\begin{matrix} z_1 x^{\rho_1} \\ \vdots \\ z_r x^{\rho_r} \end{matrix} \middle| \begin{matrix} (-R-m-\rho_2 r, \rho_1, \dots, \rho_r), \dots, \dots \\ \dots, \dots (\vartheta-m-R-\sigma_2 s, \rho_1, \dots, \dots, \rho_r) \end{matrix} \right], \qquad (4.1)$$

$\min\{\rho_{r,}\} > 0, |\arg(x/\xi)| < \pi,$

Re (m) $+\rho_r$ min {Re $(\delta_{j,}\gamma_j)\} > -1$ $(j = 1, \ldots, r)$,

$|z_r x^{\rho_r}| < r_1, |(x+\xi)^{\sigma_r} z_r| < r_2, r_1 + r_2 +, \ldots, r_n = 1$;

$$D_x^{\vartheta}\{(x-\xi)^{\lambda}(\eta-x)^{\lambda} A^{0.n_r;X}_{p_r,q_{r:Y}}\left[\begin{matrix} z_1(x-\xi)^{\rho_1} \\ \vdots \\ z_r(x-\xi)^{\rho_r}\end{matrix}\middle| \begin{matrix}\ldots,\ldots \\ \ldots,\ldots\end{matrix}\right]$$

$$\times G_1(\gamma,\delta,\delta' : Z_2, (\text{x-}\xi)^{\rho_r}, (x-\eta)^{\sigma_r} z_3, \ldots, \ldots, z_r)\}$$

$$= \sum_{r,s=0}^{\infty} \frac{(\gamma)_{r+s}(\delta)_{s-r}(\delta)'_{r-s}}{(r)!(s)!} [(z_2(-\xi)^{\rho_2}]^r [z_3\eta^{\sigma_2}]^s (\text{-}\xi)^m(\eta)^{\lambda}$$

$$\sum_{R_1=0}^{\infty}\sum_{R_2=0}^{\infty} \frac{x^{-\vartheta}(x/\xi)^{R_1}(x/\eta)^{R_2}}{(R_1)!(R_2)!} \frac{\Gamma(R_1+R_2+1)}{\Gamma(R_1+R_2-\vartheta+1)} \times \frac{\Gamma(1+\lambda+\sigma_r s)}{\Gamma(1+\sigma_r s-R_2)}$$

$$A^{0.n_r+1;X}_{p_{r+1},q_{r+1:Y}}\left[\begin{matrix} z_1(-\xi)^{\rho_1} \\ \vdots \\ z_r(-\xi)^{\rho_r}\end{matrix}\middle| \begin{matrix}(-m-\rho_2 r, \rho_1, \ldots,\ldots \rho_r), \ldots, \ldots\ldots \\ \ldots,\ldots(R_1-m-\rho_2 r_1, \rho_1, \ldots,\ldots \rho_r)(R_2-\sigma_2 s, \sigma_1, \ldots,\ldots,\sigma_r)\end{matrix}\right], \tag{4.2}$$

$\min\{\rho_{r,}\} > 0, |\arg(-x/\xi)| < \pi,$

Re (m) $+\rho_r$ min {Re $(\delta_{j,}\gamma_j)\} > -1$ $(j = 1,2, \ldots, r)$,

$|z_2(x-\xi)^{\rho_r}| < r_1, |(\eta-x)^{\sigma_r} z_3| < r_2, r_1 + r_2 + r_n = 1$;

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Some new inequalities for generalized convex functions pertaining generalized fractional integral operators and their applications

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Abstract

In this paper, authors establish a new identity for a differentiable function using generic integral operators. By applying it, some new integral inequalities of trapezium, Ostrowski and Simpson type are obtained. Moreover, several special cases have been studied in detail. Finally, many useful applications have been found.

Mathematics Subject Classification 2010: 26A51; 26A33, 26D07, 26D10, 26D15.
Keywords: Inequalities; convexity; Raina's function; special means; error estimation.

1. INTRODUCTION AND PRELIMINARIES

DEFINITION 1.1. [49] $\Lambda : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function on I, if

$$\Lambda((1-\zeta)\flat_1 + \zeta\flat_2) \le (1-\zeta)\Lambda(\flat_1) + \zeta\Lambda(\flat_2), \quad \forall \flat_1, \flat_2 \in \mathrm{I},\ \zeta \in [0,1].$$

THEOREM 1.2. (Trapezium inequality) Suppose that $\Lambda : \mathrm{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, $\flat_1, \flat_2 \in \mathrm{I}$ with $\flat_1 < \flat_2$, then

$$\Lambda\left(\frac{\flat_1+\flat_2}{2}\right) \le \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\ell)d\ell \le \frac{\Lambda(\flat_1)+\Lambda(\flat_2)}{2}. \tag{1}$$

Interested readers are referred to [4]-[6],[15; 19; 20; 22; 25; 26; 28],[33]-[38],[45; 47; 52; 53; 55; 56].

THEOREM 1.3. (Ostrowski inequality) Assume that $\Lambda : \mathrm{I} \to \mathbb{R}$ be a differentiable function on I°, $\flat_1, \flat_2 \in \mathrm{I}^\circ$ with $\flat_1 < \flat_2$. If $|\Lambda'(\ell)| \le \mathscr{K}$, then

$$\left|\Lambda(\ell) - \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \Lambda(\xi)\,d\xi\right| \le \mathscr{K}(\flat_2-\flat_1)\left[\frac{1}{4} + \frac{\left(\ell - \frac{\flat_1+\flat_2}{2}\right)^2}{(\flat_2-\flat_1)^2}\right], \quad \forall \ell \in [\flat_1, \flat_2]. \tag{2}$$

For other recent published papers about Ostrowski type inequalities, see [1]-[3],[7]-[14],[17],[29]-[32],[39]-[41],[43; 44; 50; 54; 57].

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THEOREM 1.4. (Simpson inequality) Let $\Lambda : [\flat_1, \flat_2] \to \mathbb{R}$ be four times differentiable on $(\flat_1, \flat_2)$ and suppose that

$$\|\Lambda^{(4)}\|_\infty := \sup_{\ell \in (\flat_1, \flat_2)} |\Lambda^{(4)}| < +\infty.$$

Then

$$\left| \int_{\flat_1}^{\flat_2} \Lambda(\ell) d\ell - \frac{\flat_2 - \flat_1}{3} \left[\frac{\Lambda(\flat_1) + \Lambda(\flat_2)}{2} + 2\Lambda\left(\frac{\flat_1 + \flat_2}{2}\right) \right] \right| \leq \frac{(\flat_2 - \flat_1)^5}{2880} \|\Lambda^{(4)}\|_\infty. \tag{3}$$

About Simpson type inequalities, see [27; 42; 51; 57].

In our paper we will establish some new trapezium, Ostrowski and Simpson type inequalities pertaining generalized convex functions with respect to another function. Moreover, many useful applications will be given. Hence, it is important to recall some essential facts relevant to fractional integrals.

DEFINITION 1.5. [34] Assume that $\Lambda \in \mathscr{L}[\flat_1, \flat_2]$, then κ–fractional integrals, $\eta, \kappa > 0$ with $\flat_1 \geq 0$ are

$$I_{\flat_1^+}^{\eta,\kappa} \Lambda(\xi_1) = \frac{1}{\kappa \Gamma_\kappa(\eta)} \int_{\flat_1}^{\xi_1} (\xi_1 - \xi)^{\frac{\eta}{\kappa} - 1} \Lambda(\xi) \, d\xi, \quad \flat_1 < \xi_1,$$

and

$$I_{\flat_2^-}^{\eta,\kappa} \Lambda(\xi_1) = \frac{1}{\kappa \Gamma_\kappa(\eta)} \int_{\xi_1}^{\flat_2} (\xi - \xi_1)^{\frac{\eta}{\kappa} - 1} \Lambda(\xi) \, d\xi, \quad \flat_2 > \xi_1, \tag{4}$$

respectively.

DEFINITION 1.6. [35; 36] S is called ϖ–convex set, if

$$\varpi(\jmath)\flat_2 + (1 - \varpi(\jmath))\flat_1 \in S, \;\; \forall \flat_1, \flat_2 \in S, \; \jmath \in [0,1].$$

DEFINITION 1.7. $\Lambda : S \to \mathbb{R}$ on ϖ–convex set S is called ϖ–convex function, if

$$\Lambda(\flat_1 + \jmath e^{i\varpi}(\flat_2 - \flat_1)) \leq (1 - \jmath)\Lambda(\flat_1) + \jmath\Lambda(\flat_2), \;\; \forall \flat_1, \flat_2 \in S, \; \jmath \in [0,1].$$

Raina, in [48], defined for $\rho, \delta > 0$ and $|z| < R$, the following class of functions:

$$\mathscr{F}_{\rho,\delta}^{\sigma}(z) = \mathscr{F}_{\rho,\delta}^{\sigma(0),\sigma(1),\ldots}(z) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \delta)} z^k. \tag{5}$$

Choosing $|z| \leq 1$, we take hypergeometric function. For $\sigma = (1,1,\ldots)$ with $\rho = \eta, (\Re(\eta) > 0), \delta = 1$ and $z \in \mathbb{C}$ in (5), we obtain Mittag–Leffler function

$$\mathscr{E}_\eta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1+\eta k)}.$$

DEFINITION 1.8. $S \neq \emptyset$ is called generalized convex set, if

$$m\flat_1 + \zeta \mathscr{F}^\sigma_{\rho,\delta}(\flat_2 - m\flat_1) \in S, \qquad \forall\, \flat_1, \flat_2 \in S,\ \ m \in (0,1],\ \ \zeta \in [0,1]. \tag{6}$$

DEFINITION 1.9. Λ is called generalized convex function, if

$$\Lambda(m\flat_1 + \zeta \mathscr{F}^\sigma_{\rho,\delta}(\flat_2 - m\flat_1)) \leq (1-\zeta)\Lambda(m\flat_1) + \zeta\Lambda(\flat_2), \quad \forall\, \flat_1, \flat_2 \in S,\ m \in (0,1],\ \ \zeta \in [0,1]. \tag{7}$$

REMARK 1.10. Taking $m = 1$ and $\mathscr{F}^\sigma_{\rho,\delta}(\flat_2 - \flat_1) = \flat_2 - \flat_1 > 0$, we get definition 1.1. For suitable choice of $\mathscr{F}^\sigma_{\rho,\delta}(\cdot)$, we obtain definition 1.7. This describes the reasons and motivations of newly defined notions and the relation with these known definitions.

DEFINITION 1.11. [23; 24] Assume that $\Lambda_2 : [\flat_1, \flat_2] \to \mathbb{R}$ is an increasing and positive monotone function on $[\flat_1, \flat_2]$, with continuous derivative on $(\flat_1, \flat_2)$. The left- right- fractional integrals of Λ_1 with respect to Λ_2 on $[\flat_1, \flat_2]$, $\eta > 0$ are

$$I^{\eta,\Lambda_2}_{\flat_1+}\Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{\flat_1}^{\xi_1} \frac{\Lambda_2'(\xi)\Lambda_1(\xi)}{[\Lambda_2(\xi_1) - \Lambda_2(\xi)]^{1-\eta}}\, d\xi,\ \flat_1 < \xi_1, \tag{8}$$

and

$$I^{\eta,\Lambda_2}_{\flat_2-}\Lambda_1(\xi_1) = \frac{1}{\Gamma(\eta)} \int_{\xi_1}^{\flat_2} \frac{\Lambda_2'(\xi)\Lambda_1(\xi)}{[\Lambda_2(\xi) - \Lambda_2(\xi_1)]^{1-\eta}}\, d\xi,\ \flat_2 > \xi_1, \tag{9}$$

respectively.

Function $\varpi : [0, +\infty) \to [0, +\infty)$ constructed from Sarikaya et al. in [45; 46], has the following properties:

$$\int_0^1 \frac{\varpi(\xi)}{\xi}\, d\xi < +\infty, \tag{10}$$

$$\frac{1}{A_1} \leq \frac{\varpi(\varepsilon)}{\varpi(\xi_1)} \leq A_1 \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \tag{11}$$

$$\frac{\varpi(\xi_1)}{\xi_1^2} \leq A_2 \frac{\varpi(\varepsilon)}{\varepsilon^2} \text{ for } \varepsilon \leq \xi_1, \tag{12}$$

$$\left| \frac{\varpi(\xi_1)}{\xi_1^2} - \frac{\varpi(\varepsilon)}{\varepsilon^2} \right| \leq A_3 |\xi_1 - \varepsilon| \frac{\varpi(\xi_1)}{\xi_1^2} \text{ for } \frac{1}{2} \leq \frac{\varepsilon}{\xi_1} \leq 2, \tag{13}$$

where $A_1, A_2, A_3 > 0$ are independent of $\varepsilon, \xi_1 > 0$. Moreover, Sarikaya et al. defined the following useful operators:

$$
{}_{\flat_1^+}I_{\varpi}\Lambda(\xi_1) = \int_{\flat_1}^{\xi_1} \frac{\varpi(\xi_1 - \xi)}{\xi_1 - \xi}\Lambda(\xi)\,d\xi, \quad \flat_1 < \xi_1, \tag{14}
$$

$$
{}_{\flat_2^-}I_{\varpi}\Lambda(\xi_1) = \int_{\xi_1}^{\flat_2} \frac{\varpi(\xi - \xi_1)}{\xi - \xi_1}\Lambda(\xi)\,d\xi, \quad \flat_2 > \xi_1. \tag{15}
$$

About their efficiency, see [18; 21; 45]. Finally, Farid in [16], defined the following generic operators:

$$
G_{\flat_1+}^{\varpi,\Lambda_2}\Lambda_1(\xi_1) = \int_{\flat_1}^{\xi_1} \frac{\varpi\left(\Lambda_2(\xi_1) - \Lambda_2(\xi)\right)}{\Lambda_2(\xi_1) - \Lambda_2(\xi)}\Lambda_2'(\xi)\Lambda_1(\xi)\,d\xi,\ \flat_1 < \xi_1, \tag{16}
$$

and

$$
G_{\flat_2-}^{\varpi,\Lambda_2}\Lambda_1(\xi_1) = \int_{\xi_1}^{\flat_2} \frac{\varpi\left(\Lambda_2(\xi) - \Lambda_2(\xi_1)\right)}{\Lambda_2(\xi) - \Lambda_2(\xi_1)}\Lambda_2'(\xi)\Lambda_1(\xi)\,d\xi,\ \flat_2 > \xi_1, \tag{17}
$$

respectively.

The paper is constructed in this way: In Section 2, we will find an interesting identity with parameter λ and using generic integral operators form auxiliary equality, some new integral inequalities of trapezium, Ostrowski and Simpson type will be obtain. Section 3 is devoted to useful applications.

2. MAIN RESULTS

Let $\mathscr{P} = [m\flat_1, \flat_2]$, where $\flat_1 < \flat_2$ for some fixed $m \in (0,1]$ with $\zeta \in [0,1]$.

$$
\Pi_m^{\varpi,\Upsilon}(\ell,\zeta) := \int_0^{\zeta} \frac{\varpi\left(\Upsilon\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right) - \Upsilon(m\flat_1)\right)}{\Upsilon\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right) - \Upsilon(m\flat_1)} \tag{18}
$$

$$
\times\Upsilon'\left(m\flat_1 + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1)\right)d\xi < +\infty,
$$

and

$$
\Xi_m^{\varpi,\Upsilon}(\ell,\zeta) := \int_{\zeta}^{1} \frac{\varpi\left(\Upsilon\left(m\ell + \mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right) - \Upsilon\left(m\ell + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right)\right)}{\Upsilon\left(m\ell + \mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right) - \Upsilon\left(m\ell + \xi\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell)\right)} \tag{19}
$$

$$\times \Upsilon'\left(m\ell+\xi \mathscr{F}^{\sigma}_{\rho,\lambda}(\flat_2-m\ell)\right) d\xi < +\infty.$$

The following lemma will help us to find new results.

LEMMA 2.1. Let $\Lambda : \mathscr{P} \to \mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda \in \mathbb{R}$. Assume that $\Lambda' \in \mathscr{L}(\mathscr{P})$ and $\mathscr{F}^{\sigma}_{\rho,\lambda}(\flat_2-m\flat_1) > 0$, then

$$\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)}+\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell)}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right]$$

$$+\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)}+\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\Lambda(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right]$$

$$-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left[\frac{G^{\varpi,\Upsilon}_{\left(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)^-}\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)}+\frac{G^{\varpi,\Upsilon}_{(m\ell)^+}\Lambda(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)}\right]$$

$$=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Pi^{\varpi,\Upsilon}_{m}(\ell,\zeta)-\lambda\right]\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta \quad (20)$$

$$-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Xi^{\varpi,\Upsilon}_{m}(\ell,\zeta)-\lambda\right]\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta.$$

We denote

$$T_{\Lambda,\Pi^{\varpi,\Upsilon}_{m},\Xi^{\varpi,\Upsilon}_{m}}(\lambda;\ell,\flat_1,\flat_2):=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_{m}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \quad (21)$$

$$\times\int_0^1\left[\Pi^{\varpi,\Upsilon}_{m}(\ell,\zeta)-\lambda\right]\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta$$

$$-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_{m}(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left[\Xi^{\varpi,\Upsilon}_{m}(\ell,\zeta)-\lambda\right]\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta.$$

PROOF. By integrating by parts (21), we derive

$$T_{\Lambda,\Pi_m^{\varpi,\Upsilon},\Xi_m^{\varpi,\Upsilon}}(\lambda;\ell,\flat_1,\flat_2)=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left\{\int_0^1\Pi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta-\lambda\int_0^1\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta\right\}$$

$$-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left\{\int_0^1\Xi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta-\lambda\int_0^1\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta\right\}$$

$$=\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\left\{\frac{\Pi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\right|_0^1-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}$$

$$\times\int_0^1\frac{\varpi\left(\Upsilon\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)-\Upsilon(m\flat_1)\right)}{\Upsilon\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)-\Upsilon(m\flat_1)}$$

$$\times\Upsilon'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)d\zeta$$

$$\left.-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\Lambda\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|_0^1\Bigg\}-\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left\{\frac{\Xi_m^{\varpi,\Upsilon}(\ell,\zeta)\Lambda\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\right|_0^1$$

$$-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\times\int_0^1\frac{\varpi\left(\Upsilon\left(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)-\Upsilon\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right)}{\Upsilon\left(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)-\Upsilon\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)}$$

$$\times\Upsilon'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\Lambda\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)d\zeta$$

$$\left.-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\Lambda\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|_0^1\Bigg\}$$

$$= \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda\left(m\flat_1 + \zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right) + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell)}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)}$$

$$-\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda\left(m\flat_1 + \zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)} + \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell)}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right]$$

$$+\frac{\lambda}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)} + \frac{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\Lambda(m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell))}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right]$$

$$-\frac{1}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \times \left[\frac{G^{\varpi,\Upsilon}_{\left(m\flat_1 + \mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right)^-}\Lambda(m\flat_1)}{\Pi^{\varpi,\Upsilon}_m(\ell,1)} + \frac{G^{\varpi,\Upsilon}_{(m\ell)^+}\Lambda(m\ell + \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell))}{\Xi^{\varpi,\Upsilon}_m(\ell,0)}\right].$$

REMARK 2.2. *a.* Taking $m = 1$, $\lambda = 0$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_{\Lambda}(\ell,\flat_1,\flat_2) := \Lambda(\ell) - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2}\Lambda(\zeta)\,d\zeta.$$

b. Choosing $m = 1$, $\lambda = 1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$\overline{T_{\Lambda}}(\ell,\flat_1,\flat_2) := \frac{(\ell - \flat_1)\Lambda(\flat_1) + (\flat_2 - \ell)\Lambda(\flat_2)}{\flat_2 - \flat_1} - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2}\Lambda(\zeta)\,d\zeta.$$

c. Taking $m = 1$, $\ell = \frac{\flat_1 + \flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Lemma 2.1, then

$$T_{\Lambda}(\lambda;\flat_1,\flat_2) := \lambda\left[\frac{\Lambda(\flat_1) + \Lambda(\flat_2)}{2}\right] + (1-\lambda)\Lambda\left(\frac{\flat_1 + \flat_2}{2}\right) - \frac{1}{\flat_2 - \flat_1}\int_{\flat_1}^{\flat_2}\Lambda(\zeta)\,d\zeta.$$

THEOREM 2.3. Let $\Lambda : \mathscr{P} \to \mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda \in [0,1]$. If $|\Lambda'|^q$ is generalized convex function on $\mathscr{P}$ and $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) > 0$, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt[q]{2}\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\flat_1)|^q + |\Lambda'(\ell)|^q} \tag{22}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\sqrt[q]{2}\,\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q},$$

where

$$B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p):=\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta,\ \ B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p):=\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta. \quad (23)$$

PROOF. Applying Lemma 2.1, generalized convexity of $|\Lambda'|^q$, Hölder's inequality, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|d\zeta$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|d\zeta$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)}\left(\int_0^1\left[(1-\zeta)|\Lambda'(m\flat_1)|^q+\zeta|\Lambda'(\ell)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}\left(\int_0^1\left[(1-\zeta)|\Lambda'(m\ell)|^q+\zeta|\Lambda'(\flat_2)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$= \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt[q]{2}\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\flat_1)|^q + |\Lambda'(\ell)|^q}$$

$$+ \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\sqrt[q]{2}\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)} \times \sqrt[q]{|\Lambda'(m\ell)|^q + |\Lambda'(\flat_2)|^q}.$$

COROLLARY 2.4. Taking $p = 2 = q$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\sqrt{2}\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,2)} \times \sqrt{|\Lambda'(m\flat_1)|^2 + |\Lambda'(\ell)|^2} \tag{24}$$

$$+ \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\sqrt{2}\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \sqrt{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,2)} \times \sqrt{|\Lambda'(m\ell)|^2 + |\Lambda'(\flat_2)|^2}.$$

COROLLARY 2.5. Choosing $|\Lambda'| \leq \mathscr{K}$ in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right| \leq \frac{\mathscr{K}}{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1)} \tag{25}$$

$$\times \left[\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)} + \frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)} \sqrt[p]{B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,p)}\right].$$

COROLLARY 2.6. Taking $m = 1$, $\lambda = 0$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we obtain

$$\left|T_{\Lambda}(\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(\flat_2 - \flat_1)} \tag{26}$$

$$\times \left\{(\ell - \flat_1)^2 \sqrt[q]{|\Lambda'(\flat_1)|^q + |\Lambda'(\ell)|^q} + (\flat_2 - \ell)^2 \sqrt[q]{|\Lambda'(\ell)|^q + |\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.7. Choosing $\ell = \frac{\flat_1 + \flat_2}{2}$ in Corollary 2.6, we have

$$\left|T_{\Lambda}(\flat_1,\flat_2)\right| \leq \frac{(\flat_2 - \flat_1)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{27}$$

$$\times \left\{\sqrt[q]{|\Lambda'(\flat_1)|^q + \left|\Lambda'\left(\frac{\flat_1 + \flat_2}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\flat_1 + \flat_2}{2}\right)\right|^q + |\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.8. Taking $m = 1$, $\lambda = 1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) =$

$\flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get

$$\left|\overline{T_\Lambda}(\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\sqrt[q]{2}\sqrt[p]{p+1}(\flat_2-\flat_1)} \tag{28}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{|\Lambda'(\flat_1)|^q+|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{|\Lambda'(\ell)|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.9. Choosing $m=1$, $\lambda=\frac{1}{3}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we obtain

$$\left|T_\Lambda\left(\frac{1}{3};\flat_1,\flat_2\right)\right| \leq \frac{1}{\sqrt[q]{2}(\flat_2-\flat_1)}\sqrt[p]{\frac{2^{p+1}+1}{3^{p+1}(p+1)}} \tag{29}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{|\Lambda'(\flat_1)|^q+|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{|\Lambda'(\ell)|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.10. Substituting $\lambda=0$ and $\varpi(\zeta)=\zeta$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{30}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\sqrt[p]{B^{\Upsilon}_1(\ell;p)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}\right.$$

$$\left.+\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\sqrt[p]{B^{\Upsilon}_2(\ell;p)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_1(\ell;p) := \int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^p d\zeta \tag{31}$$

and

$$B^{\Upsilon}_2(\ell;p) := \int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^p d\zeta. \tag{32}$$

COROLLARY 2.11. For $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^\alpha}{\Gamma(\alpha)}$ in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{33}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\sqrt[p]{B^{\Upsilon}_3(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}\right.$$

$$\left.+\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\sqrt[p]{B^{\Upsilon}_4(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_3(\ell;p,\alpha):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{p\alpha}d\zeta \tag{34}$$

and

$$B^{\Upsilon}_4(\ell;p,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{p\alpha}d\zeta. \tag{35}$$

COROLLARY 2.12. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\frac{\alpha}{\kappa}}}{\kappa\Gamma_\kappa(\alpha)}$ in Theorem 2.3, we obtain

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right|\leq\frac{1}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{36}$$

$$\times\left\{\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\sqrt[p]{B^{\Upsilon}_5(\ell;p,\alpha,\kappa)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}\right.$$

$$\left.+\sqrt[q]{\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\sqrt[p]{B^{\Upsilon}_6(\ell;p,\alpha,\kappa)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q}\right\},$$

where

$$B^{\Upsilon}_5(\ell;p,\alpha,\kappa):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{\frac{p\alpha}{\kappa}}d\zeta \tag{37}$$

and

$$B^{\Upsilon}_6(\ell;p,\alpha,\kappa):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\left[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{\frac{p\alpha}{\kappa}}d\zeta. \tag{38}$$

COROLLARY 2.13. For $\lambda=0,\forall\xi\in[0,\zeta]$, $\varpi_\Upsilon(\ell,\zeta)=\zeta(\Upsilon(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1))-\zeta)^{\alpha-1}$ and $\forall\xi\in[\zeta,1]$, $\varpi_\Upsilon(\ell,\zeta)=\zeta(\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\zeta)^{\alpha-1}$ in Theorem 2.3, we have

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right|\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\left[\Upsilon(m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1))-\Upsilon(m\flat_1)\right]\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{39}$$

$$\times\sqrt[p]{B^{\Upsilon}_7(\ell;p)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\left[\Upsilon^{\alpha}(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon^{\alpha}(m\ell)\right]\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\sqrt[p]{B^{\Upsilon}_8(\ell;p,\alpha)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q},$$

where

$$B^{\Upsilon}_7(\ell;p):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^p d\zeta \tag{40}$$

and

$$B^{\Upsilon}_8(\ell;p,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\Big[\Upsilon^{\alpha}(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon^{\alpha}(\zeta)\Big]^p d\zeta. \tag{41}$$

COROLLARY 2.14. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta}{\alpha}\exp(-A\zeta)$, where $A=\frac{1-\alpha}{\alpha}$, in Theorem 2.3, we get

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right|\le\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{42}$$

$$\times\sqrt[p]{B^{\Upsilon}_9(\ell;p,A)}\times\sqrt[q]{|\Lambda'(m\flat_1)|^q+|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}}{\sqrt[q]{2}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\Upsilon}_{10}(\ell;p,A)}\times\sqrt[q]{|\Lambda'(m\ell)|^q+|\Lambda'(\flat_2)|^q},$$

where

$$B^{\Upsilon}_9(\ell;p,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}\Big\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\Big\}^p d\zeta \tag{43}$$

and

$$B^{\Upsilon}_{10}(\ell;p,A):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}\Big\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))\right)\right]\Big\}^p d\zeta. \tag{44}$$

THEOREM 2.15. Let $\Lambda:\mathscr{P}\to\mathbb{R}$ be a differentiable function on $\mathscr{P}^{\circ}$ and $\lambda\in[0,1]$. If $|\Lambda'|^q$ is generalized convex on $\mathscr{P}$ and $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)>0$, then for $q\ge 1$, we have

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|\le\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}} \tag{45}$$

$$\times\sqrt[q]{\left[B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,1)-E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)\right]|\Lambda'(m\flat_1)|^q+E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda)|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda,1)-G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)\right]|\Lambda'(m\ell)|^q+G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda)|\Lambda'(\flat_2)|^q},$$

where

$$E^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda):=\int_0^1\zeta\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta,\quad G^{\varpi,\Upsilon}_{\Xi_m}(\ell;\lambda):=\int_0^1\zeta\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta. \quad (46)$$

PROOF. By Lemma 2.1, generalized convexity of $|\Lambda'|^q$ and power mean inequality, we get

$$\left|T_{\Lambda,\Pi^{\varpi,\Upsilon}_m,\Xi^{\varpi,\Upsilon}_m}(\lambda;\ell,\flat_1,\flat_2)\right|$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|d\zeta$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\times\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|d\zeta$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\flat_1+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right]^2}{\Xi^{\varpi,\Upsilon}_m(\ell,0)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}$$

$$\times\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\Xi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left|\Lambda'\left(m\ell+\zeta\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\right)\right|^q d\zeta\right)^{\frac{1}{q}}$$

$$\leq\frac{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^2}{\Pi^{\varpi,\Upsilon}_m(\ell,1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\sqrt[p]{B^{\varpi,\Upsilon}_{\Pi_m}(\ell;\lambda,p)}$$

$$\times\left(\int_0^1\left|\Pi^{\varpi,\Upsilon}_m(\ell,\zeta)-\lambda\right|\left[(1-\zeta)|\Lambda'(m\flat_1)|^q+\zeta|\Lambda'(\ell)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$+\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\sqrt[p]{B_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda,p)}$$

$$\times\left(\int_0^1\left|\Xi_m^{\varpi,\Upsilon}(\ell,\zeta)-\lambda\right|\left[(1-\zeta)|\Lambda'(m\ell)|^q+\zeta|\Lambda'(\flat_2)|^q\right]d\zeta\right)^{\frac{1}{q}}$$

$$=\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)-E_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda)\right]|\Lambda'(m\flat_1)|^q+E_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda)|\Lambda'(\ell)|^q}$$

$$+\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)-G_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda)\right]|\Lambda'(m\ell)|^q+G_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda)|\Lambda'(\flat_2)|^q}.$$

COROLLARY 2.16. For $q=1$ in Theorem 2.15, we get

$$\left|T_{\Lambda,\Pi_m^{\varpi,\Upsilon},\Xi_m^{\varpi,\Upsilon}}(\lambda;\ell,\flat_1,\flat_2)\right|\leq\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)} \tag{47}$$

$$\times\left[\left(B_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)-E_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda)\right)|\Lambda'(m\flat_1)|+E_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda)|\Lambda'(\ell)|\right]$$

$$+\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}$$

$$\times\left[\left(B_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)-G_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda)\right)|\Lambda'(m\ell)|+G_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda)|\Lambda'(\flat_2)|\right].$$

COROLLARY 2.17. Taking $|\Lambda'|\leq\mathscr{K}$ in Theorem 2.15, we have

$$\left|T_{\Lambda,\Pi_m^{\varpi,\Upsilon},\Xi_m^{\varpi,\Upsilon}}(\lambda;\ell,\flat_1,\flat_2)\right|\leq\frac{\mathscr{K}}{\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)} \tag{48}$$

$$\times\left[\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^2}{\Pi_m^{\varpi,\Upsilon}(\ell,1)}B_{\Pi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)+\frac{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^2}{\Xi_m^{\varpi,\Upsilon}(\ell,0)}B_{\Xi_m}^{\varpi,\Upsilon}(\ell;\lambda,1)\right].$$

COROLLARY 2.18. Choosing $m=1, \lambda=0, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get

$$\left|T_{\Lambda}(\ell,\flat_1,\flat_2)\right| \le \frac{1}{2\sqrt[q]{3}(\flat_2-\flat_1)} \tag{49}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{|\Lambda'(\flat_1)|^q+2|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{2|\Lambda'(\ell)|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.19. Taking $\ell=\frac{\flat_1+\flat_2}{2}$ in Corollary 2.18, we obtain

$$\left|T_{\Lambda}(\flat_1,\flat_2)\right| \le \frac{(\flat_2-\flat_1)}{8\sqrt[q]{3}} \tag{50}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\flat_1)|^q+2\left|\Lambda'\left(\frac{\flat_1+\flat_2}{2}\right)\right|^q}+\sqrt[q]{2\left|\Lambda'\left(\frac{\flat_1+\flat_2}{2}\right)\right|^q+|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.20. Choosing $m=1, \lambda=1, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have

$$\left|\overline{T_{\Lambda}}(\ell,\flat_1,\flat_2)\right| \le \frac{1}{2\sqrt[q]{3}(\flat_2-\flat_1)} \tag{51}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{2|\Lambda'(\flat_1)|^q+|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{|\Lambda'(\ell)|^q+2|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.21. Taking $m=1, \lambda=\frac{1}{3}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get

$$\left|T_{\Lambda}\left(\frac{1}{3};\flat_1,\flat_2\right)\right| \le \frac{1}{2\sqrt[q]{243}(\flat_2-\flat_1)} \tag{52}$$

$$\times\left\{(\ell-\flat_1)^2\sqrt[q]{185|\Lambda'(\flat_1)|^q+58|\Lambda'(\ell)|^q}+(\flat_2-\ell)^2\sqrt[q]{195|\Lambda'(\ell)|^q+48|\Lambda'(\flat_2)|^q}\right\}.$$

COROLLARY 2.22. Substituting $\lambda=0$ and $\varpi(\zeta)=\zeta$ in Theorem 2.15, we obtain

$$\left|T_{\Lambda,\Pi^{\Upsilon}_m,\Xi^{\Upsilon}_m}(0;\ell,\flat_1,\flat_2)\right| \le \frac{1}{\left[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{53}$$

$$\times\Big[B_1^{\Upsilon}(\ell;1)\Big]^{1-\frac{1}{q}}\sqrt[q]{\Big[B_1^{\Upsilon}(\ell;1)\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell)\Big]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\Big[B_2^{\Upsilon}(\ell;1)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_2^{\Upsilon}(\ell;1)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell)\Big]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}(\zeta-m\flat_1)(\Upsilon(\zeta)-\Upsilon(m\flat_1))\,d\zeta, \tag{54}$$

$$E_1^{\Upsilon}(\ell):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}(\zeta-m\ell)(\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta))\,d\zeta. \tag{55}$$

COROLLARY 2.23. For $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\alpha}}{\Gamma(\alpha)}$ in Theorem 2.15, we have

$$\big|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\big|\le\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)} \tag{56}$$

$$\times\Big[B_3^{\Upsilon}(\ell;1,\alpha)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_3^{\Upsilon}(\ell;1,\alpha)\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell,\alpha)\Big]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell,\alpha)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\big[\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)\big]^{\frac{q+1}{q}}\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)}\Big[B_4^{\Upsilon}(\ell;1,\alpha)\Big]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\Big[B_4^{\Upsilon}(\ell;1,\alpha)\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell,\alpha)\Big]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell,\alpha)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell,\alpha):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)}(\zeta-m\flat_1)\big[\Upsilon(\zeta)-\Upsilon(m\flat_1)\big]^{\alpha}\,d\zeta, \tag{57}$$

$$E_1^{\Upsilon}(\ell,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)}(\zeta-m\ell)\big[\Upsilon(m\ell+\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell))-\Upsilon(\zeta)\big]^{\alpha}\,d\zeta. \tag{58}$$

COROLLARY 2.24. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta^{\frac{\alpha}{\kappa}}}{\kappa\Gamma_{\kappa}(\alpha)}$ in Theorem 2.15, we get

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_5^{\Upsilon}(\ell;1,\alpha,\kappa)\right]^{1-\frac{1}{q}} \tag{59}$$

$$\times\sqrt[q]{\left[B_5^{\Upsilon}(\ell;1,\alpha,\kappa)\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell,\alpha,\kappa)\right]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell,\alpha,\kappa)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_6^{\Upsilon}(\ell;1,\alpha,\kappa)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_6^{\Upsilon}(\ell;1,\alpha,\kappa)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)-E_1^{\Upsilon}(\ell,\alpha,\kappa)\right]|\Lambda'(m\ell)|^q+E_1^{\Upsilon}(\ell,\alpha,\kappa)|\Lambda'(\flat_2)|^q},$$

where

$$C_1^{\Upsilon}(\ell,\alpha,\kappa):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}(\zeta-m\flat_1)\left[\Upsilon(\zeta)-\Upsilon(m\flat_1)\right]^{\frac{\alpha}{\kappa}}d\zeta, \tag{60}$$

$$E_1^{\Upsilon}(\ell,\alpha,\kappa):=\int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}(\zeta-m\ell)\left[\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\Upsilon(\zeta)\right]^{\frac{\alpha}{\kappa}}d\zeta. \tag{61}$$

COROLLARY 2.25. For $\lambda=0$, $\forall\xi\in[0,\zeta]$, $\varpi_{\Upsilon}(\ell,\zeta)=\zeta(\Upsilon(m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1))-\zeta)^{\alpha-1}$ and $\forall\xi\in[\zeta,1]$, $\varpi_{\Upsilon}(\ell,\zeta)=\zeta(\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\zeta)^{\alpha-1}$ in Theorem 2.15, we obtain

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Xi_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \leq \frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)} \tag{62}$$

$$\times\left[B_7^{\Upsilon}(\ell;1,\alpha)\right]^{1-\frac{1}{q}}\sqrt[q]{\left[B_7^{\Upsilon}(\ell;1,\alpha)\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)-C_1^{\Upsilon}(\ell)\right]|\Lambda'(m\flat_1)|^q+C_1^{\Upsilon}(\ell)|\Lambda'(\ell)|^q}$$

$$+\frac{1}{\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}\left[B_8^{\Upsilon}(\ell;1,\alpha)\right]^{1-\frac{1}{q}}$$

$$\times\sqrt[q]{\left[B_8^{\Upsilon}(\ell;1,\alpha)\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)-L_2^{\Upsilon}(\ell,\alpha)\right]|\Lambda'(m\ell)|^q+L_2^{\Upsilon}(\ell,\alpha)|\Lambda'(\flat_2)|^q},$$

where

$$L_2^{\Upsilon}(\ell,\alpha):=\int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}(\zeta-m\ell)\left[\Upsilon^{\alpha}(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))-\Upsilon^{\alpha}(\zeta)\right]d\zeta. \tag{63}$$

COROLLARY 2.26. Substituting $\lambda=0$ and $\varpi(\zeta)=\frac{\zeta}{\alpha}\exp(-A\zeta)$, where $A=\frac{1-\alpha}{\alpha}$

in Theorem 2.15, we have

$$\left|T_{\Lambda,\Pi_m^{\Upsilon},\Theta_m^{\Upsilon}}(0;\ell,\flat_1,\flat_2)\right| \le \frac{1}{(1-\alpha)\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)} \tag{64}$$

$$\times\left\{\left[B_9^{\Upsilon}(\ell;1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_3^{\Upsilon}(\ell,A)|\Lambda'(m\flat_1)|^q+L_4^{\Upsilon}(\ell,A)|\Lambda'(\ell)|^q}\right.$$

$$+\frac{1}{(1-\alpha)\left[\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)\right]^{\frac{q+1}{q}}\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\flat_1)}$$

$$\left.\times\left[B_{10}^{\Upsilon}(\ell;1,A)\right]^{1-\frac{1}{q}}\sqrt[q]{L_5^{\Upsilon}(\ell,A)|\Lambda'(m\ell)|^q+L_6^{\Upsilon}(\ell,A)|\Lambda'(\flat_2)|^q}\right\},$$

where

$$L_3^{\Upsilon}(\ell,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}\left(m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)-\zeta\right) \tag{65}$$

$$\times\left\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\right\}d\zeta,$$

$$L_4^{\Upsilon}(\ell,A):=\int_{m\flat_1}^{m\flat_1+\mathscr{F}_{\rho,\delta}^{\sigma}(\ell-m\flat_1)}(\zeta-m\flat_1)\left\{1-\exp\left[A\left(\Upsilon(m\flat_1)-\Upsilon(\zeta)\right)\right]\right\}d\zeta, \tag{66}$$

$$L_5^{\Upsilon}(\ell,A):=\int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}\left(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)-\zeta\right) \tag{67}$$

$$\times\left\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))\right)\right]\right\}d\zeta,$$

$$L_6^{\Upsilon}(\ell,A):=\int_{m\ell}^{m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell)}(\zeta-m\ell)\left\{1-\exp\left[A\left(\Upsilon(\zeta)-\Upsilon(m\ell+\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2-m\ell))\right)\right]\right\}d\zeta. \tag{68}$$

3. APPLICATIONS

For $\flat_1, \flat_2 \in \mathbb{R}$ and $0 < \flat_1 < \flat_2$ we recall:

(1) arithmetic mean:

$$\mathrm{A}(\flat_1, \flat_2) = \frac{\flat_1 + \flat_2}{2};$$

(2) harmonic mean:

$$\mathrm{H}(\flat_1, \flat_2) = \frac{2}{\frac{1}{\flat_1} + \frac{1}{\flat_2}};$$

(3) logarithmic mean:

$$\mathrm{L}(\flat_1, \flat_2) = \frac{\flat_2 - \flat_1}{\ln|\flat_2| - \ln|\flat_1|};$$

(4) generalized log-mean:

$$\mathrm{L}_r(\flat_1, \flat_2) = \left[\frac{\flat_2^{r+1} - \flat_1^{r+1}}{(r+1)(\flat_2 - \flat_1)}\right]^{\frac{1}{r}}; \; r \in \mathbb{R} \setminus \{-1, 0\}.$$

From the main results in Section 2, we get

PROPOSITION 3.1. Let $r, \flat_1, \flat_2 \in \mathbb{R}$ with $0 < \flat_1 < \flat_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\left|\mathrm{A}^r(\flat_1, \flat_2) - \mathrm{L}_r^r(\flat_1, \flat_2)\right| \leq \frac{r(\flat_2 - \flat_1)}{4\sqrt[p]{p+1}} \tag{69}$$

$$\times \left\{ \sqrt[q]{\mathrm{A}\left(\flat_1^{q(r-1)}, \left(\frac{\flat_1 + \flat_2}{2}\right)^{q(r-1)}\right)} + \sqrt[q]{\mathrm{A}\left(\left(\frac{\flat_1 + \flat_2}{2}\right)^{q(r-1)}, \flat_2^{q(r-1)}\right)} \right\}.$$

PROOF. Taking $m = 1$, $\lambda = 0$, $\ell = \frac{\flat_1 + \flat_2}{2}$, $\mathscr{F}_{\rho,\delta}^{\sigma}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}_{\rho,\delta}^{\sigma}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$, $\Lambda(\zeta) = \zeta^r$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ in Theorem 2.3, we get (69).

PROPOSITION 3.2. Let $r, \flat_1, \flat_2 \in \mathbb{R}$ with $0 < \flat_1 < \flat_2$, then for $r, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\left|\mathrm{A}(\flat_1^r, \flat_2^r) - \mathrm{L}_r^r(\flat_1, \flat_2)\right| \leq \frac{r(\flat_2 - \flat_1)}{4\sqrt[p]{p+1}} \tag{70}$$

$$\times\left\{\sqrt[q]{\mathrm{A}\left(\flat_1^{q(r-1)},\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathrm{A}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we have (70).

PROPOSITION 3.3. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\left|\frac{1}{\mathrm{A}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\frac{(\flat_2-\flat_1)}{4\sqrt[p]{p+1}} \tag{71}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we obtain (71).

PROPOSITION 3.4. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\frac{1}{\mathrm{H}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\frac{(\flat_2-\flat_1)}{4\sqrt[p]{p+1}} \tag{72}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.3, we get (72).

PROPOSITION 3.5. Let $r,\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$ and $r>1$, then for $q\geq1$, we obtain

$$\left|\mathrm{A}^r(\flat_1,\flat_2)-\mathrm{L}_r^r(\flat_1,\flat_2)\right|\leq\sqrt[q]{\frac{2}{3}}\frac{r(\flat_2-\flat_1)}{8} \tag{73}$$

$$\times\left\{\sqrt[q]{\mathrm{A}\left(\flat_1^{q(r-1)},2\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathrm{A}\left(2\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have (73).

PROPOSITION 3.6. Let $r,\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$ and $r>1$, then for $q\geq 1$, we get

$$\left|\mathrm{A}(\flat_1^r,\flat_2^r)-\mathrm{L}_r^r(\flat_1,\flat_2)\right|\leq\sqrt[q]{\frac{2}{3}}\frac{r(\flat_2-\flat_1)}{8}\tag{74}$$

$$\times\left\{\sqrt[q]{\mathrm{A}\left(2\flat_1^{q(r-1)},\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)}\right)}+\sqrt[q]{\mathrm{A}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{q(r-1)},2\flat_2^{q(r-1)}\right)}\right\}.$$

PROOF. Choosing $m=1$, $\lambda=1$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\zeta^r$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we obtain (74).

PROPOSITION 3.7. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q\geq 1$, we have

$$\left|\frac{1}{\mathrm{A}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\sqrt[q]{\frac{4}{3}}\frac{(\flat_2-\flat_1)}{8}\tag{75}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(2\flat_1^{2q},\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},2\flat_2^{2q}\right)}}\right\}.$$

PROOF. Taking $m=1$, $\lambda=0$, $\ell=\frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$, $\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we get (75).

PROPOSITION 3.8. Let $\flat_1,\flat_2\in\mathbb{R}$ with $0<\flat_1<\flat_2$, then for $q\geq 1$, we obtain

$$\left|\frac{1}{\mathrm{H}(\flat_1,\flat_2)}-\frac{1}{\mathrm{L}(\flat_1,\flat_2)}\right|\leq\sqrt[q]{\frac{4}{3}}\frac{(\flat_2-\flat_1)}{8}\tag{76}$$

$$\times\left\{\frac{1}{\sqrt[q]{\mathrm{H}\left(\flat_1^{2q},2\left(\frac{\flat_1+\flat_2}{2}\right)^{2q}\right)}}+\frac{1}{\sqrt[q]{\mathrm{H}\left(2\left(\frac{\flat_1+\flat_2}{2}\right)^{2q},\flat_2^{2q}\right)}}\right\}.$$

PROOF. Choosing $m=1,\lambda=1,\ell=\frac{\flat_1+\flat_2}{2},\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1,\Lambda(\zeta)=\dfrac{1}{\zeta}$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ in Theorem 2.15, we have (76).

Finally, we will find some new error estimations pertaining quadrature formula. Let denote $\mathscr{Q}:\flat_1=\varsigma_0<\varsigma_1<\ldots<\varsigma_k=\flat_2$. The following quadrature formulas are very useful in the sequel.

$$\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell=\mathrm{M}(\Lambda,\mathscr{Q})+\mathrm{E}(\Lambda,\mathscr{Q}),\quad \int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell=\mathrm{T}(\Lambda,\mathscr{Q})+\mathrm{E}^*(\Lambda,\mathscr{Q})$$

where

$$\mathrm{M}(\Lambda,\mathscr{Q}):=\sum_{j=0}^{k-1}\Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j),\quad \mathrm{T}(\Lambda,\mathscr{Q}):=\sum_{j=0}^{k-1}\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j),$$

and $\mathrm{E}(\Lambda,\mathscr{Q}),\mathrm{E}^*(\Lambda,\mathscr{Q})$ are denoted their corresponding errors.

PROPOSITION 3.9. Let $\Lambda:[\flat_1,\flat_2]\to\mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1<\flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right|\le\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{77}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. From Theorem 2.3 for $m=1,\lambda=0,\ell=\frac{\flat_1+\flat_2}{2},\mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell,\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ on $[\varsigma_j,\varsigma_{j+1}]$ $(j=0,\ldots,k-1)$ of $\mathscr{Q}$, we get

$$\left|\Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)-\frac{1}{\varsigma_{j+1}-\varsigma_j}\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell\right|\le\frac{(\varsigma_{j+1}-\varsigma_j)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{78}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

From (78), we have

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right| = \left|\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell - \mathrm{M}(\Lambda,\mathscr{Q})\right|$$

$$\leq \left|\sum_{j=0}^{k-1}\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \sum_{j=0}^{k-1}\left|\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell - \Lambda\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROPOSITION 3.10. Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1 < \flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q \geq 1$, we obtain

$$\left|\mathrm{E}(\Lambda,\mathscr{Q})\right| \leq \frac{1}{8\sqrt[q]{3}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{79}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + 2\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{2\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. The proof is analogous as to that of Proposition 3.9 taking $m=1$, $\lambda = 0$, $\ell = \frac{\flat_1+\flat_2}{2}$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\ell - m\flat_1) = \ell - m\flat_1$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\ell) = \flat_2 - m\ell$, $\mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2 - m\flat_1) = \flat_2 - m\flat_1$ and $\Upsilon(\zeta) = \zeta = \varpi(\zeta)$ using Theorem 2.15.

PROPOSITION 3.11. Let $\Lambda : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1 < \flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$\left|\mathrm{E}^*(\Lambda,\mathscr{Q})\right| \leq \frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}} \times \sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{80}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q + \left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q} + \sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q + |\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. By Theorem 2.3 for $m=1, \lambda=1, \ell=\frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ on $[\varsigma_j,\varsigma_{j+1}]$ $(j=0,\ldots,k-1)$ of $\mathscr{Q}$, we get

$$\left|\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}-\frac{1}{\varsigma_{j+1}-\varsigma_j}\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell\right| \leq \frac{(\varsigma_{j+1}-\varsigma_j)}{4\sqrt[q]{2}\sqrt[p]{p+1}} \tag{81}$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

From (81), we get

$$\left|\mathrm{E}^*(\Lambda,\mathscr{Q})\right| = \left|\int_{\flat_1}^{\flat_2}\Lambda(\ell)d\ell-\mathrm{T}(\Lambda,\mathscr{Q})\right|$$

$$\leq\left|\sum_{j=0}^{k-1}\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell-\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq\sum_{j=0}^{k-1}\left|\left\{\int_{\varsigma_j}^{\varsigma_{j+1}}\Lambda(\ell)d\ell-\frac{\Lambda(\varsigma_j)+\Lambda(\varsigma_{j+1})}{2}(\varsigma_{j+1}-\varsigma_j)\right\}\right|$$

$$\leq\frac{1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\times\sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2$$

$$\times\left\{\sqrt[q]{|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROPOSITION 3.12. Let $\Lambda:[\flat_1,\flat_2]\to\mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$, where $\flat_1<\flat_2$. If $|\Lambda'|^q$ is convex on $[\flat_1,\flat_2]$, then for $q\geq 1$, we obtain

$$\left|\mathrm{E}^*(\Lambda,Q)\right|\leq\frac{1}{8\sqrt[q]{3}}\times\sum_{j=0}^{k-1}(\varsigma_{j+1}-\varsigma_j)^2 \tag{82}$$

$$\times\left\{\sqrt[q]{2|\Lambda'(\varsigma_j)|^q+\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q}+\sqrt[q]{\left|\Lambda'\left(\frac{\varsigma_j+\varsigma_{j+1}}{2}\right)\right|^q+2|\Lambda'(\varsigma_{j+1})|^q}\right\}.$$

PROOF. The proof is analogous as to that of Proposition 3.11 taking $m=1, \lambda=1, \ell=\frac{\flat_1+\flat_2}{2}, \mathscr{F}^{\sigma}_{\rho,\delta}(\ell-m\flat_1)=\ell-m\flat_1, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\ell)=\flat_2-m\ell, \mathscr{F}^{\sigma}_{\rho,\delta}(\flat_2-m\flat_1)=\flat_2-m\flat_1$ and $\Upsilon(\zeta)=\zeta=\varpi(\zeta)$ using Theorem 2.15.

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Applying fractional calculus to analyze final consumption and gross investment influence on GDP

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Abstract

This paper points out the possibility of suitable use of Caputo fractional derivative in regression model. Fitting historical data using a regression model seems to be useful in many fields, among other things, for the short-term prediction of further developments in the state variable. Therefore, it is important to fit the historical data as accurately as possible using the given variables. Using Caputo fractional derivative, this accuracy can be increased in the model described in this paper.

Mathematics Subject Classification 2010: 26A33, 26A51, 26D15.
Keywords: Caputo derivative, Method of least squares, Regression model.

1. INTRODUCTION

Fractional (fractional-order) derivative also known as derivative with memory appears to be a powerful tool for examining the development of economic indicators (such as GDP) because of 'memory'. It has been proved that fractional models [Hilfer et al. 2000] are better then integer models, giving us a great opportunity to use it. It turns out that fractional calculus is also advantageous to use in regression models, which then provide better accuracy for modeling variable of interest based on a set of predictor variables than can be seen in [Luo et al. 2018].

In [Anghelache et al. 2015], the authors study GDP evolution for the Romanian case by using multiple linear regression model with final consumption value and the value of gross investment as independent variables via data between 1990 and 2014.

In the present paper, we go on the study of final consumption and gross investment influence on GDP for the Romanian case by using Caputo fractional derivative. We note that the Romanian case is used in this paper only for possibility to compare our achievement and proposed model with previous one.

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2. APPROXIMATION TO CAPUTO FRACTIONAL DERIVATIVE BASED ON POLYNOMIAL INTERPOLATION

From the definition of the Caputo derivative, we can find that the αth-order $(m-1<\alpha<m)$ Caputo derivative of the given function $f(t)$ can be seen as the $(m-\alpha)$th-order fractional integral of the function $f^{(m)}(t)$. In our case, we decided to use fractional rectangular formula.

For $0<\alpha<1$ we get the following formula

$$\left[{}_{C}D^{\alpha}_{0,t}f(t)\right]_{t=t_n} \approx \sum_{k=0}^{n-1} w_{n-k-1}\delta_t f(t_k), \tag{1}$$

where

$$w_i = \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\left[(i+1)^{1-\alpha} - i^{1-\alpha}\right]$$

and

$$\delta_t f(t_k) = \frac{f(t_{k+1}) - f(t_k)}{\Delta t} \approx f'(t_k).$$

3. REGRESSION MODELS WITH FRACTIONAL-ORDER DERIVATIVES

Throughout of this paper, we denote GDP by y, final consumption by x_1 and gross investments by x_2.

The modification of original model leads to

$$y(t) = \beta_0 + \beta_1\, {}_{C}D^{\alpha_1}_{t_0,t}x_1(t) + \beta_2\, {}_{C}D^{\alpha_2}_{t_0,t}x_2(t) + \varepsilon(t), \tag{2}$$

where β_i $(i=0,1,2)$ are regression coefficients, α_1, $\alpha_2 \in [0,1]$ are unknown orders and $\varepsilon(t)$ is a function of residuals.

We also denote the mean square error by MSE, the coefficient of determination by R^2, the adjusted coefficient of determination by $\bar{R}^2$, the mean absolute deviation by MAD and Akaike Information Criterion by AIC.

Define

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=0}^{n}(y_i - \hat{y}_i)^2,$$

$$R^2 = 1 - \frac{\sum_{i=0}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=0}^{n}(y_i - \bar{y})^2},$$

$$\bar{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-k-1},$$

$$\text{MAD} = \frac{1}{n}\sum_{i=0}^{n} |y_i - \hat{y}_i|,$$

$$\text{AIC} = n \ln\left(\frac{1}{n}\sum_{i=0}^{n}(y_i - \hat{y}_i)^2\right) + 2k + \frac{2k(k+1)}{n-k-1},$$

where n denotes the sample size and k denotes the number of parameters.

3.1. Results

The least squares method gives the following estimates of coefficients and orders of the fractional operators (see Table I).

	original model	model (2)
β_0	-2.14384	82.43610
β_1	1.16311	1.34269
β_2	0.32493	-10.88530
α_1	–	4.41093×10^{-17}
α_2	–	0.58188

Table I. Estimates of coefficients and orders of the fractional operators.

Results obtained by using (1)-(2) are as follows (see Table II).

	original model	model (2)
MSE	22.8812	19.9679
R^2	0.98308	0.98523
$\bar{R}^2$	0.98154	0.98389
MAD	4.21674	3.82673
AIC	85.3488	81.9441

Table II. Performance indices for the Romanian economy.

Now, when we know the coefficients and orders of the fractional operators, we are ready to give the fitting results (see Figure 1).

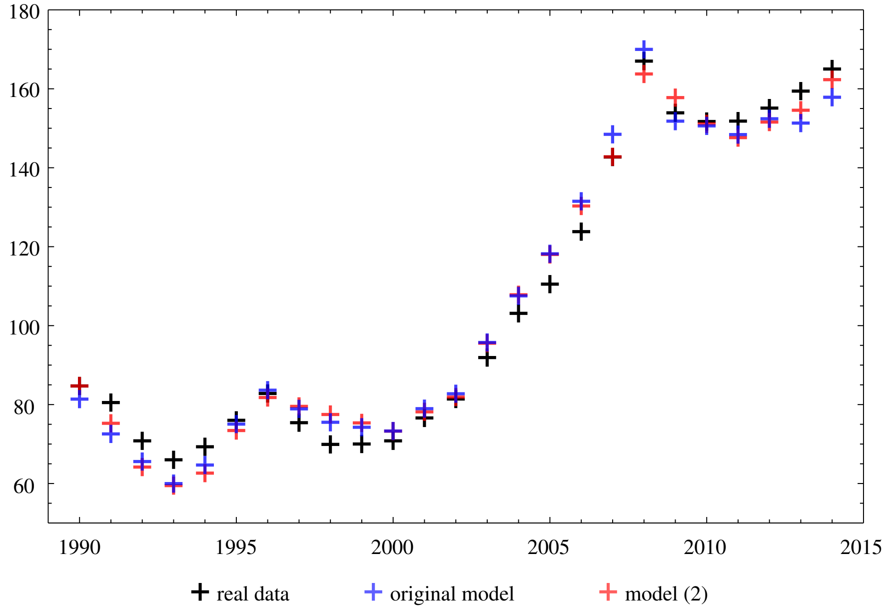

Fig. 1. Data fitting.

One can see that the simulation results of original and model (2) are close to real data. However, R^2 of modified model is closer to 1 than R^2 of original model. Thus, model (2) is better than original model.

3.2. More General Models

We consider extension of (2) in a form

$$\begin{aligned} y(t) = \beta_0 + \beta_{1,1}\, {}_C D_{t_0,t}^{\alpha_{1,1}} x_1(t) + \beta_{1,2}\, {}_C D_{t_0,t}^{\alpha_{1,2}} x_1(t) \\ + \beta_{2,1}\, {}_C D_{t_0,t}^{\alpha_{2,1}} x_2(t) + \beta_{2,2}\, {}_C D_{t_0,t}^{\alpha_{2,2}} x_2(t) + \varepsilon(t), \end{aligned} \tag{3}$$

where β_0, $\beta_{1,i}$, $\beta_{2,i}$ $(i = 0,1,2)$ are regression coefficients, $\alpha_{1,i}$, $\alpha_{2,i} \in [0,1]$ $(i = 0,1,2)$ are unknown orders and $\varepsilon(t)$ is a function of residuals. In general, we have

$$y(t) = \beta_0 + \sum_{i=1}^{p} \beta_{1,i}\, {}_C D_{t_0,t}^{\alpha_{1,i}} x_1(t) + \sum_{j=1}^{q} \beta_{2,j}\, {}_C D_{t_0,t}^{\alpha_{2,j}} x_2(t) + \varepsilon(t) \tag{4}$$

for $\alpha_{1,i}$, $\alpha_{2,j} \in [0,1]$, $i = 1,\dots,p$, $j = 1,\dots,q$. Letting $p \to \infty$ and $q \to \infty$, we get

$$y(t) = \beta_0 + \int_0^1 \beta_1(\alpha_1)\, {}_C D_{t_0,t}^{\alpha_1} x_1(t)\, d\alpha_1 + \int_0^1 \beta_2(\alpha_2)\, {}_C D_{t_0,t}^{\alpha_2} x_2(t)\, d\alpha_2 + \varepsilon(t).$$

Due to the computational complexity of the least squares method for all parameters, we chose a different method for estimating the models parameters and we get the following estimates of coefficients and orders of the fractional operators (see Table III). Note that there is no proof of the best estimate of parameters obtained using this method.

	model (3)	model (4), $p=q=3$	model (4), $p=q=4$	model (4), $p=q=5$
β_0	83.20526	84.04196	85.22125	85.15662
$\beta_{1,1}$	1.47409	19.39253	72.97881	31.97668
$\beta_{1,2}$	−1.27170	−40.21083	−169.16983	−60.53592
$\beta_{1,3}$	–	52.15342	303.43792	−9.41558
$\beta_{1,4}$	–	–	−2333.38776	−33494.91513
$\beta_{1,5}$	–	–	–	25613.50872
$\beta_{2,1}$	−4.15815	421.22459	2453.17281	−124.40979
$\beta_{2,2}$	3.28790	−103.72563	−556.75607	−30.71726
$\beta_{2,3}$	–	−341.06431	−2106.16932	264.10137
$\beta_{2,4}$	–	–	839.70531	−2685.75423
$\beta_{2,5}$	–	–	–	9747.93546
$\alpha_{1,1}$	3.1716×10^{-11}	1.4432×10^{-14}	8.2771×10^{-10}	1.49024×10^{-9}
$\alpha_{1,2}$	0.09639	0.08246	0.08246	0.08246
$\alpha_{1,3}$	–	0.35410	0.35408	0.35413
$\alpha_{1,4}$	–	–	0.99967	0.99967
$\alpha_{1,5}$	–	–	–	0.94407
$\alpha_{2,1}$	0.07450	0.07450	0.07450	0.07450
$\alpha_{2,2}$	1.0001×10^{-20}	0.00561	0.00561	0.00561
$\alpha_{2,3}$	–	0.11702	0.11702	0.11702
$\alpha_{2,4}$	–	–	0.68561	0.68561
$\alpha_{2,5}$	–	–	–	0.99984

Table III. Estimates of coefficients and orders of the fractional operators.

Results obtained by using (1)-(4) are as follows (see Table IV).

	original model	model (2)	model (3)
MSE	22.8812	19.9679	6.8990
R^2	0.98308	0.98523	0.99490
$\bar{R}^2$	0.98154	0.98390	0.99388
MAD	4.21674	3.82673	2.12068
AIC	85.3488	81.9441	58.2843

	model (4), $p = q = 3$	model (4), $p = q = 4$	model (4), $p = q = 5$
MSE	3.53354	1.51559	1.45406
R^2	0.99739	0.99888	0.99893
$\bar{R}^2$	0.99652	0.99832	0.99816
MAD	1.42666	0.96946	0.93319
AIC	48.2242	35.3951	45.0733

Table IV. Performance indices for the Romanian economy.

Now, when we know the coefficients and orders of the fractional operators, we are ready to give the fitting results (see Figure 2).

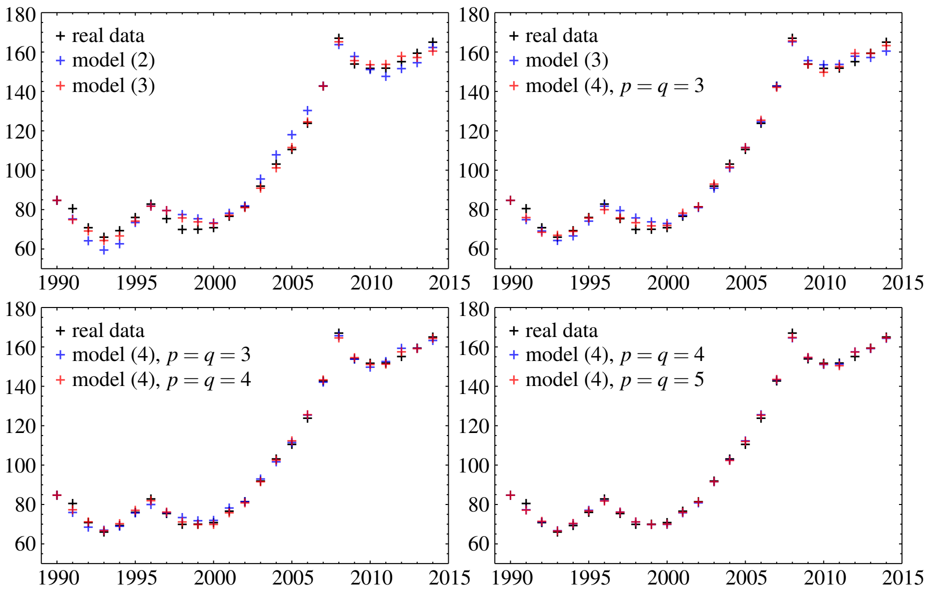

Fig. 2. Data fitting.

The simulation results of models where we estimated the parameters are close to real data. Due to the different count of parameters of models, we can use adjusted coefficient of determination for the model comparison to each other.

One can see, that data obtained by model (4) with $p = q = 4$ better fits real data than data obtained by other models, which is also indicated by values of $\bar{R}^2$ of these models. $\bar{R}^2$ of model (4) with $p = q = 4$ is closer to 1 than $\bar{R}^2$ of any other model where we estimated the parameters. Thus, model (4) with $p = q = 4$ is better then any other model where we estimated the parameters.

4. CONCLUSIONS

This paper studies a final consumption and gross investment influence on GDP for the Romanian case. Based on our results, it is shown that using the Caputo fractional derivative is convenient in this case. In addition, the data of general models are better than the data of original model from [Anghelache et al. 2015].

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Doubly stochastic matrices and the quantum channels

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Abstract

The main object of this paper is to study doubly stochastic matrices with majorization and the Birkhoff theorem. The Perron-Frobenius theorem on eigenvalues is generalized for doubly stochastic matrices. The region of all possible eigenvalues of n-by-n doubly stochastic matrix is the union of regular (n – 1) polygons into the complex plane. This statement is ensured by a famous conjecture known as the Perfect-Mirsky conjecture which is true for n = 1, 2, 3, 4 and untrue for n = 5. We show the extremal eigenvalues of the Perfect-Mirsky regions graphically for n = 1, 2, 3, 4 and identify corresponding doubly stochastic matrices. Bearing in mind the counterexample of Rivard-Mashreghi given in 2007, we introduce a more general counterexample to the conjecture for n = 5. Later, we discuss different types of positive maps relevant to Quantum Channels (QCs) and finally introduce a theorem to determine whether a QCs gives rise to a doubly stochastic matrix or not. This evidence is straightforward and uses the basic tools of matrix theory and functional analysis.

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1. INTRODUCTION

Stochastic and doubly stochastic matrices are mostly studied matrices for many years. Motivation has come from pure mathematics. The concept of stochastic matrices, a special type of nonnegative matrices, was introduced by Andrey Markov when he was working on the well-known mathematical system of the Markov chain in 1906. His intention of using these types of matrices was only for linguistic analysis and card shuffling. In 2017, the notion of steady-state is explored in connection with the long-run distribution, behavior of the Markov chain, and predictions based on Markov chains with more than two states are examined, followed by a discussion of the notion of absorbing Markov chains in Gagniuc [10]. Later Andrey Kolmogorov gave rise to developments of this type of matrices extending the possibilities of uses for continuous-time Markov processes [14].

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As they were found to have a variety of applications in the fields like probability theory, statistics, mathematical finance, and linear algebra, as well as economics, engineering, computer science, population genetics, and QI; many of the researchers were intensely motivated by this. Prashanth et. al. [25] introduced some results using eigenvalues of signed graph using the number of vertices. In addition to these applications, stochastic modeling is an interesting and challenging domain of statistics and data science, for more details see Bhuiyan [3]. Moreover, the spectral properties of stochastic matrices appeared of great interest in the various domains of mathematics. The most facetious thing about the spectral properties of those matrices is the location of eigenvalues for different n. In 1936, Romanovsky [26] first tried to characterize the location of eigenvalues of stochastic matrices in the complex plane and he managed to find the points on the circumference of the unit circle where some eigenvalues of a different order of stochastic matrices may lie. But the problem of finding the possible area covering all the proper values of stochastic matrices was first suggested by Kolmogorov in 1938.

In 1946, Dmitriev et. al. [9] tried to mark the region, denoted by Θ_n, given by the subset of the complex plane containing all possible eigenvalues of all n-by-n stochastic matrices. They managed to find that area in part. In 1938, Kolmogorov raised the question that what point of the circle of unity may serve as the characteristic root of a stochastic matrix. Later in 1946, Dmitriev and Dynkin [9] solved the problem completely for $n \leq 4$ and partially for $n > 5$. On the basis of Dmitriv and Dynkin problem, Karpelevich [13] solved a similar problem completely and found the Karpelevich's region $\mathbb{K}_n$ in 1949. Karpelevič was able to solve the problem completely, but a new statement about it was made by Ito [11] in 1997. Some time later, Perfect et. al. [23] considered the same problem for double stochastic matrices, i.e., they tried to characterize the region $\omega_n \subset C$ containing all the eigenvalues of n-by-n doubly stochastic matrices. They gave a conjecture about the possible region known as the Perfect-Mirsky conjecture. Recently, in 2015, Perfect-Mirsky Conjecture on the structure of the set of eigenvalues for all n-by-n doubly stochastic matrices in the four-dimensional case were studied by Levick J., Rajesh Pereira and David W. Kribs [19] and based on the analysis they made new conjectures for the general case. In this article, a detailed description of these results along with the diagrams of the regions are given to discuss matrices with extremal characteristic roots.

In the Quantum Information (QI) theory and Quantum system, a very important notion of quantum mechanics is the QCs are a special type of communication channel that delivers information from sender to receiver. The information conveyed by QCs is described by the quantum states of a quantum system. In addition, QCs can be regarded as a special type of positive maps on the matrix space. In this article, we discuss the idea of quantum information and the various types of positive maps concerning QCs. We also generalize the requirement that a QCs produces or does not produce a doubly stochastic matrix. An algebraic point of view, a QCs is a completely positive trace-preserving map that acts on space and gives the output of information to another space. Moreover, an euclidean formulation of relativistic quantum mechanics for systems of a finite number of degrees of freedom is considered in [27]. Based on unitarily, there may be two types of QCs: unital and non-unital. The initial QCs maintain the average of a quantum state. On the other hand, non-initial QC does not. As there is a connection between the positive trace-preserving initial maps and doubly stochastic matrices [5] there might have a connection between initial QCs and doubly stochastic matrices. In this paper, we will discuss the relationship between them. The remainder of this paper is divided into five sections. We first introduce some preliminary definitions and some relevant theorems as well as majorization illustrating with an example. In section three, we state the Perron-Frobenius theorem of nonnegative matrices. For showing the region of characteristic roots of such matrices, we first recall a theorem of Karpelevich regarding stochastic matrices. Then, we discuss the Perfect-Mirsky conjecture and show the matrices with extremal roots graphically in the star shaped region. In section four, we introduce a more general form of counterexamples for Perfect-Mirsky conjecture and compare it with that of previously found Rivard-Mashreghi counterexample for $n = 5$. Section five is devoted to the quantum channels and discusses the connection between doubly stochastic matrices and trace preserving initial positive maps. We finally generalize the condition, whether a quantum channel gives rise to doubly stochastic matrix or not by a theorem.

2. MATHEMATICAL BACKGROUND

A special type of real matrices whose all entries are positive, called positive matrices. The stochastic matrices are the simplest form of positive matrices in which certain conditions are implemented. There are two simple types of positive matrices row (column) stochastic matrices and doubly stochastic matrices. Each entry of such matrices represents a probability. In this section, we discuss doubly stochastic matrices and the famous theorem of Birkhoff is regarding those types of matrices. We define majorization for one dimensional and multidimensional case and formulate a lemma connecting doubly stochastic matrices and majorization. (For more details, see Armandnejad et. al. [1].)

DEFINITION 1. A matrix $A=(a_{ij}) \in M_n(\Re)$ is called row (column) stochastic if the sum of each of its rows (columns) equals to 1. That is,

$(i)\ a_{ij} \geq 0,\ for\ i, j = 1, \ldots \ldots \ldots, n.$

$(ii) \sum_{j=1}^{n} a_{ij} = 1,\ for\ i = 1, \ldots \ldots \ldots, n \ \left(\sum_{i=1}^{n} a_{ij} = 1\ for\ j = 1, \ldots \ldots \ldots, n \right).$

The set of all $n \times n$ row (column) stochastic matrices is denoted by $\Omega_n^{row}(\Omega_n^{column})$.

DEFINITION 2. A matrix $A \in M_n(\Re)$ is called doubly stochastic if the sum of each of its rows and columns equals to 1 i. e. $(i) a_{ij} \geq 0,$ for $i, j = 1, \ldots n$, and $(ii) \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij} = 1, for i, j = 1, \ldots n.$

Doubly stochastic matrices are also known as Bistochastic matrices, Schur transformation. The set of all $n \times n$ doubly stochastic matrices is called Birkhoff polytope and is denoted by Ω_n. A special example of doubly stochastic matrices are the permutation matrices, the square matrices whose entries are all either 0 or 1 and which contain exactly a 1 in each row and each column.

THEOREM 1. (Mehlum [22]) The set of all $n \times n$ doubly stochastic matrices is convex.

THEOREM 2. (Birkhoff [2]) The set Ω_n of doubly stochastic matrices of size n-by-n is the convex hull of the n-by-n permutation matrices.

To get a better understanding of the Birkhoff theorem of doubly stochastic matrices and their importance for this paper, we present an example to illustrate how the doubly stochastic matrices can be expressed as the convex combination of permutation matrices. This is illustrated by the following example:

EXAMPLE 1. Illustrate the doubly stochastic matrix $\begin{pmatrix} \mathbf{\frac{1}{10}} & \mathbf{\frac{3}{10}} & \mathbf{\frac{3}{5}} \\ \mathbf{\frac{3}{5}} & \mathbf{\frac{1}{10}} & \mathbf{\frac{3}{10}} \\ \mathbf{\frac{3}{10}} & \mathbf{\frac{3}{5}} & \mathbf{\frac{1}{10}} \end{pmatrix}$ for Birkhoff theorem.

Proof: To illustrate the Birkhoff theorem, set $A_0 = \begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{5} & \frac{1}{10} \end{pmatrix}$, then we have the following:

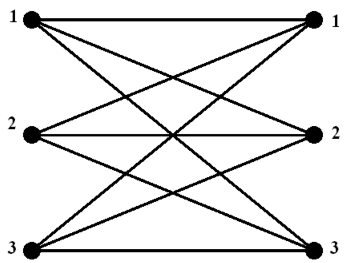

Fig. 1: Bipartite graph associated to A_0 matrix

A perfect matching is $\{(1,1),(2,3),(3,2)\}$ and the corresponding permutation matrix is $P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The smallest entry of A_0 corresponding to the non-zero entries of P_0 is $\alpha_0 = \frac{1}{10}$ (If $\alpha_0 = 1$, then we would have $A_0 = P_0$), then we get the following doubly stochastic matrix,

$$A_1 = \frac{1}{1-\alpha_0}\left(A_0 - \alpha_0 P_0\right) = \frac{10}{9}\left[\begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{5} & \frac{1}{10} \end{pmatrix} - \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & 0 & \frac{1}{10} \\ 0 & \frac{1}{10} & 0 \end{pmatrix}\right] == \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{5}{9} & \frac{1}{9} \end{pmatrix}$$ and we have the bipartite graph associated to A_1 in the following:

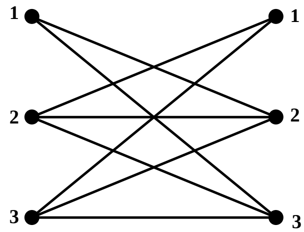

Fig. 2: Bipartite graph associated to A_1 matrix

A perfect matching is $\{(1,2),(2,1),(3,3)\}$, and the corresponding permutation matrix is $P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then we have $A_2 = \frac{1}{1-\alpha_1}(A_1 - \alpha_1 P_1) = \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{5}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{3}{8} & \frac{5}{8} & 0 \end{pmatrix}$ of A_1 corresponding to the non-zero entries of P_1 is $\alpha_1 = \frac{1}{9}$. Bipartite graph associated to A_2 is

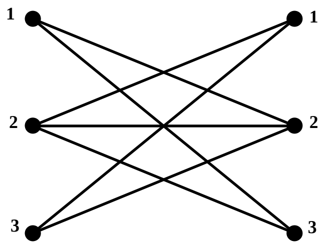

Fig. 3: Bipartite graph associated to A_2 matrix

A perfect matching is $\{(1,3),(2,2),(3,1)\}$ and the corresponding permutation matrix is $P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We then get $A_3 = \frac{1}{1-\alpha_2}(A_2 - \alpha_2 P_2) = \begin{pmatrix} 0 & \frac{2}{7} & \frac{5}{7} \\ \frac{5}{7} & 0 & \frac{2}{7} \\ \frac{2}{7} & \frac{5}{7} & 0 \end{pmatrix}$ by choosing the smallest entry of A_2 and then Bipartite graph associated to A_3 is following.

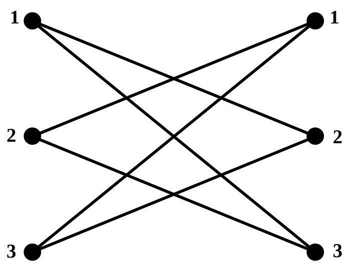

Fig. 4: Bipartite graph associated to A_3 matrix

A perfect matching is $\{(1,3),(2,1),(3,2)\}$ and the corresponding permutation matrix is $P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The smallest entry of A_3 to the non-zero entries of P_3 is $\alpha_3 = \frac{5}{7}$, then we get $A_4 = \frac{1}{1-\alpha_2}(A_2 - \alpha_2 P_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = P_4$. Thus,

$A_3 = \alpha_3 P_3 + (1-\alpha_3)A_4 = \frac{5}{7}P_3 + \frac{2}{7}P_4, \quad A_4 = P_4$,

$A_2 = \alpha_2 P_2 + (1-\alpha_2)A_3 = \frac{1}{8}P_2 + \frac{5}{8}P_3 + \frac{2}{8}P_4$,

$A_1 = \alpha_1 P_1 + (1-\alpha_1)A_2 = \frac{1}{9}P_1 + \frac{1}{9}P_2 + \frac{5}{9}P_3 + \frac{2}{9}P_4$, and finally we get

$A_0 = \alpha_0 P_0 + (1-\alpha_0)A_1 = \frac{1}{10}P_0 + \frac{1}{10}P_1 + \frac{1}{10}P_2 + \frac{5}{10}P_3 + \frac{2}{10}P_4$. Therefore, we finally have

$$\begin{pmatrix} \frac{1}{10} & \frac{3}{10} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{3}{5} & \frac{1}{10} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{5}{10}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \frac{2}{10}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$.

We now describe how two vectors X and Y are related to each other by majorization and doubly stochastic matrices. Normally, it measures which of the vectors of X or Y is "more or less spread out". For example, in economics, the majorization is used to compare the income distribution of two groups of the population.

DEFINITION 3. Let $X = (x_1, x_2, \dots\dots\dots, x_n)$ and $Y = (y_1, y_2, \dots\dots\dots, y_n)$ be the n-tuples of real numbers. Then we say that X is majorized by Y, written $X \prec Y$, if $\sum_{i=1}^{k} x_i^{\downarrow} = \sum_{i=1}^{k} y_i^{\downarrow}$; $1 \rightleftarrows\leq k \leq n$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$. The notation $X^{\downarrow}$ means that the entries of X are arranged in descending order. We present now a real life example for explaining the notion of majorization in our own way as follows:

EXAMPLE 2. Consider two Cricket teams in Oklahoma State University, Stillwater, OK, USA namely, Cowboy cricket club A, and Cowboy cricket club B. Both teams scored 150 runs in a $20-20$ over match, resulted in a tie. But someone may try to identify which team was better. The question is: can we really identify properly which team was better?

Two teams played 20 overs, scored 150, but they scored different runs in different phase of the game. Let us divide the phases in terms of overs. That is, the scoring rate in each over is not equally distributed. We re-arrange the runs scored in each over in an ascending sequence. x_i and y_i represent the runs in the i^{th} over of team A and B respectively. Now we compute the relative runs of the i^{th} over to be sum of the i^{th} smallest overs divided by total runs. We represent this numbers by $\overline{x_i}$ and $\overline{y_i}$ respectively. We also define n_i to be the proportion of overs having runs $x_i(y_i)$ or less, such that n_n denotes the whole overs. Then we plot the pairs $(n_i, \overline{x_i})$ and $(n_i, \overline{y_i})$, $i = 1, \dots\dots\dots, 20$ in the same axes of coordinates. We add a straight line which represents the overs where scores are at level. The associated graph shown in the following:

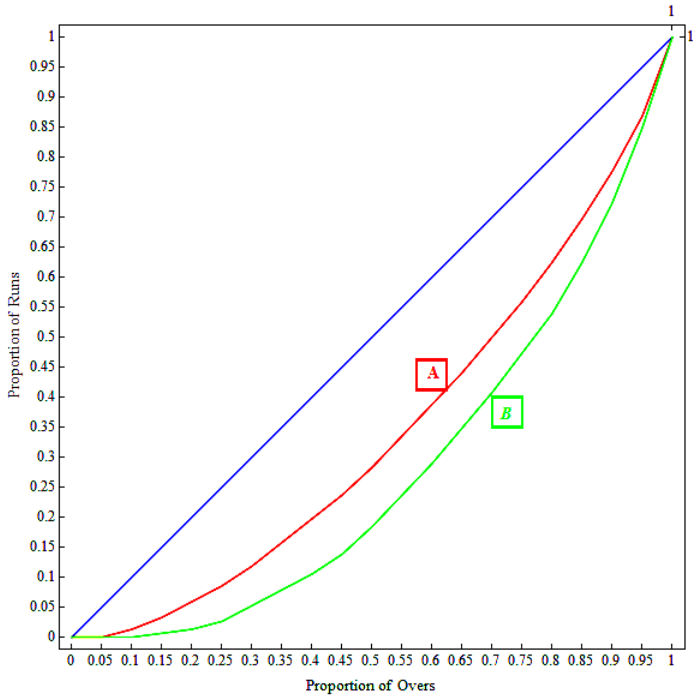

Fig. 5: Comparison of two *Cowboy Cricket Clubs* over by over

we observe that the graph of A is less convex than the graph of B or B spread out more than A Using this concept, we can conclude that A is more evenly distributed than B or A is a better team than B Although the idea of majorization comes from the comparison of two distinct objectives such as their income, height or other things, this concept is extended for a vector of more objects or points. But the problem for solving the comparison for a multidimensional array is more complicated than that of the single vector case. For a multidimensional case or simply for a matrix, the majorization is characterized differently using the concept of doubly stochastic matrices.

THEOREM 3. Let $\boldsymbol{X} = \{\boldsymbol{x}: \boldsymbol{X} \prec \boldsymbol{Y}\}$. Then the set X is the convex hull of points generated by permuting the points of Y. (Marshall [20])

DEFINITION 4. Let A and B be two $m \times n$ real matrices. Then we say that A is majorized by B if there exists at least one doubly stochastic matrix D of order $m \times m$ such that $A = BD$.

The definition above is equivalent to saying that A is row (column) majorized by B if each row (column) of A is majorized by the corresponding row (column) of B. For more details characterization of row or column majorization one can see Armandnejad et. al. [1], and Dahl [8]. Linking up theorem 3 and definitions above, we have the following relation between majorization and doubly stochastic matrices.

LEMMA 1. Let $x, y \in \Re^n_{\geq 0}$. Then the followings are equivalent:

1. $x \prec y$
2. $x \in Conv\{y\}$
3. $x = yD$ for some doubly stochastic matrix $D \in \Omega_n$.

3. REGIONS OF CHARACTERISTIC ROOTS OF DOUBLY STOCHASTIC MATRICES

One of the most important theorems in matrix analysis is the Perron-Frobenius theorem, a theorem regarding the eigenvalues and corresponding eigenvectors of non-negative matrices. This theorem also has a similar assertion for different classes of non-negative matrices. In 2012, Cheng et.al. [4], provide a simple proof for the Perron-Frobenius theorem concerned with positive matrices using a homotopy technique. In this section, we give our observation of the Perron-Frobenius theorem in the case of stochastic and doubly stochastic matrices. We discuss the Perfect-Mirsky conjecture for doubly stochastic matrices and show the extremal characteristic roots of ω_n for $n \geq 2$.

THEOREM 4. (Perron-Frobenius theorem, (Cheng et. Al. [4])) If $A = \left(a_{ij}\right)$ is a real $n \times n$ non-negative matrix, then *(i)* A has a non-negative eigenvalue λ which is equal to the spectral radius of A

(ii) There is a unique eigenvector corresponding to λ whose all entries are non-negative.

In our case, the simple observation of Perron-Frobenius theorem is that

(i) Every stochastic and doubly stochastic matrix has spectral radius 1.

(ii) The eigenvector corresponding to the spectral radius is $(1,1,\dots\dots\dots,1)^T$.

Perfect-Mirsky Conjecture.

It was observed that the subset of C which may contain the eigenvalues of an $n \times n$ doubly stochastic matrix is a subset of the characteristic region of an $n \times n$ stochastic matrix. Also, the characteristic region of $(n-1) \times (n-1)$ doubly stochastic matrices is a subset of the characteristic region of $n \times n$ doubly stochastic matrices.

DEFINITION 5. Let A be an $n \times n$ doubly stochastic matrix and λ be any complex number inside the unit circle. Then the set of eigenvalues of A consists of the collection of all $\lambda's$ if there exists a v corresponding to λ such that $Av = \lambda v$.

$$i.e, \quad \omega_n = \{\lambda \in C : \exists A \in \Omega_n, \ Av = \lambda v\}.$$

DEFINITION 6. The region $\prod_n$ is the closed region whose boundary is the regular n-gon circumscribed in the unit circle with vertices at $\left\{1, e^{\frac{i2\pi}{n}}, e^{\frac{i4\pi}{n}}, \dots\dots\dots, e^{\frac{i2(n-1)\pi}{n}}\right\}$. That is,

$$\prod_n = Conv\left\{e^{\frac{i2\pi k}{n}} : 0 \le k \le n-1\right\}.$$

CONJECTURE 1. (Perfect and Mirsky [23]) $\omega_n = \bigcup_{i=1}^{n} \prod_i$.

The Perfect-Mirsky conjecture is trivially true for $n = 1,2$ and Perfect and Mirsky proved the conjecture for $n = 3$. In 2015 Levick et. al. [19], Levick [18] in his Ph.D.

thesis, and latter in 2016 Levick [17] proved this conjecture for n = 4, and the conjecture is false for n = 5. Still for the higher values of n, the conjecture is unknown.

Matrices with Extremal Eigenvalues

According to the Perfect-Mirsky conjecture the region ω_n is the union of the n-gons in the unit circle. For $n = 1$, the matrix $A = [1]$ has the requirement of doubly stochastic matrices and its only eigenvalue is 1, located at the point $(1,0)$ on the arc of the unit circle. To discuss the matrices with extremal characteristic roots for $n \geq 2$, we first define doubly stochastic circulant matrix slightly modifying the definition of circulant matrix in Kra and Simanca [15] as follows:

DEFINITION 7. Let us consider a row vector

$$v = \left\{(v_0, v_1, \dots \dots \dots, v_{n-1}) \in \Re^n : v_i \in [0, 1], \sum_{i=0}^{n-1} v_i = 1\right\}.$$

and define a shift operator $\sigma(v_0, v_1, \dots\dots\dots, v_{n-1}) = (v_{n-1}, v_0, \dots\dots\dots, v_{n-2})$. Then the doubly stochastic circulant matrix $A = cir(v)$ associated to the vector $v \in \Re^n$ is the $n \times n$ matrix whose rows are generated by the shift operator σ defined on v; its i^{th} row is $\sigma^{i-1}(v), i = 1, \dots\dots\dots, n$:

$$A = \begin{pmatrix} v_0 & v_1 & \cdots & \cdots & \cdots & v_{n-2} & v_{n-1} \\ v_{n-1} & v_0 & \cdots & \cdots & \cdots & v_{n-3} & v_{n-2} \\ \vdots & \vdots & & & & \vdots & \vdots \\ \vdots & \vdots & & & & \vdots & \vdots \\ \vdots & \vdots & & & & & \vdots \\ v_2 & v_3 & \cdots & \cdots & \cdots & v_0 & v_1 \\ v_1 & v_2 & \cdots & \cdots & \cdots & v_{n-1} & v_0 \end{pmatrix}$$

PROPOSITION 1. For $n \geq 3$, the n-gon described in definition 6 corresponds $\frac{n}{2}(when\ n\ is\ even)$ or $\frac{n+1}{2}(when\ n\ is\ odd)$ doubly stochastic circulant matrices.

The sides connected with the point 1 represent a trace non-zero doubly stochastic circulant matrix with at most two non-zero adjacent entries. The other sides of the

n-gon represent trace zero doubly stochastic circulant matrices with at most two non-zero adjacent entries. For example, in the heptagon ($\prod_7$) for $n = 7$, we get

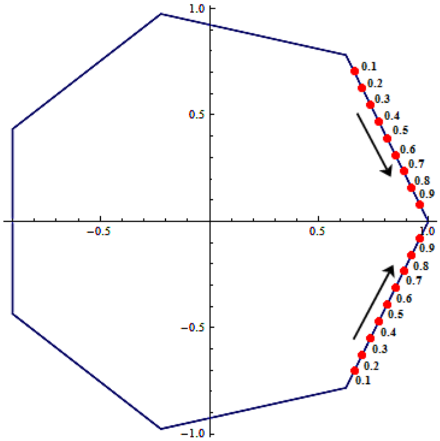

Fig. 6: The region $\prod_7$.

where the lines connected to the point 1 is represented by the trace non-zero doubly stochastic circulant matrix

$$\begin{pmatrix} t & 1-t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 1-t & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 1-t & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 1-t & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 1-t \\ 1-t & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix};\ t \in [0,1].$$

A. The Region ω_2

The region$\omega_2 = \prod_1 \cup \prod_2$ consists of the line joining the points 1 and $e^{i\pi}$. The 2×2 doubly stochastic circulant matrix with eigenvalues located on this line is of the form $A = \begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}$, $t \in [0,1]$. Note that, all roots on this line are extremal and as t varies we get

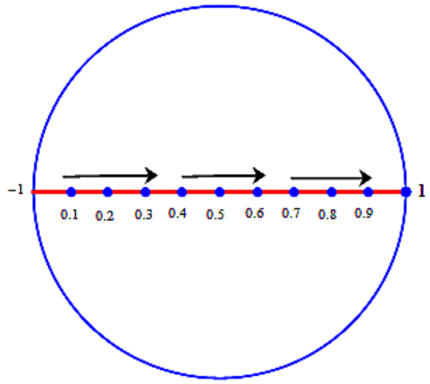

Fig. 7: Extremal roots of ω_2

B. The Region ω_3

The region $\omega_3 = \prod_1 \cup \prod_2 \cup \prod_3$ consists of a regular triangle with vertices at $1, e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ and of the closed interval $\left[-1, -\frac{1}{2}\right]$.

(1) The 3×3 doubly stochastic trace non-zero circulant matrix with extremal eigenvalues located on the lines joining the point 1 to the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ is of the form $A_{31} = \begin{pmatrix} t & 1-t & 0 \\ 0 & t & 1-t \\ 1-t & 0 & t \end{pmatrix}$ and the following figure shows the location of the roots as t varies:

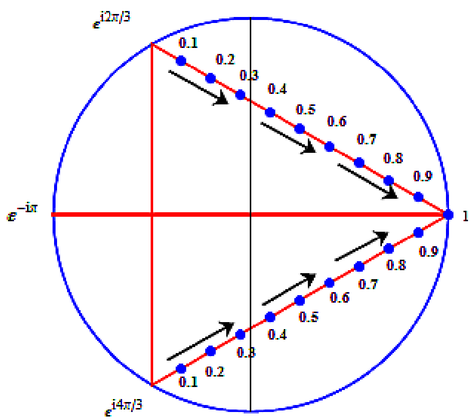

Fig. 8: Extremal roots of ω_3 between $\left\langle 1, e^{\frac{i2\pi}{3}} \right\rangle$ and $\left\langle 1, e^{\frac{i4\pi}{3}} \right\rangle$.

(2) The 3×3 doubly stochastic trace zero circulant matrix with extremal eigenvalues located on the lines joining the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ is of the form:

$$A_{32} = \begin{pmatrix} 0 & t & 1-t \\ 1-t & 0 & t \\ t & 1-t & 0 \end{pmatrix}.$$

Following figure shows the location of the roots as t varies:

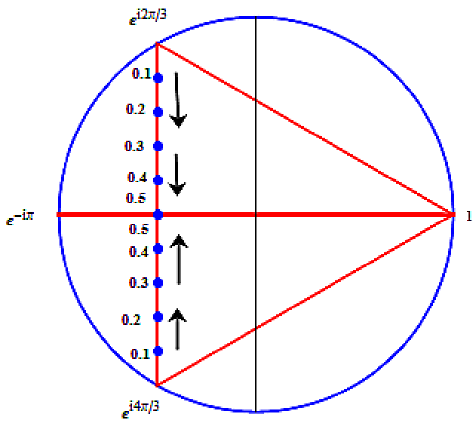

Fig. 9: Extremal roots of ω_3 between $\left\langle e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}} \right\rangle$

(3) For the closed interval $\left[-1, -\frac{1}{2}\right]$ this line has the characteristic polynomial multiplied by the factor $\lambda - 1$ to the characteristic polynomial of A in 2×2 case. So the 3×3 matrix is obtained simply by adding a new column and a new row with 3^{rd} entry 1 to the matrix A. The 3×3 doubly stochastic matrix with extremal eigenvalues located $\left[-1, -\frac{1}{2}\right]$ on the real line is of the form $A_{33} = \begin{pmatrix} t & 1-t & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and following figure shows the location of the roots as t varies:

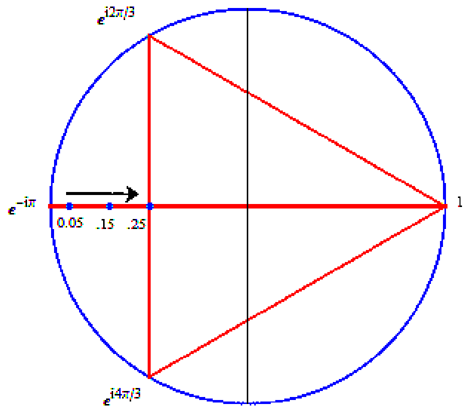

Fig. 10: Extremal roots of ω_3 between $\left[-1,-\frac{1}{2}\right]$

C. The Region ω_4

The region ω_4 intersects the unit circle at six points, $1, e^{\frac{i\pi}{2}}, e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}}, e^{i\pi}$ and $e^{\frac{i3\pi}{2}}$, consisting of $\prod_1 \cup \prod_2 \cup \prod_3 \cup \prod_4$.

(1) The 4×4 doubly stochastic trace non-zero circulant matrix with extremal eigenvalues located on the lines joining the points 1 to the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ is of the form $A_{41} = \begin{pmatrix} t & 1-t & 0 & 0 \\ 0 & t & 1-t & 0 \\ 0 & 0 & t & 1-t \\ 1-t & 0 & 0 & t \end{pmatrix}$, and following figure shows the location of roots as t varies

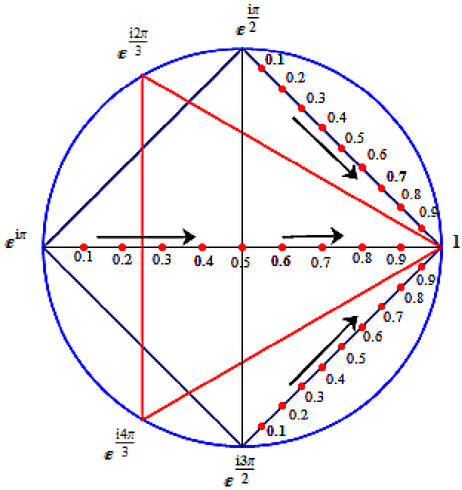

Fig. 11: Extremal roots of $\omega_{4.}$ between $\left\langle 1, e^{\frac{i\pi}{2}} \right\rangle \left\langle 1, e^{\frac{i\pi}{2}} \right\rangle$ and $\left\langle 1, e^{\frac{i3\pi}{2}} \right\rangle$

(2) Following figure shows the location of the root as t varies:

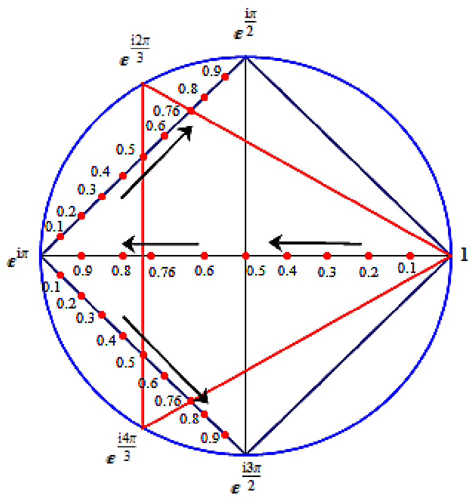

Fig. 12: Extremal roots of $\omega_{4.}$ between $\left\langle e^{i\pi}, e^{\frac{i\pi}{2}} \right\rangle$ and $\left\langle e^{i\pi}, e^{\frac{i3\pi}{2}} \right\rangle$

Where, the 4×4 doubly stochastic trace zero circulant matrix with extremal eigenvalues located on the lines joining the points $e^{i\pi}$ to the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ is of the form

$$A_{42} = \begin{pmatrix} 0 & t & 1-t & 0 \\ 0 & 0 & t & 1-t \\ 1-t & 0 & 0 & t \\ t & 1-t & 0 & 0 \end{pmatrix}.$$

(3) For the lines joining the point 1 to the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ respectively, these lines have the characteristic polynomial multiplied by the factor $\lambda - 1$ to the characteristic polynomial of the matrix A_{31} in 3×3 case. So the 4×4 doubly stochastic trace non-zero matrix is obtained simply by adding a new column and a new row with 4^{th} entry 1 to the matrix A_{31}. The 4×4 doubly stochastic matrix with extremal eigenvalues are of the form $A_{43} = \begin{pmatrix} t & 1-t & 0 & 0 \\ 0 & t & 1-t & 0 \\ 1-t & 0 & t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and following figure shows the location of the roots as t varies:

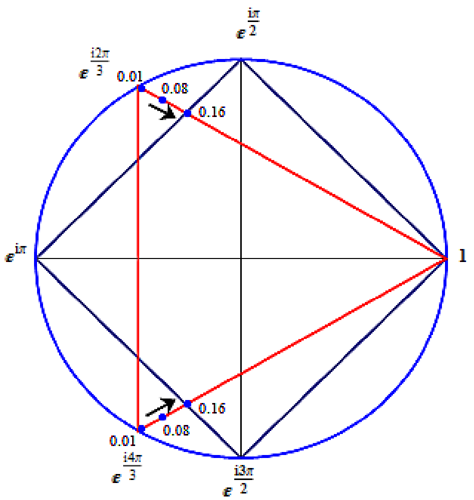

Fig. 13: Extremal roots of ω_4 between $\left\langle 1, e^{\frac{i2\pi}{3}} \right\rangle$ and $\left\langle 1, e^{\frac{i4\pi}{3}} \right\rangle$

(4) For the line joining the points $e^{\frac{i2\pi}{3}}$ and $e^{\frac{i4\pi}{3}}$ has the characteristic polynomial multiplied by the factor $\lambda - 1$ to the characteristic polynomial of the matrix A_{32} in 3×3 case. Thus 4×4 doubly stochastic matrix is obtained simply by adding a new column and a new row with 4^{th} entry 1 to the matrix A_{32}. The 4×4 doubly stochastic matrix with extremal eigenvalues is of the form $A_{44} = \begin{pmatrix} 0 & t & 1-t & 0 \\ 1-t & 0 & t & 0 \\ t & 1-t & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and following figure shows the location of the roots as t varies:

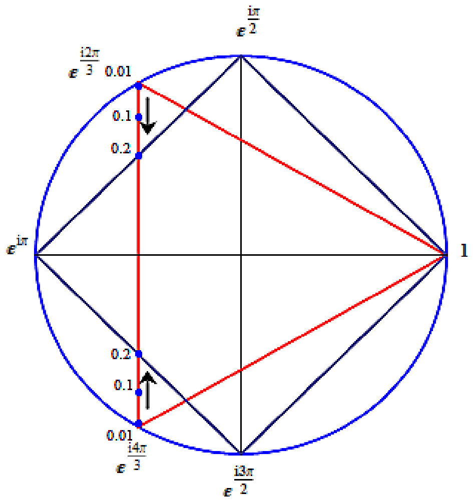

Fig. 14: Extremal roots of ω_4 between $\left\langle e^{\frac{i2\pi}{2}}, e^{\frac{i4\pi}{3}} \right\rangle$

4. COUNTEREXAMPLE FOR $n = 5$

In this section, we give a more general structure of counterexample of the Perfect-Mirsky conjecture for $n = 5$ than that of Rivard-Mashreghi and at the end, we produce some particular cases for counterexample to make reasonable comparison.

Rivard-Mashreghi Counterexample.

In 2007, Mashregi et. al. [21] found a counterexample, showing that $\bigcup_{i=1}^{5} \prod i \subsetneq \omega_5$.

$$P_t = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & 1-t \\ 0 & t & 1-t & 0 & 0 \\ 0 & 1-t & 0 & 0 & t \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

They numerically showed that the matrix has an eigenvalue for $t \in [0.5 - \varepsilon, 0.5 + \varepsilon]$ outside the Perfect-Mirsky region. They showed that the matrix

$$P_{0.5} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has two complex eigenvalues $\lambda \approx 0.28 \pm 0.76i$, lying outside the region ω_5.

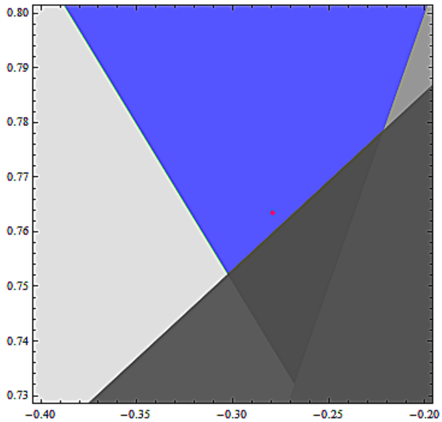

Fig. 15: The Rivard-Mashregi Counterexample for ω_5

New Counterexamples for $n = 5$.

We propose a new matrix of the form

$$\begin{pmatrix} r & 0 & 0 & 1-r & 0 \\ 0 & s & t & 0 & 1-s-t \\ 0 & t & 1-s-t & 0 & s \\ 0 & 1-s-t & s & 0 & t \\ 1-r & 0 & 0 & r & 0 \end{pmatrix} \quad (1)$$

For $r \in [0-\varepsilon, 0+\varepsilon], s \in [0-\eta, 0+\eta]$ and $t \in [0.5-\upsilon, 0.5+\upsilon]$, the matrix has some eigenvalues outside the Perfect-Mirsky region.

CASE (I). For $r = 0, s = 0,$ and $t = \frac{1}{2}$, we get the Rivard-Mashreghi counterexample from our proposed structure of matrix.

CASE (II). Again, from equation (1), for $r = 0.001, s = 0.001, t = 0.494$, we have the matrix

$$A = \begin{pmatrix} 0.001 & 0 & 0 & 0.999 & 0 \\ 0 & 0.001 & 0.494 & 0 & 0.505 \\ 0 & 0.494 & 0.505 & 0 & 0.001 \\ 0 & 0.505 & 0.001 & 0 & 0.494 \\ 0.999 & 0 & 0 & 0.001 & 0 \end{pmatrix} \quad (2)$$

has two roots outside the region ω_5, and geometrically we have

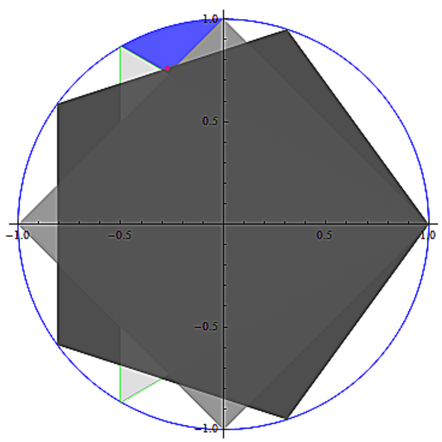

Fig. 16: Eigenvalue location of the matrix A

CASE (II) MATHEMATICAL EXPLAINATION. The characteristic polynomial of A is

$Det(\lambda I - A) = -\lambda^5 + 0.507\lambda^4 + 0.243519\lambda^3 + 0.492774\lambda^2 + 0.00480328\lambda - 0.248096.$ One root of the polynomial is 1, so we factor out the term $\lambda - 1$ and let the remaining part be denoted by $f(\lambda) = -\lambda^4 - 0.493\lambda^3 - 0.249481\lambda^2 + 0.243292\lambda + 0.248096.$ The characteristic polynomial has two more real roots and two complex roots which are (using the Mathematica software) approximately $\lambda_1 \approx 0.642647$, $\lambda_2 \approx -0.587367$, and $a \pm ib \approx -0.27414 \pm 0.762959i$. Let $\lambda_1, \lambda_2 \in \Re$ and $a \pm ib$ be the other roots of the equation $f(\lambda) = -\lambda^4 - 0.493\lambda^3 - 0.249481\lambda^2 + 0.243292\lambda + 0.248096 = 0$. Then the sum of the roots is

$$\lambda_1 + \lambda_2 + 2a = -0.493 \qquad (3)$$

and the product of the roots is

$$\lambda_1\lambda_2\left(a^2 + b^2\right) = -0.248096 \qquad (4)$$

Now we show that the roots $a \pm ib$ to the forbidden region denoted by Δ.

We find that $f(0.642646) > 0$, and $f(0.642648) < 0$. Hence, by the Intermediate Value, we have $\lambda_1 \in [0.642646, 0.642648]$. In a similar way, $f(-0.587366) > 0$, and $f(-0.587368) < 0$ and we find $\lambda_2 \in [-0.587366, -0.587368]$.

Hence, by (3) , we get $a \in [-0.27413, -0.27415]$, and, by (4) , we get $b \in [0.762958, 0.762960]$,

This is same to saying that

$$a + ib \in [-0.27413, -0.27415] \times [0.762958, 0.762960] = Z.$$

Now, the equation of the line joining the points $(-1,0)$ and $(0, 1)$ is $y - x - 1 = 0$, the equation of the line joining the points $(1,0)$ and $e^{\frac{i4\pi}{3}}$ is $\sqrt{3}y + x - 1 = 0$, and the equation of the line joining the points $e^{\frac{i2\pi}{5}}$ and $e^{\frac{i4\pi}{5}}$ is

$$\frac{x - Cos\left(\frac{2\pi}{5}\right)}{Cos\left(\frac{4\pi}{5}\right) - Cos\left(\frac{2\pi}{5}\right)} - \frac{y - Sin\left(\frac{2\pi}{5}\right)}{Sin\left(\frac{4\pi}{5}\right) - Sin\left(\frac{2\pi}{5}\right)}.$$

Let $F(x, y) = y - x - 1 = 0$, $G(x, y) = \sqrt{3}y + x - 1 = 0$,

$$H(x,y)=\frac{x-Cos\left(\frac{2\pi}{5}\right)}{Cos\left(\frac{4\pi}{5}\right)-Cos\left(\frac{2\pi}{5}\right)}-\frac{y-Sin\left(\frac{2\pi}{5}\right)}{Sin\left(\frac{4\pi}{5}\right)-Sin\left(\frac{2\pi}{5}\right)}.$$

Then it is easy to verify that $F(x,y)>0, G(x,y)>0, H(x,y)>0$ for the four extreme corner points of Z and ensure that the rectangle Z lies in the forbidden region Δ.

CASE (III). For $r=0, s=0.01, t=0.52$, from equation $(4.5.1)$, we have the matrix

$$B=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0.01 & 0.52 & 0 & 0.47 \\ 0 & 0.52 & 0.505 & 0 & 0.01 \\ 0 & 0.47 & 0.01 & 0 & 0.52 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (5)$$

has two roots $\lambda \approx -0.298683 \pm 0.755402$ outside the region ω_5.

Note that, the following matrices A and B are a Convex Combination of 3 Permutation Matrices, where

$$A=\begin{pmatrix} 0.001 & 0 & 0 & 0.999 & 0 \\ 0 & 0.001 & 0.494 & 0 & 0.505 \\ 0 & 0.494 & 0.505 & 0 & 0.001 \\ 0 & 0.505 & 0.001 & 0 & 0.494 \\ 0.999 & 0 & 0 & 0.001 & 0 \end{pmatrix}, \text{ and } B=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0.01 & 0.52 & 0 & 0.47 \\ 0 & 0.52 & 0.505 & 0 & 0.01 \\ 0 & 0.47 & 0.01 & 0 & 0.52 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof: Let us consider the matrix B. We set it as

$$B_0=\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0.01 & 0.52 & 0 & 0.47 \\ 0 & 0.52 & 0.47 & 0 & 0.01 \\ 0 & 0.47 & 0.001 & 0 & 0.52 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (6)$$

The bipartite graph associated to B_0 is the following:

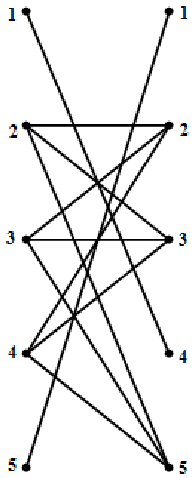

Fig. 17: Bipartite graph associated to B_0 matrix

A perfect matching is $(1,4),(2,3),(3,2),(4,5),(5,1)$ and the corresponding permutation matrix is

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (7)$$

The smallest entry of B_0 corresponding to the non-zero entries of P_0 is $\alpha_0 = 0.52$.
Then, we get

$$B_1 = \frac{1}{1-\alpha_0}\left(B_0 - \alpha_0 P_0\right) = \frac{1}{0.48}\begin{pmatrix} 0 & 0 & 0 & 0.48 & 0 \\ 0 & 0.01 & 0 & 0 & 0.47 \\ 0 & 0 & 0.47 & 0 & 0.01 \\ 0 & 0.47 & 0.01 & 0 & 0 \\ 0.48 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{48} & 0 & 0 & \frac{47}{48} \\ 0 & 0 & \frac{47}{48} & 0 & \frac{1}{48} \\ 0 & \frac{47}{48} & \frac{1}{48} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

.

The bipartite graph associated to B_1 is the following:

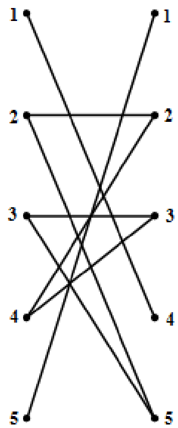

Fig. 18: Bipartite graph associated to B_1 matrix

A perfect matching is $(1,4), (2,2), (3,5), (4,3), (5,1)$ and the corresponding permutation matrix is

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (8)$$

The smallest entry of B_1 corresponding to the non-zero entries of P_1 is $\alpha_1 = \frac{1}{48}$.

Then, we get

$$B_2 = \frac{1}{1-\alpha_1}\left(B_1 - \alpha_1 P_1\right) = \frac{48}{0.47}\begin{pmatrix} 0 & 0 & 0 & \frac{47}{48} & 0 \\ 0 & 0 & 0 & 0 & \frac{47}{48} \\ 0 & 0 & \frac{47}{48} & 0 & 0 \\ 0 & \frac{47}{48} & 0 & 0 & 0 \\ \frac{47}{48} & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = P_2$$

We now go backward, and then

$$B_1 = \alpha_1 P_1 + (1-\alpha_1)B_2$$

$$B_0 = \alpha_0 P_0 + (1-\alpha_0)B_1$$

$$= \alpha_0 P_0 + (1-\alpha_0)\{\alpha_1 P_1 + (1-\alpha_1)P_2\}$$

$$= 0.52P_0 + 0.01P_1 + 0.47P_2 \cdot$$

Finally we have

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0.01 & 0.52 & 0 & 0.47 \\ 0 & 0.52 & 0.47 & 0 & 0.01 \\ 0 & 0.47 & 0.001 & 0 & 0.52 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = 0.52 \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + 0.01 \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + 0.47 \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

as a consequence of Birkhoff theorem which says that B_0 is a convex combination of 3 permutation matrices.

Comparing our counterexamples to that of Rivard-Mashreghi.

We note that the Rivard-Mashreghi counterexample is a convex combination of just two permutation matrices; but our new counterexamples are convex combination of three permutation matrices.

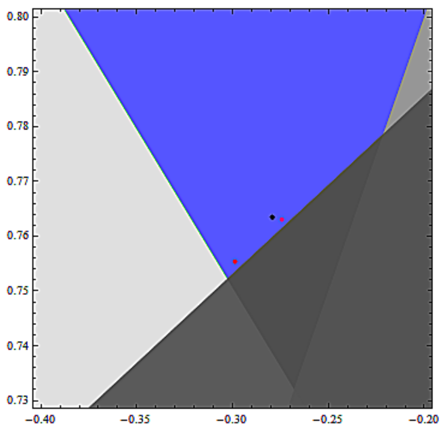

Fig. 19: Comparison of counterexamples of case (i), (ii) and (iii)

Comparing our counterexamples to that of Rivard-Mashreghi, we see that for more terms in the convex combination, the eigenvalues are more close to the real boundary of ω_5. Since the region ω_n approaches the full circle, the forbidden region inside the circle becomes smaller. And since each vertex on the unit circle represents a permutation matrix, so if there exists any counterexample to the Perfect-Mirsky conjecture for higher values of n, they are expected to be a convex combination of fewer permutation matrices. For deeper understanding, one can see Levick [18].

5. QUANTUM CHANNELS

In this section, we introduce a theorem to determine whether a QCs gives rise to a doubly stochastic matrix or not.

DEFINITION 8. A symmetric matrix A is called positive semidefinite if $x^*Ax \geq 0$ for all column vector x in $\mathbb{C}^n$ (or, x in $\Re^n$ for the real matrices). If $x^*Ax > 0$, then A is said to be positive definite. We note that the eigenvalues of all positive semidefinite matrices are nonnegative whereas the eigenvalues of all positive definite matrices are positive.

DEFINITION 9. A positive semidefinite matrix ρ of $n \times n$ complex entries is called a density matrix if $Tr(\rho) = 1$. Mathematically, density matrices can be formulated as $\rho = \sum p_i \, |\psi_i\rangle\langle\psi_i|$ (see details Das[7] for some remarkable results depending on density matrices). Thus, any general density matrix is a convex combination of pure state density matrices; hence density matrices are generally considered to be as mixed states.

DEFINITION 10. Let M_n be a set of $n \times n$ complex matrices. A positive map is a linear map defined as $\Phi: M_{n_1} \to M_{n_2}$ such that it maps positive semidefinite matrix $X \in M_{n_1}$ into a positive semidefinite matrix $\Phi(X) \in M_{n_2}$. We recall that if the matrix X is positive, then we have

$$\langle\psi|X|\psi\rangle \geq 0 \tag{9}$$

for all $\psi \in C^n$, and the map Φ is positive if it satisfies

$$\langle \psi | \Phi(|\phi\rangle\langle\phi|)|\psi\rangle \geq 0. \qquad (10)$$

for $\varphi \in C^{n_1}$ and $\psi \in C^{n_2}$.

DEFINITION 11. Let $id_n: M_n \to M_n$ be an identity map for which $id_n(X) = X$ for any $X \in M_n$. Then a linear map $\Phi: M_{n_1} \to M_{n_2}$ is said to be *n*-positive if $\Phi \otimes id_n: M_{n_1} \otimes M_n \to M_{n_2} \otimes M_n$ is positive.

DEFINITION 12. A linear map $\Phi: M_{n_1} \to M_{n_2}$ is said to be completely positive if it is *n*-positive for all $n = 1,2,\ldots..$

we note that due to Kraus et. al. [16], any completely positive map $\Phi: M_{n_1} \to M_{n_2}$ can be expressed as

$$\Phi(X) = \sum_i K_i X K_i^* \qquad (11)$$

The representation (11) is called the Kraus decomposition of the map Φ, and the operators $\{K_i\}$ are called the Kraus operators of Φ. Every Kraus operator is an $n_1 \times n_2$ matrix such that (11) holds for all $X \in M_{n_1}$. We recall that similar to the definition 10, the condition (9) for the positivity of the matrix X reduces to the simple spectral condition: all eigenvalues of the matrix X have to be nonnegative. But this is not true for the condition (10). Because the map Φ may be checked $1-$positive, 2-positive etc; that condition does not ensure the positivity for all $n = 1,2,\ldots\ldots..$ To get a simple spectral condition for complete positivity , let us define the Choi-Jamiolkowsky matrix ([6], [10])

$$C_\Phi := \sum_{i,j=1}^{n_1} |e_i\rangle\langle e_j| \otimes \Phi\left(|e_i\rangle\langle e_j|\right), \qquad (12)$$

where $(e_1, e_2, \ldots\ldots\ldots, e_{n_1})$ is an orthonormal basis in C^{n_1}. The Choi matrix $C_\Phi \in M_{n_1 n_2}(C)$ in (5) can also be written as

$$C_\Phi = \sum_{i,j=1}^{n_1} E_{ij} \otimes \Phi(E_{ij}) \qquad (13)$$

Now, for n-positivity of the map Φ can be rewritten from condition (2) as follows:

$$\langle\psi|C_\Phi|\psi\rangle \geq 0 \qquad (14)$$

EXAMPLE 3. Let us consider a map $\Phi: M_n \to M_n$ defined by

$$\Phi(X) = X^T.$$

This map is trace preserving and trivially positive; since for $n = 2$, we have

$$X = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

has spectrum $\{1,1,0,0\}$ and after performing the operation transposition the spectrum is not changed. But

$$(I_2 \otimes \Phi)(X) = \Phi_2\left(\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}\right)$$

$$= \begin{pmatrix} \Phi(E_{11}) & \Phi(E_{12}) \\ \Phi(E_{21}) & \Phi(E_{22}) \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

has spectrum $\{-1,1,1,1\}$, shows that the matrix is indefinite. Hence, Φ is a positive map but not completely positive.

The following theorem is the generalized condition of completely positivity for all trivially positive maps:

THEOREM 5. (Choi theorem [17]) A map $\Phi: M_{n_1}(C) \to M_{n_2}(C)$ is completely positive if and only if its Choi matrix

$$C_\Phi = \sum_{i,j=1}^{n_1} E_{ij} \otimes \Phi(E_{ij})$$

$$= \left(\begin{array}{c|c|ccc|c} \Phi(E_{11}) & \Phi(E_{12}) & \ldots & \ldots & \ldots & \Phi(E_{1n_1}) \\ \hline \Phi(E_{21}) & \Phi(E_{22}) & \ldots & \ldots & \ldots & \Phi(E_{2n_1}) \\ \hline \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & \vdots \\ \hline \Phi(E_{n_11}) & \Phi(E_{n_12}) & \ldots & \ldots & \ldots & \Phi(E_{n_1n_1}) \end{array}\right)$$

is positive semidefinite.

DEFINITION 13. A completely positive map $\Phi: M_{n_1} \to M_{n_2}$ is said to be trace preserving if $Tr(\Phi(X)) = Tr(X)$ for all $X \in M_{n_1}$.

A completely positive trace preserving map sends QI of a n_1 – dimensional quantum system to a n_2 – dimensional quantum system. Thus, a completely positive trace preserving map is considered to be a QC.

DEFINITION 14. A QC $\Phi: M_{n_1} \to M_{n_2}$ is called unital if $\Phi(I_{n_1}) = I_{n_2}$. Otherwise, it will be said non-unital.

Relation Between QCs and Doubly Stochastic Matrices.

Positive maps acting on a space of nonnegative elements yield another nonnegative elements of a second space. Linear positive maps which are trace preserving and unital give rise to doubly stochastic matrices. Let $\{u_1, u_2, \ldots\ldots\ldots, u_n\}$ and $\{v_1, v_2, \ldots\ldots\ldots, v_n\}$ be to orthonormal basis in C^n. Then, we define

$$D_{ij} = \langle u_i | \Phi(|v_j\rangle\langle v_j|)|u_i\rangle. \tag{15}$$

Now, by definition Φ is positive if $D_{ij} \geq 0$. Taking sum over i, we obtain

$$\sum_{i=1}^{n} D_{ij} = \sum_{i=1}^{n} \langle u_i | \Phi(|v_j\rangle\langle v_j|)|u_i\rangle = Tr(\Phi(|v_j\rangle\langle v_j|)), \tag{16}$$

Since Φ is trace preserving, we have $Tr(\Phi(X)) = Tr(X)$. Using this fact, we get $\sum_{i=1}^{n} D_{ij} = Tr\left(\Phi(|v_j\rangle\langle v_j|)\right) = 1$, shows that D_{ij} is a row stochastic Matrix. Again, taking sum over j, we obtain $\sum_{j=1}^{n} D_{ij} = \langle u_i | \sum_{j=1}^{n} \Phi(|v_j\rangle\langle v_j|)|u_i\rangle$.

Here, $\sum_{j=1}^{n} \Phi(|v_j\rangle\langle v_j|) = \Phi(\sum_{j=1}^{n} |v_j\rangle\langle v_j|) = \Phi(I_n)$, and for unitality of Φ, $\Phi(I_n) = I_n$. Using these facts, we obtain

$$\begin{aligned} \sum_{j=1}^{n} D_{ij} &= \langle u_i | \sum_{j=1}^{n} \Phi(|v_j\rangle\langle v_j|)|u_i\rangle \\ &= \langle u_i | \Phi(I_n)|u_i\rangle = \langle u_i | I_n | u_i\rangle = 1, \end{aligned}$$

shows that D_{ij} is column stochastic.

THEOREM 6. (Poon [24]) Let $\Phi: M_{n_1} \to M_{n_2}$ and $n \geq 1$. Then the followings are equivalent:

(i) Φ is n −positive

(ii) Φ is completely positive

(iii) The Choi matrix C_{Φ} is positive semidefinite

(iv) Φ admits an operator-sum representation

$$\Phi(X) = \sum_{i=1}^{n} K_i X K_i^{*}. \qquad (17)$$

If a map Φ is completely positive, then it is positive. Hence, a completely positive trace preserving unital map or unital QC, also gives rise to doubly stochastic matrices.

EXAMPLE 4. (Gagniuc [10]) Let us consider a family of Choi maps in M_n defined by

$$\Phi_t(X) = I_n \, Tr(X) - t\, X, \quad t \leq 1. \qquad (18)$$

Now, if $\frac{1}{k+1} \leq t \leq \frac{1}{k}$, then Φ_t is k positive but not completely positive. Hence, Φ_t is completely positive if $t \leq \frac{1}{n}$. For $n = 2$, the Choi matrix of Φ_t is

$$C_{\Phi} = \left(\begin{array}{cc|cc} 1-t & 0 & 0 & -t \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ -t & 0 & 0 & 1-t \end{array}\right)$$

which has the spectrum $\{1 - 2t, 1,1,1\}$. Hence, Φ_t is completely positive if $t \leq \frac{1}{2}$. Then the corresponding doubly stochastic matrix is

$$D_{ij} = (1-t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

That is , convex combination of two permutation matrices and hence a doubly stochastic matrix. For $t \leq \frac{1}{2}$, Φ_t is a unital QC yielding a doubly stochastic matrix.

By the motivation of the above remarkable results we present the following theorem in this article. We now generalize the condition whether a QC gives rise to a doubly stochastic matrix or not.

THEOREM 7. Let $\Phi: M_{n_1} \to M_{n_2}$ be a completely positive map with Kraus operator $\{K_i\}$. Then the map Φ gives rise to doubly stochastic matrix if and only if $\sum_{i=1}^{n_1} K_i^* K_i = I$ and $\sum_{i=1}^{n_1} K_i K_i^* = 1$.

Or, Let $\Phi: M_{n_1} \to M_{n_2}$ be a QC with Kraus operator $\{K_i\}$. Then the QC Φ gives rise to doubly stochastic matrix if and only if $\sum_{i=1}^{n_1} K_i K_i^* = 1$.

Proof: We already know that every completely positive map $\Phi: M_{n_1} \to M_{n_2}$ can be expressed as

$$\Phi(X) = \sum_{i=1}^{n_1} K_i X K_i^*, \tag{19}$$

Now, the map Φ gives rise to doubly stochastic matrix if and only if Φ is trace preserving and unital.

By definition, Φ is unital $\Leftrightarrow \Phi(I_{n_1}) = I_{n_2} \Leftrightarrow \sum_{i=1}^{n_1} K_i I_{n_1} K_i^* = I_{n_2} \Leftrightarrow \sum_{i=1}^{n_1} K_i K_i^* = I_{n_2}$ and Φ is trace preserving $\Leftrightarrow Tr(\Phi(X)) = Tr(X)$

$$\Leftrightarrow Tr\left(\sum_{i=1}^{n_1} K_i X K_i^*\right) = Tr(X) \text{ for all } X \in M_{n_1}$$

$$\Leftrightarrow Tr\left(\sum_{i=1}^{n_1} X K_i^* K_i\right) = Tr(X) \text{ for all } X \in M_{n_1}$$

$$\Leftrightarrow Tr\left(\sum_{i}^{n_1} X K_i^* K_i\right) - Tr(X) = 0 \text{ for all } X \in M_{n_1}$$

$$\Leftrightarrow Tr\left(X\left(\sum_{i=1}^{n_1} K_i^* K_i - I_{n_1}\right)\right) = 0 \text{ for all } X \in M_{n_1}$$

$$\Leftrightarrow \left(\sum_{i=1}^{n_1} K_i^* K_i - I_{n_1}\right) = 0; \text{ since } Tr(X) \neq 0$$

$$\Leftrightarrow \sum_{i=1}^{n_1} K_i^* K = I_{n_1}.$$

6. CONCLUSION

We studied the relationship between doubly stochastic matrices and majorization along with the structures of marginal doubly stochastic matrices in the Perfect-Mirsky region simply. We extended the Rivard-Mashreghi counterexample into a more general form for $n = 5$ of the conjecture. Since the new conjecture for the characteristic region of doubly stochastic matrices given by Levick [18] contains a large part of the forbidden region for $n = 5$ it needs a better analysis of the case $n \geq 5$. For a better idea of the location of eigenvalues in the Perfect-Mirsky region for $n \geq 5$ these counterexamples may be helpful to the readers for future research in the field. Every completely positive map has an operator-sum representation. Using these, we considered, in general, how can we get a doubly stochastic matrix corresponding to a QCs. The reader can go for further investigation into the relationship between doubly stochastic matrices and private QCs as well as the connection between spectra of doubly stochastic matrices and QCs. The connections between majorization, QCs, and private QCs are to be investigated. There are also many interesting connections between QCs, private QCs, and probability distribution.

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