An Alternative Proof For the Minimum Fisher Information of Gaussian Distribution

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Abstract

Fisher information is of key importance in estimation theory. It is used as a tool for characterizing complex signals or systems, with applications, e.g. in biology, geophysics and signal processing. The problem of minimizing Fisher information in a set of distributions has been studied by many researchers. In this paper, based on some rather simple statistical reasoning, we provide an alternative proof for the fact that Gaussian distribution with finite variance minimizes the Fisher information over all distributions with the same variance.

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Keywords: Fisher information, Gaussian distribution, Minimum risk equivariant estimator.

1. INTRODUCTION

The role of Fisher information as a way of measuring information in a distribution is well established in the literature. Fisher information is used in estimation theory for constructing a basic bound, known as Cramer-Rao lower bound (CRLB), on the variance of an estimator ([Khoolenjani and Alamatsaz (2016)]). Applications of Fisher information in geophysics ([Balasco et al. (2008)]), biology ([Frank (2009)]), analysing complex signals or systems ([Martin et al. (2009)], [Nagy (2003)]), signal processing ([Vignat and Bercher (2003)], [Zivojnovic and Noll (1997)]), computing the asymptotic covariance matrix of the models ([Hussin et al. (2010)] and [Mamun et al. (2013)]) and obtaining performance bounds ([Xu et al. (2008)]) are discussed in the literature. It is also used in statistical physics and biology as a way of inference and understanding ([Frieden (2009)]). Recently, [Dulek and Gezici (2014)] studied the maximization of Fisher information in presence of a constraint on the cost of measurements. [Neri et al. (2013)] studied the theoretical evaluation of the achievable performance using Fisher information.

Gaussian distribution is one of the most well-known and widely applied distributions in many fields such as statistics, engineering and physics. One of the major reasons why Gaussian distribution has become so prominent is because of the Central Limit Theorem (CLT) and the fact that the distribution of noise in numerous

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engineering systems is well fitted by Gaussian distribution. It is well known that Gaussian distribution minimizes the Fisher information, which equals to the inverse of Cramer-Rao lower bound, (see [Shao (1999)]). This fact is established in [Park et al. (2013)]. Especially, when there is no information about the distribution of observations, Gaussian assumption appears as the most traditional choice. Therefore, optimization of estimation methods based on the CRLB that holds under Gaussian distribution yields the best CRLB-related performance. [Stoica and Babu (2011)] provided a general proof of result that the largest CRLB is achievable by the Gaussian distribution. In this note, we provide a simple alternative proof for the fact that Gaussian distribution yields the minimum Fisher information. Using certain standard statistical reasoning, we believe that there is a value – in general, and also here – in presenting alternative proofs for fundamental theorems. Such alternative proofs can shed new light on the statement being proven, introduce new arguments that can be useful elsewhere, or yield different generalizations and applications.

2. MAIN RESULT

Let us first review the fundamental notions and basic definitions used in the paper.

Suppose that X is a random observable taking on values in a sample space $\mathscr{X}$ according to a probability distribution from the family $\mathscr{F} = \{f(x;\theta) : \theta \in \Theta\}$, in which θ is a deterministic parameter.

DEFINITION 1. The Fisher information $I(F)$ of a distribution F on the real line is defined as

$$I(F) = \int_{-\infty}^{\infty} \left(\frac{d \ln f(x;\theta)}{dx} \right)^2 f(x;\theta) dx, \tag{1}$$

where f denotes the density of F.

DEFINITION 2. An *estimator* is a real-valued function δ defined over the sample space $\mathscr{X}$. It is used to estimate an estimand, $g(\theta)$, a real-valued function of the parameter.

Quite generally, suppose that the consequences of estimating $g(\theta)$ by a value d are measured by $L(\theta,d)$. Of the *loss function* L, we shall assume that $L(\theta,d) \geq 0$ for all θ,d and $L[\theta,g(\theta)] = 0$ for all θ, so that the loss is zero when the correct value is estimated. The accuracy, or rather inaccuracy, of an estimator θ is then measured by the *risk function*

$$R(\theta,\delta) = E_\theta\{L[\theta,\delta(X)]\}. \tag{2}$$

DEFINITION 3. A set of functions $\{g(x) : g \in \mathscr{G}\}$ from the sample space $\mathscr{X}$ onto $\mathscr{X}$ is called a *group of transformations* of $\mathscr{X}$ if

i. (Inverse) For every $g \in \mathscr{G}$ there is a $g' \in \mathscr{G}$ such that $g'(g(x)) = x$ for all $x \in \mathscr{X}$.

ii. (Composition) For every $g \in \mathscr{G}$ and $g' \in \mathscr{G}$ there exists $g'' \in \mathscr{G}$ such that $g'(g(x)) = g''(x)$ for all $x \in \mathscr{X}$.

iii. (Identity) The identity, $e(x)$, defined by $e(x) = x$ is an element of $\mathscr{G}$.

DEFINITION 4. Let $\mathscr{G}$ be a group of transformations of the sample space $\mathscr{X}$. Then, the family $\mathscr{F} = \{f(x;\theta) : \theta \in \Theta\}$ is *invariant under the group* $\mathscr{G}$ if for every $\theta \in \Theta$ there exists a unique $\theta' \in \Theta$ such that $Y = g(X)$ has the distribution $f(y;\theta')$ if X has the distribution $f(x;\theta)$.

The θ' uniquely determined by θ is denoted by $\bar{g}(\theta)$.

DEFINITION 5. An estimation problem (Θ,δ,L) is said to be *invariant under the group* $\mathscr{G}$ if the family $\mathscr{F} = \{f(x;\theta) : \theta \in \Theta\}$ of distributions is invariant under $\mathscr{G}$ and if the loss function is invariant under $\mathscr{G}$ in the sense that for every $g \in \mathscr{G}$ and every δ in the class of estimators D, there exists a unique $\delta^* \in D$ such that

$$L(\theta,\delta) = L(\bar{g}(\theta),\delta^*) \quad \forall\theta \in \Theta. \tag{3}$$

The δ^* uniquely determined by g and δ is denoted by $\tilde{g}(\delta)$.

In an invariant estimation problem, an estimator δ is said to be *equivariant* if for all $g \in \mathscr{G}$

$$\delta(g(x)) = \tilde{g}(\delta(x)). \tag{4}$$

If an equivariant estimator exists and minimizes the risk function, it is called the *minimum risk equivariant* (MRE) estimator.

THEOREM 1. Let $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be distributed as

$$f(\mathbf{y}-\theta) = f(y_1-\theta,\ldots,y_n-\theta),$$

$X_i = Y_i - Y_n$ and $\mathbf{X} = (X_1,\ldots,X_{n-1})$. Suppose that the loss function is given by $L(\theta,W) = (W-\theta)^2$ and that there exists a location invariant estimator δ_0 of θ with finite risk. Then, the minimum risk equivariant estimator of θ exists and is given by

$$\delta^*(\mathbf{Y}) = \delta_0(\mathbf{Y}) - E_0[\delta_0(\mathbf{Y}) \mid \mathbf{x}]$$

PROOF. See [Lehmann and Casella (1998)].

THEOREM 2. (Cramer-Rao Inequality) Let $Y_1, ..., Y_n$ be independent random variables with a common probability density $f_\theta(y)$ and $W(Y_1, ..., Y_n)$ be an unbiased estimator of θ. Then, under the regularity conditions we have

$$Var(W) \geq \frac{1}{nI(F)}. \tag{5}$$

PROOF. See [Lehmann and Casella (1998)].

THEOREM 3. Among all densities with mean θ and finite variance σ^2, Fisher information is minimized by Gaussian density.

PROOF. Let F be a univariate distribution with density f and fixed finite variance σ^2 and $Y_1, ..., Y_n$ be independently identically distributed random variables with density $f_\theta(y)$, where $\theta = E(Y_i)$ is a location parameter. Assume that $s_n(F)$ is the risk of the minimum risk equivariant estimator of θ under squared error loss $L(\theta, W) = (W - \theta)^2$. For Gaussian distribution with mean θ and finite variance σ^2, if we let $\delta_0 = \bar{Y}$ in Theorem 1, it follows that δ_0 is independent of $\mathbf{X}$ and hence $E_0[\bar{Y} \mid \mathbf{x}] = 0$. Thus, the minimum risk equivariant estimator of θ becomes $\bar{Y}$ with risk $E_\theta(\bar{Y} - \theta)^2 = \frac{\sigma^2}{n}$. On the other hand, we obtain Fisher information in the Gaussian case as

$$\begin{aligned}
I(N) &= \int_{-\infty}^{\infty} \left(\frac{d \ln f(y;\theta)}{dy} \right)^2 f(y;\theta) dy \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dy} \ln \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\theta)^2} \right)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\theta)^2} dy \\
&= \frac{1}{\sigma^4} \int_{-\infty}^{\infty} \frac{(y-\theta)^2}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y-\theta)^2} dy \\
&= \frac{1}{\sigma^2}.
\end{aligned} \tag{6}$$

Therefore, we have

$$E_\theta(\bar{Y} - \theta)^2 = \frac{1}{nI(N)}. \tag{7}$$

We know that for any distribution F, $\bar{Y}$ is an unbiased estimator of θ with risk given in (7). So, the risk of the minimum risk equivariant estimator for any distribution F must be less than $1/(nI(N))$. Now, let b be the constant bias of the MRE estimator δ^*. Then, $\delta_1(y) = \delta^*(y) - b$ is a location invariant estimator of θ and the risk of δ_1 under squared error loss becomes

$$R_{\delta_1} = E[\delta^*(y) - b - \theta]^2 = Var(\delta^*) \leq Var(\delta^*) + b^2 = R_{\delta^*}.$$

Since δ^* is the MRE estimator, $b = 0$, i.e., δ^* is unbiased (see [Shao (1999)], p. 215). Therefore, by using Theorem 2, we have

$$s_n(F) \geq \frac{1}{nI(F)}.$$

Thus, $I(N) \leq I(F)$ and the proof is complete.

3. CONCLUSION

This paper focuses on deriving an alternative proof for the fact that the Fisher information is minimized by Gaussian distribution. The risk of the sample mean in Gaussian density is used to obtain an upper bound for the risk of the minimum risk equivariant estimator for any other distribution F. Then, applying Cramer-Rao inequality, a lower bound is obtained for the risk of the minimum risk equivariant estimator, regardless of F. Combining these two bounds, the result is concluded.

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Study of Incomplete Elliptic Integrals Pertaining to ${}_p\psi_q$ Function

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Abstract

Elliptic-type integral plays a major role in the study of different problems of physics and technology including fracture mechanics. Many papers have been written for various families of elliptic-type integrals. Due to their applications here, we are presenting an organized study of certain generalized family of incomplete elliptic integral. The obtained results are basic in nature have various generalizations. While using the fractional integral operator of Riemann-Liouville type, we found several obvious hyper geometric representations. Which are further used to originate many definite integrals relating to their modules and amplitude of elliptic type generalized incomplete integrals.

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1. INTRODUCTION AND DEFINITIONS

The incomplete elliptic integrals having a keen interest of mathematician form a long time. In this way Legendre's normal form of incomplete elliptic integrals of the first and second kind are given [1-6]:

$$\mathrm{F}(\phi,\mathrm{k}) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1-\mathrm{k}^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{\mathrm{dt}}{\sqrt{\left(1-\mathrm{t}^2\right)\left(1-\mathrm{k}^2\mathrm{t}^2\right)}}, \left(\left|\mathrm{k}^2\right| < 1; 0 \le \phi \le \frac{\pi}{2}\right) \quad (1)$$

and

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2 sin^2\theta}\, d\theta = \int_0^{sin\,\phi} \frac{\sqrt{\left(1-k^2t^2\right)}}{\sqrt{\left(1-t^2\right)}}\, dt, \qquad \left(\left|k^2\right| < 1; 0 \le \phi \le \frac{\pi}{2}\right), \quad (2)$$

with $|\mathrm{k}|$ modulus and amplitude ϕ.

In this paper, we take necessary constraint $\left|k^2\right| < 1$ rather than $0 \le k < 1$. Here the amplitude ϕ may attend complex values. Specially, when $\phi = \frac{\pi}{2}$, the equations (1) and (2) provides the corresponding complete elliptic integrals. It is very useful in radiation physics, nuclear technology fracture mechanics etc. (see [7-17]).

We have generalized elliptic function of third kind [6]

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$$R\,(\phi,k,\xi\,;\alpha\,,\gamma)\;=\;\int_0^{\phi}\frac{1}{\left(1+\xi\,\sin^2\theta\right)^{\alpha}\left(1-k^2\sin^2\theta\right)^{1/2-\gamma}}\,d\theta, \tag{3}$$

$$\mathrm{R}\,(\phi,\mathrm{k},\xi\,;\alpha\,,\gamma)\;=\;\int_0^{\sin\phi}\frac{1}{\left(1+\xi\,\mathrm{v}^2\right)^{\alpha}\sqrt{(1-\mathrm{v}^2)}\left(1-\mathrm{k}^2\mathrm{v}^2\right)^{1/2-\gamma}}\,\mathrm{dv},\quad\begin{pmatrix}\left|\mathrm{k}^2\right|<1\,;0\le\phi\le\frac{\pi}{2}\,;\\ \gamma\in\mathrm{C},\,\alpha\ge0\end{pmatrix} \tag{4}$$

where ξ is elliptic characteristic and $\xi > -1$.
Also we have elliptic function

$$\mathrm{I}\,(\phi,\mathrm{k},\xi\,;\gamma)\;=\;\int_0^{\phi}\frac{1}{\left(1+\xi\sin^2\theta\right)\left(1-\mathrm{k}^2\sin^2\theta\right)^{1/2-\gamma}}\,\mathrm{d}\theta, \tag{5}$$

$$\mathrm{I}\,(\phi,\mathrm{k},\xi\,;\gamma)\;=\;\int_0^{\sin\phi}\frac{1}{(1+\xi\,\mathrm{v}^2)\sqrt{(1-\mathrm{v}^2)}\,(1-\mathrm{k}^2\,\mathrm{v}^2)^{1/2-\gamma}}\,\mathrm{dv}\,,\quad\left(\left|\mathrm{k}^2\right|<1\,;0\le\phi\le\tfrac{\pi}{2}\,;\gamma\ge0\right) \tag{6}$$

It is seen that by assigning some particular values of α, γ and ϕ, the above defined results reduce into known elliptic integral (see [2,6-17]).
The multivariable hyper geometric function defined by Srivastava & Daoust ([16-17])

$$\begin{aligned}&\mathrm{F}^{\mathrm{p}:\mathrm{q}_1;\ldots;\mathrm{q}_\mathrm{r}}_{l:\mathrm{m}_1;\ldots;\mathrm{m}_\mathrm{r}}\left[\begin{matrix}\left(\mathrm{a_j}\,;\alpha'_\mathrm{j},\ldots\alpha_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{p}}:\left(\mathrm{c}'_\mathrm{j},\gamma'_\mathrm{j}\right)_{1,\mathrm{q}_1};\ldots;\left(\mathrm{c}_\mathrm{j}^{(\mathrm{r})},\gamma_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{q}_\mathrm{r}};\\ \left(\mathrm{b_j}\,;\beta'_\mathrm{j},\ldots\beta_\mathrm{j}^{(\mathrm{r})}\right)_{1,l}:\left(\mathrm{d}'_\mathrm{j},\delta'_\mathrm{j}\right)_{1,\mathrm{m}_1};\ldots;\left(\mathrm{d}_\mathrm{j}^{(\mathrm{r})},\delta_\mathrm{j}^{(\mathrm{r})}\right)_{1,\mathrm{m}_\mathrm{r}};\end{matrix}\;\mathrm{z}_1,\ldots,\mathrm{z}_\mathrm{r}\right]\\ &=\sum_{\mathrm{n}_1,\ldots,\mathrm{n}_\mathrm{r}=0}^{\infty}\frac{\prod_{\mathrm{j}=1}^{\mathrm{p}}(\mathrm{a_j})_{\mathrm{n}_1\alpha'_\mathrm{j}+\ldots+\mathrm{n}_\mathrm{r}\alpha_\mathrm{j}^{(\mathrm{r})}}\prod_{\mathrm{j}=1}^{\mathrm{q}_1}(\mathrm{c}'_\mathrm{j})_{\mathrm{n}_1\gamma'_\mathrm{j}}\cdots\prod_{\mathrm{j}=1}^{\mathrm{q}_\mathrm{r}}\left(\mathrm{c}_\mathrm{j}^{(\mathrm{r})}\right)_{\mathrm{n}_\mathrm{r}\gamma_\mathrm{j}^{(\mathrm{r})}}}{\prod_{\mathrm{j}=1}^{l}(\mathrm{b_j})_{\mathrm{n}_1\beta'_\mathrm{j}+\ldots+\mathrm{n}_\mathrm{r}\beta_\mathrm{j}^{(\mathrm{r})}}\prod_{\mathrm{j}=1}^{\mathrm{m}_1}(\mathrm{d}'_\mathrm{j})_{\mathrm{n}_1\delta'_\mathrm{j}}\cdots\prod_{\mathrm{j}=1}^{\mathrm{m}_\mathrm{r}}\left(\mathrm{d}_\mathrm{j}^{(\mathrm{r})}\right)_{\mathrm{n}_\mathrm{r}\gamma_\mathrm{j}^{(\mathrm{r})}}}\,\frac{\mathrm{z}_1^{\mathrm{n}_1}}{\mathrm{n}_1!}\cdots\frac{\mathrm{z}_\mathrm{r}^{\mathrm{n}_\mathrm{r}}}{\mathrm{n}_\mathrm{r}!}\,,\end{aligned} \tag{7}$$

With variable and parametric constraints the above mentioned series is absolutely convergent.
We have by the definition of well-known Riemann-Liouville operator $D_z^{\mu}f\,(z)$ of fractional calculus (see [6,17]):

$$\begin{aligned}&\mathrm{D}_\mathrm{z}^{\lambda-\mu}\left\{\mathrm{z}^{\lambda-1}\prod_{\mathrm{j}=1}^{\mathrm{r}}\left\{\left(1-\mathrm{a_j}\,\mathrm{z}^{\mu_\mathrm{j}}\right)^{-\alpha_\mathrm{j}}\right\}\right\}\\ &=\frac{\Gamma(\lambda)}{\Gamma(\mu)}\,\mathrm{z}^{\mu-1}\,\mathrm{F}^{1:1;\ldots;1}_{1:0;\ldots;0}\left[\begin{matrix}(\lambda\,;\mu_1,\ldots\mu_\mathrm{r}):(\alpha_1,1);\ldots;(\alpha_\mathrm{r},1);\\ (\mu\,;\mu_1,\ldots\mu_\mathrm{r}):---;\ldots;---;\end{matrix}\;\mathrm{a}_1\mathrm{z}^{\mu_1},\ldots,\mathrm{a}_\mathrm{r}\mathrm{z}^{\mu_\mathrm{r}}\right],\\ &\left[\mathrm{R}\,(\lambda)>0;\,\mu_\mathrm{j}>0\;\;(\mathrm{j}=1,\ldots,\mathrm{r})\,;\max\left\{\left|\mathrm{a}_1\mathrm{z}^{\mu_1}\right|,\ldots,\left|\mathrm{a}_\mathrm{r}\mathrm{z}^{\mu_\mathrm{r}}\right|\right\}<1\right],\end{aligned} \tag{8}$$

where the D_z^{υ} is the Riemann-Liouville fractional differintegral operator (see [3,18-20])

$$D_z^{\upsilon}\,\{f\,(z)\}\;=\;\begin{cases}\frac{1}{\Gamma(-\upsilon)}\int_0^z(z-\zeta)^{-\upsilon-1}f(\zeta)\,d\zeta\,, & [R(\upsilon)<0]\\ \frac{d^n}{dz^n}D_z^{\upsilon-n}\{f(z)\}\,, & [0\le R(\upsilon)<n:n\in N_0]\end{cases} \tag{9}$$

which shows the defining integral in (9) exists.

Equation (3) reduces the following result by using the definition (8) applying r = 3 with, $\lambda = \mu - 1 = 1$ and z = sin ϕ, we find that

$$R\,(\phi,k,\xi\,;\alpha,\gamma) \;=\; \sin\phi\, F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1:2,2,2):(1/2-\gamma,1);(1/2,1);(\alpha,1);\\(2:2,2,2):\quad\;\;;\quad;\quad;\end{matrix}\,k^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right], \tag{10}$$

conditions are already defined in the (3) and (4).

In the same manner

$$I\,(\phi,k,\xi\,;\gamma) \;=\; \sin\phi\, F^{1:1;1;1}_{1:0;0;0}\left[\begin{matrix}(1;2,2,2):(1/2-\gamma,1);(1/2,1);(1,1);\\(2;2,2,2):\quad\;\;;\quad;\quad;\end{matrix}\,k^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right], \tag{11}$$

for details see [6].

By using the definition of Pochhammer symbol, we have

$$\frac{(1)_{2l+2m+2n}}{(2)_{2l+2m+2n}} \;=\; \frac{\Gamma(2l+2m+2n+1)}{\Gamma(2l+2m+2n+2)} \;=\; \frac{\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{2}\right)}\,\frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}} \;=\; \frac{\left(\frac{1}{2}\right)_{l+m+n}}{\left(\frac{3}{2}\right)_{l+m+n}}, \tag{12}$$

where we used the duplication formula defined bellow:

$$\Gamma(2z) \;=\; \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \tag{13}$$

with the help of above relation defined by equation (10), we can write the relation as

$$R\,(\phi,k,\xi\,;\alpha,\gamma) \;=\; \sin\phi\,.F_1\left[\frac{1}{2}\,:\,\frac{1}{2}-\gamma,\,\frac{1}{2},\,\alpha\,;\,\frac{3}{2}\,;\,k^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right], \tag{14}$$

$$\left(\left|k^2\right| < 1\,;0 \le \phi \le \frac{\pi}{2}\,;\gamma\in C,\,\alpha \ge 0\right),$$

and similarly $I\,(\phi,k,\xi\,;\gamma)$ can be defined in the similar manner, where F_1 denotes the particular case of the multivariable hypergeometric function given by Srivastava-Daoust for three variables defined in (7).

We know the definition of binomial expansion:

$$(1-z)^{-\lambda} \;=\; \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,z^n, \qquad (|z| < 1\,;\lambda\in C) \tag{15}$$

and

$$(1+z)^{-\lambda} \;=\; \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,(-z)^n, \qquad (|z| < 1\,;\lambda\in C)\,. \tag{16}$$

By the help of the binomial expansion, we can say that

$$\left(1 - k^2 \sin^2\theta\right)^{\gamma - \frac{1}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - \gamma\right)_n}{n!} k^{2n} \sin^{2n}\theta , \tag{17}$$

and

$$\left(1 + \xi \sin^2\theta\right)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \left(-\xi \sin^2\theta\right)^n . \tag{18}$$

We have by the definition of Beta function B (α, β):

$$\int_0^{\pi/2} \sin^{2\alpha-1}\theta \cos^{2\beta-1}\theta \, d\theta = \frac{1}{2} B(\alpha,\beta), \qquad \left(\min\{R(\alpha) . R(\beta)\} > 0 : B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right) \tag{19}$$

with help of these formulas and relations, we can easily establish Theorem 1.

2. THEOREMS AND COROLLARIES.

THEOREM 1. If ${}_p\psi_q$ is a Wright function [21], whose series representation is given by

$$ {}_p\psi_q \left[{}^{(a_1,\alpha_1),\ldots,(a_p,\alpha_p);}_{(b_1,\beta_1),\ldots,(b_q,\beta_q);} x\right] = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j r)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j r)} \frac{x^r}{r\,!}, \tag{20}$$

where α_i and β_j $(i = 1, \ldots, p; j = 1, \ldots, q)$ are real and positive, and $1 + \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0$.

Also consider that $\{\tau, \eta, \lambda, \mu\} \geq 0, (\tau + \eta > 0; \lambda + \mu > 0)$ and

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^{1/2(1+\rho)}} \right| < \infty, \qquad [\tau = 0; R(\rho) > -1] \tag{21}$$

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^{1+\sigma/2}} \right| < \infty, \qquad [\eta = 0; R(\sigma) > -2] \tag{22}$$

then

$$\begin{aligned} &\int_0^1 \mathrm{k}^{\rho} \left(\sqrt{(1-\mathrm{k}^2)}\right)^{\sigma} {}_{\mathrm{p}}\psi_{\mathrm{q}} \left[{}^{(\mathrm{a}_1,\alpha_1),\ldots,(\mathrm{a}_\mathrm{p},\alpha_\mathrm{p});}_{(\mathrm{b}_1,\beta_1),\ldots,(\mathrm{b}_\mathrm{q},\beta_\mathrm{q});} \mathrm{z}\mathrm{k}^{\tau} \left(\sqrt{(1-\mathrm{k}^2)}\right)^{\eta}\right] \mathrm{R}\left(\phi, \zeta\,\mathrm{k}^{\lambda}\left(\sqrt{(1-\mathrm{k}^2)}\right)^{\mu}, \xi\,; \alpha\,, \gamma\right) \mathrm{dk} \\ &= \frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)} \frac{\sin\phi}{2} \mathrm{B}\left(\frac{\rho+1}{2}, \frac{\sigma+2}{2}\right) . \mathrm{F}^{3:n;1;1;1}_{2:n;0;0;0} \left[{}^{\left(\frac{1}{2}:0,1,1,1\right), \left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right), \left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):}_{\left(\frac{3}{2}:0,1,1,1\right), \left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):} \right. \\ &\qquad \left. {}^{(a_1,\alpha_1),\ldots,(a_n,\alpha_n); \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (\alpha,1);}_{(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;} z, \zeta^2 \sin^2\phi, \sin^2\phi, -\xi \sin^2\phi \right], \end{aligned} \tag{23}$$

where $R(\rho) > -1$ *and* $|\zeta| < 1$ $[$*or* $|\zeta| = 1$ *and* $R(\rho + 2\lambda) > -1]$.

PROOF. To establish the result defined in Theorem 1, we use the values of $R\left(\phi,\zeta k^{\lambda}\kappa^{\mu},\xi;\alpha,\gamma\right)$ and ${}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix}x\right]$ from equations (3) and (18) respectively, we get the required result after simplification by using the formulas which are defined above.

COROLLARY 1. With the help of definition new elliptic function defined in equation (5), we can establish the following result

$$\int_0^1 k^{\rho}\left(\sqrt{(1-k^2)}\right)^{\sigma}{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix}zk^{\tau}\left(\sqrt{(1-k^2)}\right)^{\eta}\right] I\left(\phi,\zeta k^{\lambda}\left(\sqrt{(1-k^2)}\right)^{\mu},\xi;\gamma\right)dk$$
$$=\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}\frac{\sin\phi}{2}B\left(\frac{\rho+1}{2},\frac{\sigma+2}{2}\right).F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{matrix}\left(\frac{1}{2}:0,1,1,1\right),\left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right),\left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):\\\left(\frac{3}{2}:0,1,1,1\right),\left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):\end{matrix}\right.$$
$$\left.\begin{matrix}(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1);\\(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;\end{matrix}z,\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right],\tag{24}$$

with help of Theorem 1 we can determine the above Corollary 1, by putting $\alpha = 1$.

COROLLARY 2. With the help of elliptic integral of third kind (see [6]), we can establish the following result

$$\int_0^1 k^{\rho}\left(\sqrt{(1-k^2)}\right)^{\sigma}{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix}zk^{\tau}\left(\sqrt{(1-k^2)}\right)^{\eta}\right]\Pi\left(\phi,\zeta k^{\lambda}\left(\sqrt{(1-k^2)}\right)^{\mu},\xi\right)dk$$
$$=\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}\frac{\sin\phi}{2}B\left(\frac{\rho+1}{2},\frac{\sigma+2}{2}\right).F^{3:n;1;1;1}_{2:n;0;0;0}\left[\begin{matrix}\left(\frac{1}{2}:0,1,1,1\right),\left(\frac{\rho+1}{2}:\frac{\tau}{2},\lambda,0,0\right),\left(\frac{\sigma+2}{2}:\frac{\eta}{2},\mu,0,0\right):\\\left(\frac{3}{2}:0,1,1,1\right),\left(\frac{\rho+\sigma+3}{2}:\frac{\eta+\tau}{2},\lambda+\mu,0,0\right):\end{matrix}\right.$$
$$\left.\begin{matrix}(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1);\\(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;\end{matrix}z,\zeta^2\sin^2\phi,\sin^2\phi,-\xi\sin^2\phi\right],\tag{25}$$

with help of Theorem 1 we can determine the above Corollary 2, by putting $\alpha = 1$ and $\gamma = 0$.

THEOREM 2. The following families of integrals hold true

$$\int_0^{\pi/2}\sin^{2(a-1)}\phi\cos^{2b-1}\phi\,{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\(b_1,\beta_1),\ldots,(b_q,\beta_q);\end{smallmatrix}zk\right]R\left(\phi,k,\xi;\alpha,\gamma\right)d\phi=\frac{1}{2}\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}B\left(a,b\right)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{matrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\ldots,(a_n,\alpha_n);\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(\alpha,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\ldots,(b_n,\beta_n);-;-;-;\end{matrix}zk,k^2,1,-\xi\right],\tag{26}$$

$$\left[\left|k^2\right|<1:\min\left\{R\left(a\right),R\left(b\right),R(a_i),R(b_i)\right\}>0,i=1,2,\ldots;\gamma\in C\right].$$

and

$$\int_0^w x^{2(a-1)}\left(w^2-x^2\right)^{b-1}{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}zk\right]R\left(arc\sin\frac{x}{w},k,\xi;\alpha,\gamma\right)dx$$
$$=\frac{1}{2}w^{2a+2b-3}\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}B(a,b)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\dots,(a_n,\alpha_n);\ \left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);\ (\alpha,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\dots,(b_n,\beta_n);-;-;-;\end{smallmatrix}zk,\ k^2,1,-\xi\right],\tag{27}$$

only if the second member of each of the integral formulas defined in equations (26) and (27) occurs.

PROOF. After replacing the $R(\phi,k,\xi;\alpha,\lambda)$ from equation (10) and the value of Wright function from equation (20) in to the integral of the affirmation equation (26) of Theorem 2, if we use the trigonometric integral (19) as it is, we can find the integral formula (26) as given above.

COROLLARY 1. With help of definition of new elliptical function defined in equation (5), we can establish the following result

$$\int_0^{\pi/2}\sin^{2(a-1)}\phi\cos^{2b-1}\phi\,{}_p\psi_q\left[\begin{smallmatrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p);\\(b_1,\beta_1),\dots,(b_q,\beta_q);\end{smallmatrix}zk\right]I(\phi,k,\xi;\gamma)\,d\phi=\frac{1}{2}\frac{\prod_{n=1}^{\infty}\Gamma(a_n)}{\prod_{n=1}^{\infty}\Gamma(b_n)}B(a,b)$$
$$F^{2:1;1;1}_{2:0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):(a_1,\alpha_1),\dots,(a_n,\alpha_n);\ \left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);\ (1,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):(b_1,\beta_1),\dots,(b_n,\beta_n);-;-;-;\end{smallmatrix}zk,\ k^2,1,-\xi\right],\tag{28}$$

or

$$\int_0^{\pi/2}\sin^{2(a-1)}(\phi)\cos^{2b-1}(\phi)\,I(\phi,k,\xi;\gamma)\,d\phi$$
$$=\frac{1}{2}B(a,b)\,F^{2;1;1;1}_{2;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):\left(\frac{1}{2}-\gamma,1\right);\left(\frac{1}{2},1\right);(1,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):-;-;-;\end{smallmatrix}k^2,1,-\xi\right].\tag{29}$$

This Corollary can be found with help of Theorem 2 by putting $\alpha=1$.

COROLLARY 2. With help of elliptical integral of third kind (see[6]), we can establish the following result

$$\int_0^{\pi/2}\sin^{2(a-1)}(\phi)\cos^{2b-1}(\phi)\,\Pi(\phi,k,\xi)\,d\phi$$
$$=\frac{1}{2}B(a,b)\,F^{2;1;1;1}_{2;0;0;0}\left[\begin{smallmatrix}\left(\frac{1}{2}:1,1,1\right),(a:1,1,1):\left(\frac{1}{2},1\right);\left(\frac{1}{2},1\right);(1,1);\\\left(\frac{3}{2}:1,1,1\right),(a+b:1,1,1):-;-;-;\end{smallmatrix}k^2,1,-\xi\right].\tag{30}$$

This Corollary can be find with the help of Theorem 2 by putting $\alpha=1$ and $\gamma=0$.

REMARK 2. On changing the following variables

$$\phi = arc\sin x \qquad and \qquad d\phi = \frac{dx}{\sqrt{1-x^2}} \qquad with\, x \in (0,1) \tag{31}$$

equation (26) can be rewritten as

$$\begin{aligned} &\int_0^1 x^{2(a-1)} \left(1-x^2\right)^{b-1} R\left(arc\sin x, k, \xi\,; \alpha, \gamma\right) dx \\ &= \frac{1}{2} B\left(a,b\right) F^{2:1;1;1}_{2:0;0;0} \left[\begin{array}{l} \left(\frac{1}{2}:1,1,1\right), (a:1,1,1): \ \left(\frac{1}{2}-\gamma,1\right); \left(\frac{1}{2},1\right); (\alpha,1); \\ \left(\frac{3}{2}:1,1,1\right), (a+b:1,1,1): -; -; -; \end{array} k^2, 1, -\xi \right], \end{aligned} \tag{32}$$

$$\left[\left|k^2\right| < 1 : \min\left\{R\left(\alpha\right), R\left(\beta\right)\right\} > 0; \gamma \in C\right]$$

Here equation (32) can be equated with equation (27) stated by Theorem 2.

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On Some New Classes of Bi-univalent Functions

M. DARUS AND S. SINGH

Abstract

In the present paper, we introduce and investigate two new subclasses $\mathscr{Q}_{\Sigma}(n,\gamma,k)$ and $\mathscr{B}_{\Sigma}(n,\beta,k)$ of bi-valent functions in the unit disk $\mathbb{U}$. For functions belonging to the classes $\mathscr{Q}_{\Sigma}(n,\gamma,k)$ and $\mathscr{B}_{\Sigma}(n,\beta,k)$, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{A}$ be the class of analytic functions defined on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized condition $f(0) = 0 = f'(0) - 1$. Let $\mathscr{S}$ be the class of all functions $f \in \mathscr{A}$ which are univalent in Δ. So $f(z) \in \mathscr{S}$ has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

Let $f^{-1}(z)$ be inverse of the function $f(z)$ and it is well known that every function $f \in \mathscr{S}$ has an inverse $f^{-1}(z)$, defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad for \quad |w| < r_0(f); r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(w)$ are univalent in U.
Let Σ denote the class of bi-univalent functions in U given by (1).
Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions (including also

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several open problems and conjectures involving bounds of the coefficients of the functions in Σ), can be found in the earlier work studied by Lewin[17], Brannan and Clunie [16], Netanyahu[18] and others. They introduced subclasses of Σ, like class of bi-starlike and convex functions, bi-strongly starlike and convex functions similar to the well-known subclasses $\mathscr{S}^*(\alpha)$ and $\mathscr{K}^*(\alpha)$ of starlike and convex functions of order α $(0 < \alpha < 1)$, respectively (see [15]) and obtained non-sharp estimates on the initial coefficients in the Taylor-Maclaurin series expansion (1) see[16; 9; 10]. More recently, Srivastava et al. [8; 12; 13], Frasin and Aouf [11], R.M. Ali et al. [14] and Porwal and Darus [6] introduced some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1).
Motivated by the work of Porwal and Darus [6], we introduce a new subclass $\mathscr{Q}_\Sigma(k,n,\alpha,\gamma)$.

DEFINITION 1.1. A function f given by (1) is said to be in the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ if the following conditions are satisfied:
For $n \in Z$, $0 \leq \gamma < 1$, $\alpha \geq 1, \lambda \geq 0$ we introduce the subclass $\mathscr{Q}_\Sigma(n,\gamma,k)$ of S of functions of the form (1) satisfying the condition

$$f \in \Sigma \quad and \quad \left| arg\left(\frac{(1-\alpha)I_\lambda^n f(z) + \alpha I_\lambda^{n+1} f(z)}{z}\right)\right| < \frac{\gamma\pi}{2} \quad z \in U, \tag{3}$$

$$f \in \Sigma \quad and \quad \left| arg\left(\frac{(1-\alpha)I_\lambda^n g(w) + \alpha I_\lambda^{n+1} g(w)}{w}\right)\right| < \frac{\gamma\pi}{2} \quad z \in U, \tag{4}$$

where

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

And

$$I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} (1+\lambda(k-1))^n a_k z^k, \quad z \in \Delta, \quad \lambda \geq 0, \quad n \in \mathbb{Z}.$$

is generalized *Sălăgean* derivative defined by [2].

This generalized operator is studied by many and mentioned again by [3]. For $k = 1$, this class is introduced and investigated in [6]. For $n = 0$ and $\lambda = 1$ the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ reduces to H_Σ^α introduced and studied by Srivastava et al. [8] and for $n = 0$ the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ reduces to $\mathscr{B}_\Sigma(\alpha,\lambda)$ introduced and studied by Frasin and Aouf

[11]. In this paper, we investigate the estimates for the initial coefficients a_2 and a_3 of bi-univalent functions belonging to the class $\mathcal{Q}_{\Sigma}(n,\gamma,k)$. Our results generalize several well-known results in [1; 4; 5; 10] and these are pointed out. In order to prove our main result we need the following lemma:

LEMMA 1.1. [3] If $p \in \mathcal{P}$, then $|c_k| \le 2$ for each k, where $\mathcal{P}$ is the family of all functions $p(z)$ analytic in U for which $\operatorname{Re} p(z) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ for $z \in U$.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{Q}_{\Sigma}(N,\gamma,K)$

THEOREM 2.1. Let $f(z)$ given by (1) be in the class $\mathcal{Q}_{\Sigma}(n,\gamma,k)$, $k \in \mathbb{N}$, $n \in Z$, $0 \le \gamma < 1$, $\alpha \ge 1$,$\lambda \ge 0$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{(1+\lambda)^{2n}(1+\lambda\alpha)^2 + \gamma[2(1+2\lambda)^n(1+2\lambda\alpha) - (1+\lambda)^{2n}(1+\lambda\alpha)^2]}}, \quad (5)$$

and

$$|a_3| \le \frac{2\gamma}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]} + \frac{4\gamma^2}{2[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]}. \quad (6)$$

PROOF. It follows from (3) and (4) that

$$\frac{(1-\alpha)I_\lambda^n f(z) + +\alpha I_\lambda^{n+1} f(z)}{z} = (p(z))^\gamma \quad (7)$$

$$\frac{(1-\alpha)I_\lambda^n g(w) + +\alpha I_\lambda^{n+1} g(w)}{w} = (q(w))^\gamma \quad (8)$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ in $\mathcal{P}$. Now on equating the coefficients in (7) and (8), we have

$$[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = \gamma p_1 \quad (9)$$

$$[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]a_3 = \gamma p_2 + \frac{\gamma(\gamma-1)}{2} p_1^2 \quad (10)$$

$$-[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = \gamma q_1 \quad (11)$$

and

$$[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](2a_2^2-a_3)=\gamma q_2+\frac{\gamma(\gamma-1)}{2}q_1^2. \tag{12}$$

From (9) and (11) we get

$$p_1=-q_1 \tag{13}$$

and

$$2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]a_2^2=\gamma^2(p_1^2+q_1^2) \tag{14}$$

From (10), (12) and (14), we get

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]a_2^2$$

$$=(p_2+q_2)\gamma+\frac{\gamma(\gamma-1)}{2}(p_1^2+q_1^2)$$

$$=(p_2+q_2)\gamma+\frac{\gamma(\gamma-1)}{2}\frac{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]}{\alpha^2}a_2^2.$$

Therefore, we have

$$a_2^2=\frac{\gamma^2(p_2+q_2)}{(1+\lambda)^{2n}(1+\lambda\alpha)^2+\gamma[2(1+2\lambda)^n(1+2\lambda\alpha)-(1+\lambda)^{2n}(1+\lambda\alpha)^2]} \tag{15}$$

Applying Lemma 1.1 for (15), we get

$$|a_2|\le\frac{2\alpha}{\sqrt{(1+\lambda)^{2n}(1+\lambda\alpha)^2+\gamma[2(1+2\lambda)^n(1+2\lambda\alpha)-(1+\lambda)^{2n}(1+\lambda\alpha)^2]}}.$$

which gives us desired estimate on $|a_2|$ as asserted in (5).

Next in order to find the bound on $|a_3|$, by subtracting (12) from (10), we get

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](a_3-a_2^2)=\gamma(p_2-q_2)+\frac{\gamma(\gamma-1)}{2}(p_1^2-q_1^2) \tag{16}$$

It follows from (13), (14) and (16)

$$a_3=\frac{\gamma(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}+\frac{\gamma^2(p_1^2+q_1^2)}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]} \tag{17}$$

Applying Lemma 1.1 for (17), we get

$$|a_3| \leq \frac{2\gamma}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}+\frac{4\gamma^2}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]}.$$

This completes the proof of Theorem 2.1.

3. COEFFICIENT BOUNDS FOR THE FUNCTION $\mathscr{B}_\Sigma(N,\beta,K)$

DEFINITION 3.1. A function f given by (1) is said to be in the class $\mathscr{B}_\Sigma(n,\beta,k)$ if the following conditions are satisfied:
For $n \in Z$, $0 \leq \beta < 1$, $\alpha \geq 1, \lambda \geq 0$, we introduce the subclass $\mathscr{B}_\Sigma(n,\beta,k)$ of S of functions of the form (1) satisfying the condition

$$f \in \Sigma \quad and \quad \mathscr{R}e\left(\frac{(1-\alpha)I_\lambda^n f(z)+\alpha I_\lambda^{n+1} f(z)}{z}\right) > \beta \quad z \in U, \tag{18}$$

$$f \in \Sigma \quad and \quad \mathscr{R}e\left(\frac{(1-\alpha)I_\lambda^n g(w)+\alpha I_\lambda^{n+1} g(w)}{w}\right) > \beta \quad z \in U, \tag{19}$$

where

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

And $I_\lambda^n f(z)$ is generalized *Sălăgean* derivative defined by [2].

For $k = 1$ and $n = 0$, the class $\mathscr{B}_\Sigma(n,\beta,k)$ reduces the class $\mathscr{H}_\Sigma(n,\beta,\lambda)$ and $\mathscr{H}_\Sigma(\beta,\lambda)$ studied by Porwal and Darus [6] and Frasin and Aouf [11], respectively. For $n = 0$, $\lambda = 1$, this class reduces to $\mathscr{H}_\Sigma(\lambda)$ studied by Srivastava et al. [8].

THEOREM 3.1. Let $f(z)$ given by (1) be in the class $\mathscr{B}_\Sigma(n,\beta,k)$, $n \in Z$, $0 \leq \beta < 1$, $\alpha \geq 1$, $\lambda \geq 0$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta))}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]^2}} \tag{20}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]^2}+\frac{2(1-\beta)}{[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}. \tag{21}$$

PROOF. It follows from (18) and (19) that there exists $p(z) \in P$ and $q(z) \in P$

$$\frac{(1-\alpha)I_\lambda^n f(z) + +\alpha I_\lambda^{n+1} f(z)}{z} = \beta + (1-\beta)p(z) \tag{22}$$

$$\frac{(1-\alpha)I_\lambda^n g(w) + +\alpha I_\lambda^{n+1} g(w)}{w} = \beta + (1-\beta)q(w) \tag{23}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ in $\mathscr{P}$. Now on equating the coefficients in (22) and (23), we have

$$[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = (1-\beta)p_1 \tag{24}$$

$$([(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]a_3 = (1-\beta)p_2 \tag{25}$$

$$-[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = (1-\beta)q_1 \tag{26}$$

and

$$[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}](2a_2^2 - a_3) = (1-\beta)q_2 \tag{27}$$

From (24) and (26) we get

$$p_1 = -q_1 \tag{28}$$

and

$$2[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \tag{29}$$

From (25) and (27), we get

$$2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2 a_2^2 = (p_2 + q_2)(1-\beta) \tag{30}$$

From (29) and (30), we get

$$|a_2|^2 \le \frac{(1-\beta)(|p_2|^2 + |q_2|^2)}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2} \tag{31}$$

and

$$|a_2^2| \le \frac{2(1-\beta))}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2}. \tag{32}$$

Which is the bound on $|a_2|$ as given in (20).

Next, in order to find the bound on $|a_3|$ by subtracting (29) from (25), we obtain

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](a_3-2a_2^2)=(1-\beta)(p_2-q_2) \tag{33}$$

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]} \tag{34}$$

On substituting the value $|a_2^2|$ from (31), we have

$$a_3=\frac{(1-\beta)^2(|p_2|^2+|q_2|^2)}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]^2}+\frac{(1-\beta)(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]} \tag{35}$$

On applying Lemma 1.1 for the coefficients p_1, q_1, p_2 and q_2, we obtain

$$|a_3|\leq\frac{4(1-\beta)^2}{[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]^2}+\frac{2(1-\beta)}{[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}. \tag{36}$$

This completes the proof of Theorem 3.1.

ACKNOWLEDGEMENT

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Some Cubic Rank Transmuted Distributions

N. CELIK

Abstract

In this article, we introduce some examples of cubic rank transmuted distributions proposed by Granzatto et al. (2017). The statistical aspects of the introduced distributions such as probability density functions, hazard rate functions and reliability functions are studied. The maximum likelihood estimation method is used in order to estimate the parameters of interest. Finally, real data examples are applied for the illustration of these distributions.

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Keywords: Cubic Rank Transmution, Frechet Distribution, Gumbel Distribution, Gombertz Distribution, Maximum Likelihood.

1. INTRODUCTION

In order to obtain more flexible statistical models, generalization of the well-known distributions have been widely used. Firstly, Amoroso (1925) introduced the generalized gamma distribution in order to model the distribution of income rate. Since then various authors have discussed the generalizations of the distributions. Good (1953), for example, proposed the inverse Gaussian distribution. Ljubo (1965), Pickands (1975) and Hoskings and Wallis (1987) made generalization of Pareto distribution. The generalized beta of the first and second kind was introduced by McDonald (1984) to study the distribution of income.

Shaw and Buckley (2007) proposed a new generalization method called transmution mapping. According to them a ranking quadratic transmutation (QRT) map is

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, |\lambda| < 1 \tag{1}$$

where $G(x)$ is the cumulative distribution function (cdf) of the base distribution. It should be noted that, when $\lambda = 0$, the new distribution becomes the original distribution.

This method have been used by many researchers to obtain new distributions, see Aryal and Tsokos (2011), Aryal (2013), Elbatal and Aryal (2013) and Merovci (2013). Recently, Granzatto et al. (2017) introduced a new family of transmuted distributions, the cubic rank transmutation (CRT) map distribution and to demonstrate the usefulness

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of this method CRT Weibull and log-logistic distribution are used in their article. This new method enables to fit complex data sets with bimodal hazard rates. The cdf and the probability density function (pdf) of a CRT distribution are given

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)[G(x)]^2 + (1-\lambda_2)[G(x)]^3 \tag{2}$$

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)[G(x)]^2] \tag{3}$$

respectively. Here , $\lambda_1 \in [0,1]$ $\lambda_2 \in [-1,1]$ and $g(x)$ is the pdf of the base distribution. The proofs and the further details can be found in Granzatto et al. (2017).

In this paper, we are motivated to generate a new family of the distributions in order to get more flexible fitting. For this reason further examples of CRT distributions are introduced. The rest of the paper organizes as follows, in Section 2-4, we offer Frechet, Gumbel and Gombertz distributions which are commonly used as life-time distributions in survival analysis. The cubic rank transmutation method is applied to these distributions and some mathematical and statistical properties of these new distributions are derived. The maximum likelihood estimations of the parameters of interest are obtained. In Section 5, real data examples, which were previously studied with Frechet, Gumbel and Gombertz distribution are fitted into the cubic rank transmuted version of the base distributions. A conclusion is given at the end of this paper.

2. CUBIC RANK TRANSMUTED FRECHET DISTRIBUTION

Generalized extreme value (GEV) distribution covers the well known probability distributions developed within extreme value theory and it combines Gumbel, Frechet and Weibull families. It is proposed by Jenkinson (1955) in order to model extreme values based on Fisher-Tippet theorem. The class of GEV distributions is very flexible, since it can be represented by single shape parameter (ξ) which controls the tail behaviour with three different distribution families. If $\xi = 0$, then the distribution has thin tail behaviour and is called Gumbel type distribution. When $\xi > 0$, then the distribution has fat tail and is called Frechet type distribution which includes well known fat tailed distribution such as Pareto, Student-t and Cauchy. Finally, if $\xi < 0$, the distribution class converts to Weibull which has short tail behaviour and includes uniform and beta distribution.

A random variable X is said to have a Frechet distribution with parameters $\mu > 0$ and $\sigma > 0$ if its pdf is given by

$$g(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \tag{4}$$

The cdf of Frechet distribution is

$$G(x) = e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \tag{5}$$

Mahmoud and Mandouh (2013) introduced the QRT Frechet distribution and studied its statistical properties. Now using (2) the cdf of cubic rank transmuted Frechet (CRTF) distribution with parameters μ, σ, λ_1 and λ_2 takes the form

$$F(x) = \lambda_1 e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^3 \tag{6}$$

and the pdf of CRT Frechet distribution becomes

$$f(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \left(\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right) \tag{7}$$

Figure 1 shows the pdfs of the CRT Frechet distributions for different λ_1 and λ_2 values.

It can be seen from the Figure 1, for some special λ_1 and λ_2 values the distribution become the bimodal distribution.

The hazard rate function for the CRT Frechet distribution is given by

$$h(x) = \frac{\frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \left(\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right)}{1 - \lambda_1 e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^3} \tag{8}$$

Figure 2 illustrates some of the possible shapes of the hazard function of a CRT Frechet distribution for selected values of the parameters λ_1 and λ_2.

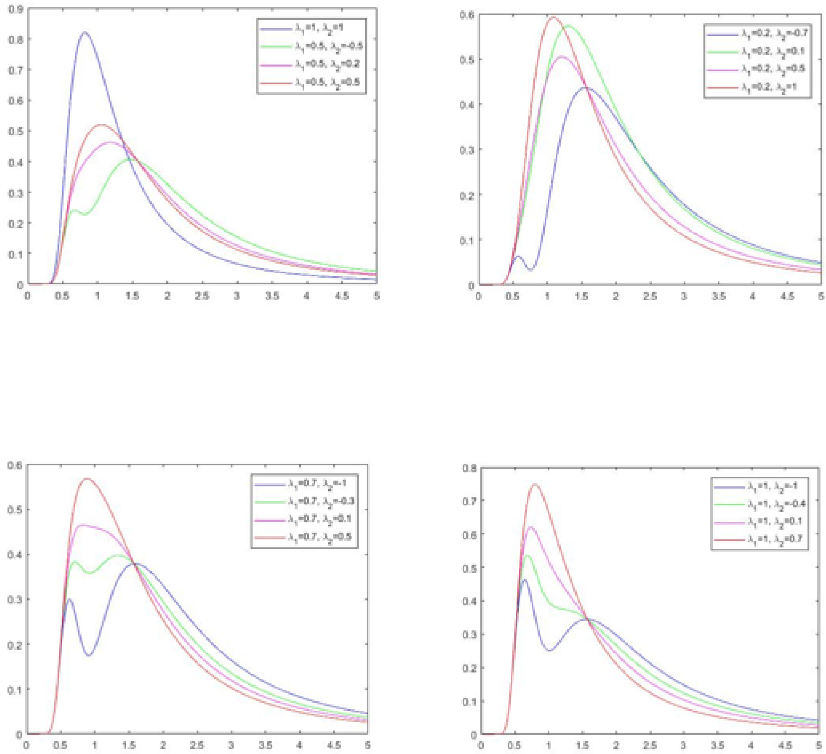

Fig. 1. The pdfs of CRT Frechet distribution, $\alpha = 2$, $\sigma = 1$.

The moments of the proposed distribution can be found easily by using the following integration

$$E(X^k) = \int x^k \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{-1-\alpha} e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} \left(\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{x}{\sigma}\right)^{-\alpha}\right)}\right]^2\right) dx \quad (9)$$

Taking $t = \left(\frac{x}{\sigma}\right)^{-\alpha}$, we can obtain the general formula of the moments of the distribution as

$$E(X^k) = \Gamma\left(1 - \frac{k}{\alpha}\right)\left[\lambda_1 + (\lambda_2 - \lambda_1)2^{1/\alpha} + (1-\lambda_2)3^{1/\alpha}\right]\sigma \quad (10)$$

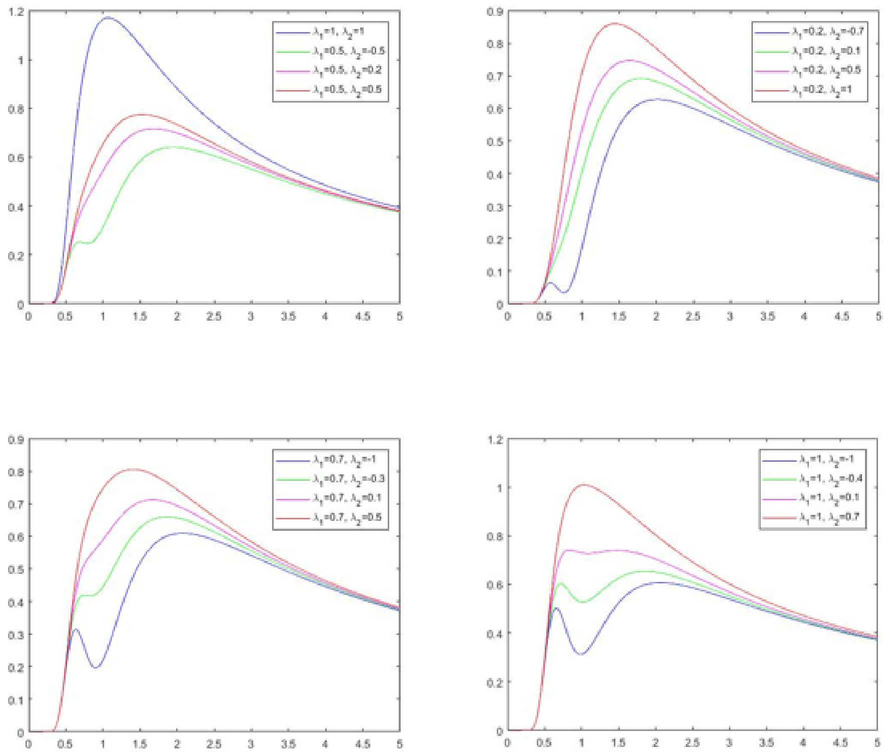

Fig. 2. The hazard rate functions of CRT Frechet distribution, $\alpha = 2$, $\sigma = 1$.

For generating random numbers from the distribution, one can use the method of inversion. After simple calculation this yields

$$x = \sigma\Big\{-ln\Big[\Big(q+\big(q^2+(r-p^2)^3\big)^{1/2}\Big)^{1/3} + \Big(q-\big(q^2+(r-p^2)^3\big)^{1/2}\Big)^{1/3} + p\Big]\Big\}^{-1/\alpha} \quad (11)$$

where $p = -\frac{\lambda_2-\lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2-\lambda_1)+3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Suppose $X_1, X_2, ..., X_n$ are random samples from a CRT Frechet distribution defined in (7), then the likelihood function is given by

$$L = \Big(\frac{\alpha}{\sigma}\Big)^n e^{\left(-\Sigma_{i=1}^n \left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)} \prod_{i=1}^n \Big(\frac{x_i}{\sigma}\Big)^{-1-\alpha} \prod_{i=1}^n \Big(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)} + 3(1-\lambda_2)\Big[e^{\left(-\left(\frac{x_i}{\sigma}\right)^{-\alpha}\right)}\Big]^2\Big) \quad (12)$$

and the log-likelihood function is

$$lnL = nln(\alpha) - nln(\sigma) - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\alpha} - (1+\alpha)\sum_{i=1}^{n} ln\left(\frac{x_i}{\sigma}\right) + \sum_{i=1}^{n} ln\left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\right) \tag{13}$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero, we obtain the following likelihood equations.

$$\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^{\alpha} ln\left(\frac{\sigma}{x_i}\right) + \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right) \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\left(\frac{\sigma}{x_i}\right)^{\alpha} ln\left(\frac{\sigma}{x_i}\right) + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\left(\frac{\sigma}{x_i}\right)^{\alpha} ln\left(\frac{\sigma}{x_i}\right)}{\left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\right)} \\
\frac{\partial \ln L}{\partial \sigma} &= -\frac{n}{\sigma} - \sum_{i=1}^{n}\frac{\sigma}{x_i}\left(\frac{\sigma}{x_i}\right)^{\alpha} + (\alpha+1)\frac{n}{\sigma} \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\alpha\left(\frac{\sigma}{x_i}\right)^{\alpha-1}\frac{1}{x_i} + 6(1-\lambda_2)[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}]^2\alpha\left(\frac{\sigma}{x_i}\right)^{\alpha-1}\frac{1}{x_i}}{\left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\right)} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1-2e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}}{\left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\right)} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} - 3[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}]^2}{\left(\lambda_1 + 2(\lambda_2-\lambda_1)e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)} + 3(1-\lambda_2)\left[e^{\left(-\left(\frac{\sigma}{x_i}\right)^{\alpha}\right)}\right]^2\right)}
\end{aligned} \tag{14}$$

Solutions of these equations are called ML estimates. However, the equations must be solved with numerical methods such as Newton Raphson or iteratively reweighting algorithm.

3. CUBIC RANK TRANSMUTED GUMBEL DISTRIBUTION

A random variable X is said to have a Gumbel distribution with parameters if its pdf and cdf is given by

$$g(x) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \tag{15}$$

and

$$G(x) = e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \tag{16}$$

respectively.

Now using (2) the cdf of cubic rank transmuted Gumbel (CRT Gumble) distribution with parameters is

$$F(x) = \lambda_1 e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^3 \tag{17}$$

and the pdf of CRT Gumbel distribution takes the form

$$f(x) = \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2\right\} \tag{18}$$

Figure 3 and Figure 4 show the pdfs and the hazard rate functions of the CRT Gumbel distribution for representative λ values respectively.

The hazard rate function for the distribution is

$$h(x) = \frac{\frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \left\{\lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2\right\}}{1 - \left\{\lambda_1 e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + (\lambda_2 - \lambda_1)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^2 + (1-\lambda_2)\left[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right]^3\right\}} \tag{19}$$

The moments of CRT Gumbel distribution can be found as

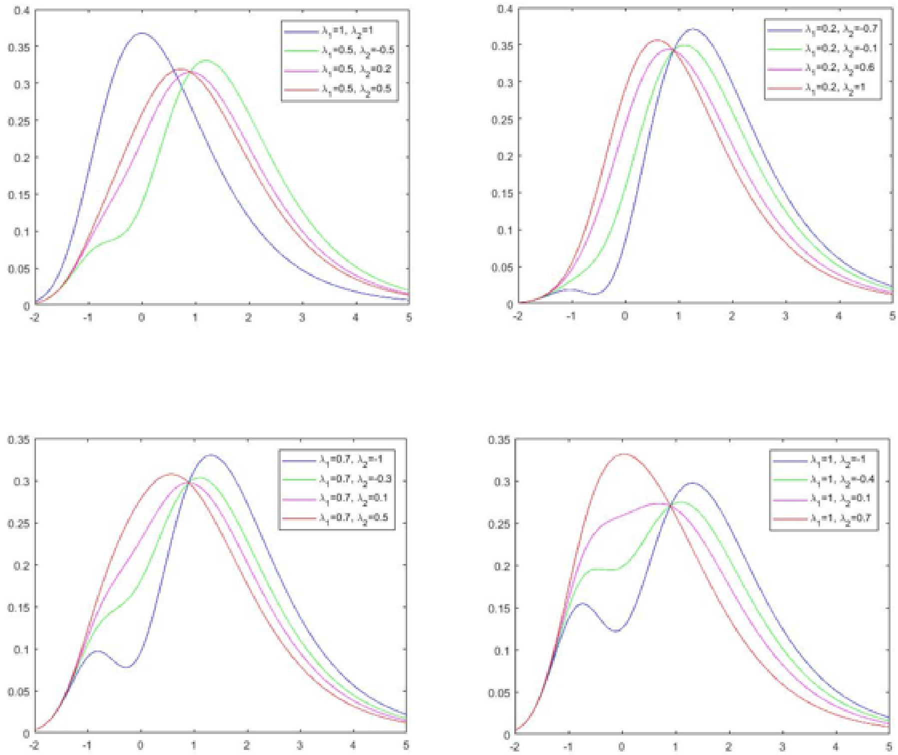

Fig. 3. The pdfs of CRT Gumbel distribution, $\mu = 0$, $\sigma = 1$.

$$E(X^k) = \int_0^\infty x^k \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sigma} + e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \Big\{ \lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} \Big]^2 \Big\} dx \tag{20}$$

By taking $y = exp\left(-\frac{x-\mu}{\sigma}\right)$ the moments can be obtained like

$$E(X^k) = \sum_{i=0}^{n} (-1)^n \binom{n}{i} \sigma^i \mu^{n-i} \Big[\lambda_1 \frac{\partial i}{\partial \nu^i} \Gamma(\nu) + 2(\lambda_2 - \lambda_1) \frac{\partial i}{\partial \nu^i} (2^{-\nu}\Gamma(\nu)) + 3(1-\lambda_2) \frac{\partial i}{\partial \nu^i} (3^{-\nu}\Gamma(\nu)) \Big] |_{\nu=1} \tag{21}$$

For generation random numbers the following formula can be used.

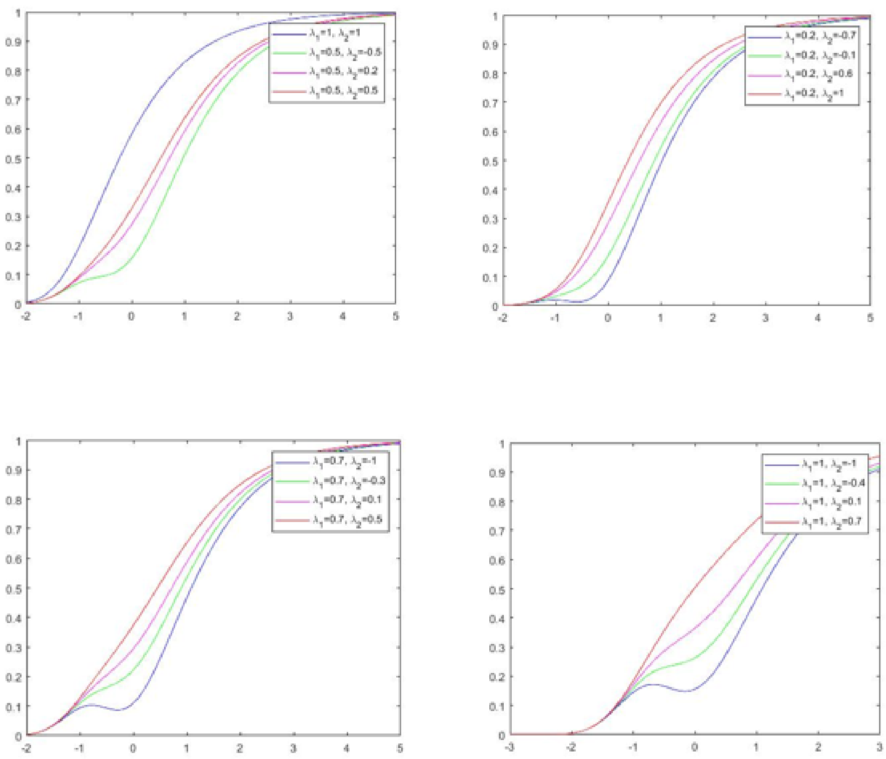

Fig. 4. The hazard rate functions of CRT Gumbel distribution, $\mu = 0$, $\sigma = 1$.

$$x = \mu - \sigma\Big(ln\Big\{-ln\Big[\Big(q + (q^2 + (r - p^2)^3)^{1/2}\Big)^{1/3} + \Big(q - (q^2 + (r - p^2)^3)^{1/2}\Big)^{1/3} + p\Big]\Big\}\Big) \tag{22}$$

where $p = -\frac{\lambda_2 - \lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2 - \lambda_1) + 3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Now, in order to obtain the ML estimators of the parameters, the likelihood function

$$L = \sigma^{-n} e^{-\Sigma_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma} + e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} \prod_{i=1}^{n}\Big\{\lambda_1 + 2(\lambda_2 - \lambda_1)e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2\Big\} \tag{23}$$

and the log-likelihood function

$$lnL = -nln(\sigma) - \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} + e^{\left(-\frac{x_i-\mu}{\sigma}\right)} \right) + \sum_{i=1}^{n} ln\Big\{ \lambda_1 + 2(\lambda_2 - \lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} \Big]^2 \Big\} \tag{24}$$

can be obtained respectively. And the likelihood equations are

$$\begin{aligned}
\frac{\partial \ln L}{\partial \mu} &= n\mu - \sum_{i=1}^{n} e^{-\left(\frac{x_i-\mu}{\sigma}\right)} \qquad (25) \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} e^{\left(-\frac{x-\mu}{\sigma}\right)} + 6(1-\lambda_2)\left(e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right)^2 e^{\left(-\frac{x-\mu}{\sigma}\right)}}{\lambda_1 + 2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \\
\frac{\partial \ln L}{\partial \sigma} &= -n + \sum_{i=1}^{n} \frac{x_i-\mu}{\sigma} - \sum_{i=1}^{n} e^{-\left(\frac{x_i-\mu}{\sigma}\right)} \frac{x_i-\mu}{\sigma} \\
&+ \sum_{i=1}^{n} \frac{2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} e^{\left(-\frac{x-\mu}{\sigma}\right)} \frac{x_i-\mu}{\sigma} + 6(1-\lambda_2)\left(e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\right)^2 e^{\left(-\frac{x-\mu}{\sigma}\right)} \frac{x_i-\mu}{\sigma}}{\lambda_1 + 2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1 - 2e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}}{\lambda_1 + 2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)} - 3\Big[e^{\left(-e^{\left(-\frac{x-\mu}{\sigma}\right)}\right)}\Big]^2}{\lambda_1 + 2(\lambda_2-\lambda_1) e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)} + 3(1-\lambda_2)\Big[e^{\left(-e^{\left(-\frac{x_i-\mu}{\sigma}\right)}\right)}\Big]^2}
\end{aligned}$$

By equating them to zero and solving the equations the ML estimators of the unknown parameters can be obtained.

4. CUBIC RANK TRANSMUTED GOMPERTZ DISTRIBUTION

The Gompertz distribution has been widely used in actuarial sciences especially in calculation of adult deaths. The pdf and the cdf of Gompetz distribution are given

$$g(x) = \alpha\beta e^{\alpha x} e^{\beta} exp\Big(- \beta e^{\alpha x} \Big) \tag{26}$$

$$G(x) = 1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big) \tag{27}$$

respectively.

Following the idea of (2), the CRT Gompertz distribution is obtained as follows,

$$\begin{aligned} F(x) = {} & \lambda_1\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) + (\lambda_2 - \lambda_1)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2 \\ & + 3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^3 \end{aligned} \tag{28}$$

and the corresponding pdf is defined

$$\begin{aligned} f(x) = {} & \alpha\beta e^{\alpha x} e^{\beta} exp\Big(-\beta e^{\alpha x}\Big)\Big[\lambda_1 + 2(\lambda_2 - \lambda_1)\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) \\ & + 3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2\Big] \end{aligned} \tag{29}$$

Figure 5 shows different pdfs of CRT Gompertz distribution for plausible alternatives of λ_1 and λ_2.

The hazard rate function for the CRT Gombertz distribution is given by

$$h(x) = \frac{\alpha\beta e^{\alpha x} e^{\beta} e^{(-\beta e^{\alpha x})}\Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big(1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big) + 3(1-\lambda_2)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^2\Big]}{1 - \lambda_1\Big(1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big) + (\lambda_2-\lambda_1)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^2 + 3(1-\lambda_2)\Big[1 - e^{\left(-\beta(e^{\alpha x}-1)\right)}\Big]^3} \tag{30}$$

The possible shapes of hazard rate functions can be seen from Figure 6.

The moment generating function of CRT Gombertz distribution is

$$\begin{aligned} M_X(t) = {} & \int_0^\infty e^{tx}\alpha\beta e^{\alpha x} e^{\beta} exp\Big(-\beta e^{\alpha x}\Big)\Big[\lambda_1 + 2(\lambda_2 - \lambda_1)\Big(1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big) \\ & + 3(1-\lambda_2)\Big[1 - exp\Big(-\beta\left(e^{\alpha x} - 1\right)\Big)\Big]^2\Big]dx \end{aligned} \tag{31}$$

By taking $y = \beta\left(e^{\alpha x} - 1\right)$ and $z = \frac{y+\beta}{\beta}$ then, the mgf of CRT Gombertz distribution becomes

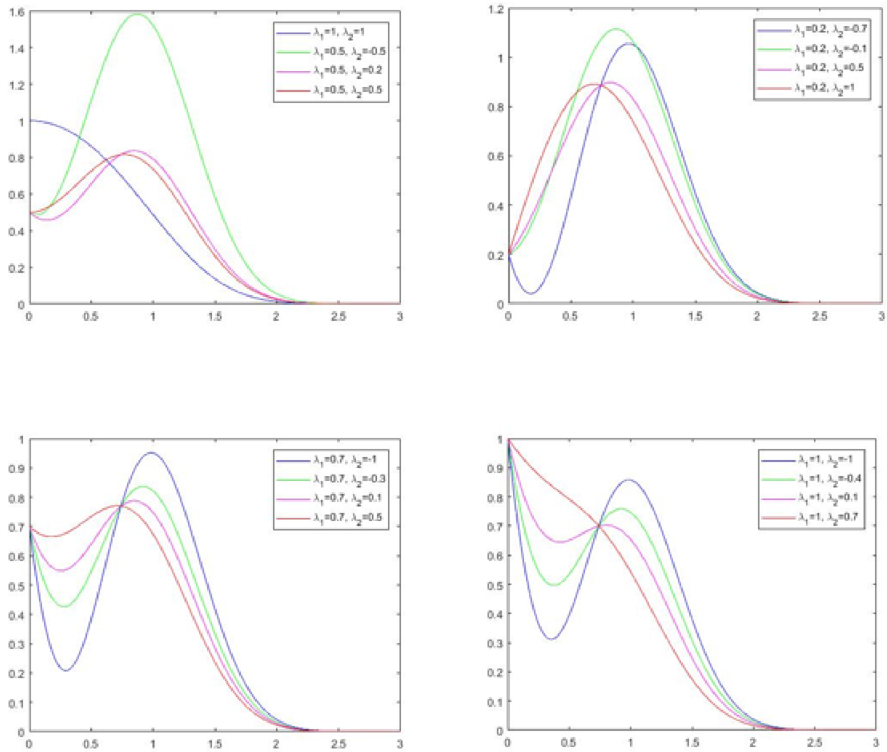

Fig. 5. The pdfs of CRT Gompertz distribution, $\alpha = 1$, $\beta = 1$.

$$
\begin{aligned}
M_X(t) = \beta^{-t/\alpha}\lambda_1 e^{\beta}\Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i\beta^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big] + (2\beta)^{-t/\alpha}(\lambda_2-\lambda_1)e^{2\beta} \\
\Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i(2\beta)^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big] + (3\beta)^{-t/\alpha}(1-\lambda_2)e^{3\beta} \\
\Big[\Gamma\Big(\frac{t}{\beta}+1\Big) - \sum_{i=0}^{\infty}\frac{(-1)^i(3\beta)^{t/\alpha+1+i}}{i!(t/\alpha+1+i}\Big]
\end{aligned}
\tag{32}
$$

For generation random numbers from the distribution, one can use the method of inversion. After simple calculation this yields

$$
x = \frac{1}{\alpha} ln\Big\{\frac{-ln\Big[1+\Big(q+(q^2+(r-p^2)^3)^{1/2}\Big)^{1/3}+\Big(q-(q^2+(r-p^2)^3)^{1/2}\Big)^{1/3}+p\Big]}{\beta}+1\Big\} \tag{33}
$$

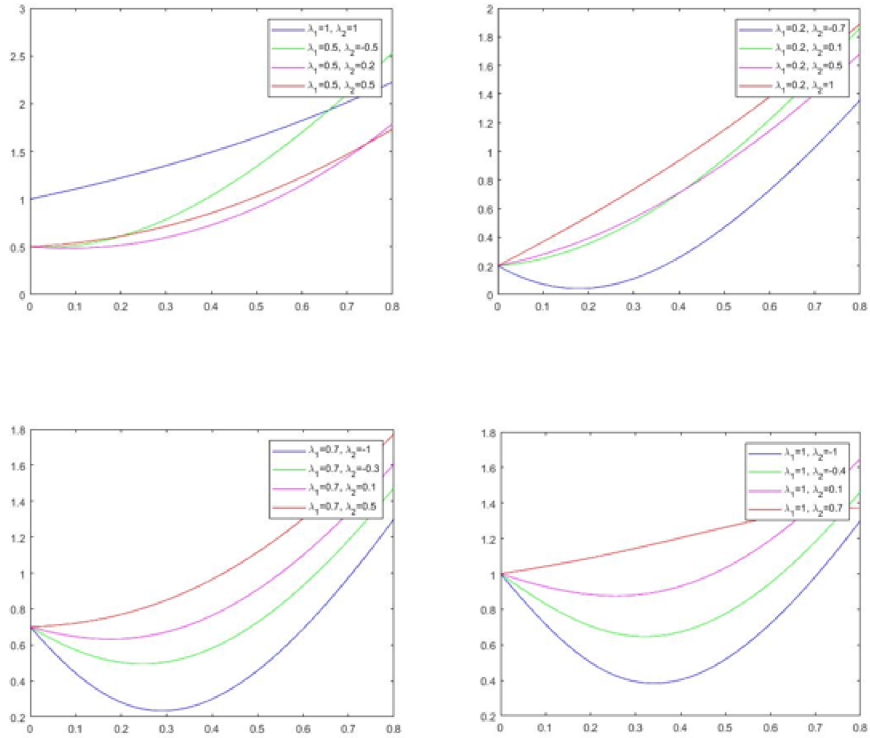

Fig. 6. The hazard rate functions of CRT Gompertz distribution, $\alpha = 1$, $\beta = 1$.

where $p = -\frac{\lambda_2-\lambda_1}{3(1-\lambda_2)}$, $q = p^3 + \frac{\lambda_1(\lambda_2-\lambda_1)+3u(1-\lambda_2)}{6(1-\lambda_2)^2}$, $r = \frac{\lambda_1}{3(1-\lambda_2)}$ and u is uniformly distributed random variable.

Suppose $X_1, X_2, ..., X_n$ are random samples from a CRT Gompertz distribution defined in (27), then the likelihood function is given by

$$L = \alpha^n \beta^n e^{\alpha \sum_{i=1}^n x_i} e^{\beta n} e^{-\beta \sum_{i=1}^n e^{\alpha x_i}} \prod_{i=1}^n \Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big(1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big) + 3(1-\lambda_2)\Big[1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big]^2\Big] \quad (34)$$

and the log-likelihood function is

$$lnL = nln(\alpha) + nln(\beta) + \alpha\sum_{i=1}^n x_i + \beta n - \beta\sum_{i=1}^n e^{\alpha x_i} + 3(1-\lambda_2)\Big[1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big]^2\Big] + \sum_{i=1}^n ln\Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big(1-exp\Big(-\beta\big(e^{\alpha x_i}-1\big)\Big)\Big) \quad (35)$$

By differentiating the log-likelihood function with respect to the unknown parameters and equating them to zero, we obtain the following likelihood equations.

$$
\begin{aligned}
\frac{\partial \ln L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i e^{\alpha x_i} \\
&+ \frac{2(\lambda_2-\lambda_1)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right) + 6(1-\lambda_2)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)\left(1-e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \qquad (36) \\
\frac{\partial \ln L}{\partial \beta} &= \frac{n}{\beta} + n - \sum_{i=1}^{n} e^{\alpha x_i} \\
&+ \frac{2(\lambda_2-\lambda_1) e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right) + 6(1-\lambda_2)\beta x_i e^{\alpha x_i}\left(e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)\left(1-e^{\left(-\beta(e^{\alpha x_i}-1)\right)}\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \\
\frac{\partial \ln L}{\partial \lambda_1} &= \sum_{i=1}^{n} \frac{1-2\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right)}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]} \\
\frac{\partial \ln L}{\partial \lambda_2} &= \sum_{i=1}^{n} \frac{2\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) - 3\left[1-exp\left(-\beta\left(e^{\alpha x}-1\right)\right)\right]^2}{\left[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right) + 3(1-\lambda_2)\left[1-exp\left(-\beta\left(e^{\alpha x_i}-1\right)\right)\right]^2\right]}
\end{aligned}
$$

Solutions of these equations are called ML estimates. However, the equations must be solved with numerical methods such as Newton Raphson or iteratively reweighting algorithm.

5. APPLICATION

In this section, we applied each of the new distribution families to the real data for demonstrating the behaviour of the distributions. Determining an appropriate model from a population problem has been widely discussed by several authors. However, Nelson (1982) suggested that *after fitting the general model to the data, then one seeks to find which special case is suitable*. For this reason, we used the data sets taken from the literature which has been fitted by the original distributions.

5.1. Wind speed data

The data used for the present study were obtained from a yearly published book at Permerhatian Cuaca Harian Pusat Pengajian Sosial, Pembangunan and Persekitaran (PPSPP), Fakulti Sains Sosial, Kemanusiaan (FSSK), Universiti Kebangsaan Malaysia (UKM) during the year 2004 to 2006, Zaharim et al. (2009). This data was collected from Malaysia and wind speeds were observed every 10 seconds and averaged over 5 minutes period. The 5-minutes averaged data were further averaged over one hour. At the end of each hour, the hourly mean wind speed was calculated and stored sequentially in a permanent memory.

Elbatal et al. (2014) fit the data into the Frechet (F), Transmuted Frechet (TF) and Transmuted Exponentiated Frechet (TEF) distributions. We also propose CRT Frechet (CRTF) distribution and Table (1) shows the comparison results based on the estimated parameter values.

Table I. Estimated Parameters of Frechet, Transmuted Frechet, Transmuted Exponentiated Frechet and Cubic Rank Transmuted Frechet Distributions.

Distribution	*ParameterEstimates*					*Log − Likelihood*
	β	θ	α	λ_1	λ_2	
F	1.922	1.024	–	–	–	−19.43
TF	2.014	2.581	–	0.746	–	−11.44
TEF	1.913	3.481	9.88	0.380	–	−6.65
$CRTF$	1.887	3.041	–	0.591	0.115	−5.97

5.2. Water Quality Data

This water quality data were obtained from the Department of Chemistry, Gauhati University. Various water quality parameters were estimated for the project entitled Assessment of Toxic Element in Water of Semi-Under Area of Assam and Investigation of the Disease Related Contaminants during 2009 for three administration sub-divisions of Nogaon district of Assam, India. Deka et al (2017) proposed Transmuted Exponentiated Gumbel (TEG) for this data set and compared the results with Gumbel (G) and Transmuted Gumbel (TG) distributions. We fit the data into the CRT Gumbel (CRTG) distribution and the results are given in Table (2).

Table II. Estimated Parameters of Gumbel, Transmuted Gumbel, Transmuted Exponentiated Gumbel and Cubic Rank Transmuted Gumbel Distributions.

Distribution	*ParameterEstimates*					*Log − Likelihood*
	β	θ	α	λ_1	λ_2	
G	1.063	0.769	–	–	–	−40.80
TG	1.001	0.855	–	0.711	–	−40.14
TEG	0.259	0.185	0.181	0.530	–	−39.70
$CRTG$	0.715	0.625	–	0.856	0.211	−38.95

5.3. Failure Data

Abdul-Maniem and Seham (2015) used the data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. For this data set Gombertz (Go) and Transmuted Gombertz (TGo) distributions were proposed in this paper. We, now propose CRT Gombertz (CRTGo) distribution for the failure rate data and the results are shown in Table (3).

Table III. Estimated Parameters of Gombertz, Transmuted Gombertz, Transmuted Exponentiated Gombertz and Cubic Rank Transmuted Gombertz Distributions.

$Distribution$			$Parameter Estimates$			$Log-Likelihood$
	β	θ	α	λ_1	λ_2	
Go	0.121	3.385	–	–	–	−87.20
TGo	0.187	1.148	–	0.819	–	−64.25
$TEGo$	0.895	3.128	0.521	0.985	–	−63.25
$CRTGo$	0.135	1.568	–	0.851	0.119	−61.13

6. CONCLUSION

In this paper, we introduce some examples of cubic rank transmutation mapping. Frechet, Gumbel and Gombertz distributions are used as the base distribution. The properties of these distributions such as the density functions, the medians, hazard rate functions and the quantile functions are examined. Also, maximum likelihood estimations are obtained. In the application section of the paper, real data set examples are used to illustrate better fit than the distributions which have been used before. For all real data sets introduced distributions provides better fittings than the corresponding distributions.

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A New Compound Lifetime Distribution: Model, Characterization, Estimation and Application

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Abstract

There are diverse lifetime models available to the researchers to predict the uncertain behavior of random events but at times they fail to provide adequate fit for some complex and new data sets. New probability distributions are emerging as lifetime models to meet this ever growing demand of modeling complex real world phenomena from different sciences with better efficiency. Here, in this manuscript we shall compose Ailamujia distribution with that of power series distribution. This newly developed distribution called Ailamujia power series distribution reduces to four new special lifetime models on simple specific function parametric setting. Apart from this some important mathematical properties in the form of propositions will also be discussed. Furthermore, characterization and some statistical properties that include mgf, moments, and parameter estimation have also been discussed. Finally, the potency of newly proposed model has been analyzed statistically and graphically and it has been established from the statistical analysis that newly proposed model offers a better fit when it comes to model some lifetime data set.

Mathematics Subject Classification 2000: 62EXX
Key Words: Ailamujia Distribution, Power Series Distribution, Compounding and Order Statistics.

1. INTRODUCTION

Lifetime distributions play an important role in almost every field of science be it engineering, industrial, medical or similar biological science. The events of interest such as death, appearance of some disease and system failure are of major concern for statisticians because of their uncertain behavior. And there are so many probability distributions such as exponential, Weibull, gamma and log normal that can be used as lifetime models to predict this uncertain behavior of random events but due to varying pattern of different data sets, these probability models can not be used adequately because of some serious limitations. To overcome these limitations researchers have developed many lifetime distributions by using different techniques such as compounding, transmutation etc. For instance, Adamidis and Loukas [1], Tahmasbi [10] and Morais and Baretto Souza [9] developed several lifetime distributions through compounding mechanisms that proved to be very effective in modeling the lifetime data having different characteristics. The efforts of Adil and

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Jan ([2], [3] [4], [5], [6]) got materialized when they obtained many compound distributions that exhibited with clinical precision to be superior in comparison to existing lifetime models.

Let us take a series system with N components, where N, the number of components is itself a discrete random variable with support N =1,2,…, The lifetime of i^{th} component in this set up can be portrayed by any suitable lifetime distribution like viz; exponential, gamma, Weibull, Lindley, Ailamujia etc. And N, the discrete random variable may have any of the ascribed distribution such as geometric, zero truncated Poisson or power series distribution in general. The lifetime for this kind of system in series combination will be denoted by $Y = \min\left\{X_i\right\}_{i=1}^{N}$. In this paper we will consider the lifetime of i^{th} component to be distributed as Ailamujia distribution and the index N itself as powers series distribution. The new lifetime distribution that is obtained by compounding Ailamujia distribution with that of powers series distribution will be known as Ailamujia power series distribution. The present paper is organized as follows: In section (2) we present the construction of the proposed lifetime distribution. Density, survival, hazard rate functions and some of the properties of the proposed family are given in section (3). Moment generating function of proposed distribution is given in section (4). Order statistics, their moments and parameter estimation are discussed in detail respectively in section (5) and (6). Special cases that include new lifetime distributions have been given in section (7). Finally, real application and conclusion about new findings are respectively given in section (8) and (9).

2. CONSTRUCTION OF THE CLASS

Let X_i, $i = 1, 2, ..., N$ be independent and identically distributed (iid) random variables following Ailamujia distribution with CDF

$$G(x;\beta) = 1 - (1 + 2\beta x)e^{-2\beta x} \tag{1}$$

Here, the index N is itself a discrete random variable following power series distribution that have been truncated at zero with probability function given by

$$P(N=n)=\frac{a_n\lambda^n}{C(\lambda)}, \quad n=1,2,...$$

where a_n depends only on n, $C(\lambda)=\sum_{n=1}^{\infty}a_n\lambda^n$ and $\lambda>0$ is such that $C(\lambda)$ is finite. Table 1 is very informative and it will be helpful for obtaining the special cases of the proposed model on specific function setting.

Table 1: Useful quantities of Some Power Series Distribution

Distribution	a_n	$C(\lambda)$	$C^{'}(\lambda)$	$C^{''}(\lambda)$	$C^{-1}(\lambda)$	λ
Poisson	$n!^{-1}$	$e^{\lambda}-1$	e^{λ}	e^{λ}	$\log(\lambda+1)$	$\lambda\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\lambda)$	$(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$1-e^{-\lambda}$	$\lambda\in(0,1)$
Geometric	1	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$\lambda(\lambda+1)^{-1}$	$\lambda\in(0,1)$
Binomial	$\binom{m}{n}$	$(\lambda+1)^m-1$	$m(\lambda+1)^{m-1}$	$\frac{m(m-1)}{(\lambda+1)^{2-m}}$	$(\lambda-1)^{1/m}-1$	$\lambda\in(0,\infty)$

Let $X_{(1)}=\min\{X_i\}_{i=1}^{N}$. The conditional cumulative distribution function of $X_{(1)}\mid N=n$ is given by

$G_{X_{(1)}|N=n}(x)=1-\left[\bar{G}(x)\right]^n$, where $G(x)$ is the cdf of Ailamujia distribution

$$=1-\left[(1+2\beta x)e^{-2\beta x}\right]^n$$

The joint probability function is

$$P\left(X_{(1)}\le x, N=n\right)=\frac{a_n\lambda^n}{C(\lambda)}\left\{1-\left[(1+2\beta x)e^{-2\beta x}\right]^n\right\}, \ x>0,\ n\ge 1.$$

By using the compounding technique, the CDF of newly proposed lifetime distribution is

$$F(x)=\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{C(\lambda)}\left\{1-\left[(1+2\beta x)e^{-2\beta x}\right]^n\right\}$$

$$=1-\frac{C\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)}, \quad x>0 \qquad (2)$$

The newly proposed distribution will be called Ailamujia power series distribution and will be symbolically represented as $\text{APSD}(X;\beta,\lambda)$.

3. DENSITY, SURVIVAL AND HAZARD RATE FUNCTION

Since PDF is essentially a derivative of CDF, therefore probability density function of APSD $(X;\beta,\lambda)$ can be obtained as

$$f(x)=\frac{dF(x)}{dx}$$

$$=\frac{d}{dx}\left\{1-\frac{C\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)}\right\}$$

$$f(x)=4\lambda\beta^2 xe^{-2\beta x}\frac{C'\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)} \tag{3}$$

$$S(x)=1-F(x)\quad=\frac{C\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)}$$

and the hazard function is

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{4\lambda\beta^2 xe^{-2\beta x}C'\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}\quad ,x>0$$

Next, we will explore some important properties of $\text{APSD}(X;\beta,\lambda)$ through the following propositions.

PROPOSITION 1. The Ailamujia distribution is the limiting case of the APSD $(X;\beta,\lambda)$ whenever $\theta\to 0^+$.

PROOF. The cumulative distribution function of $\text{APSD}(X;\beta,\lambda)$

$$\lim_{\theta\to 0^+}F(x)=1-\lim_{\theta\to 0^+}\frac{C\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)},\quad x>0$$

In view of the fact

$$C(\lambda)=\sum_{n=1}^{\infty} a_n \lambda^n$$

$$\lim_{\lambda\to 0^+} F(x)=1-\lim_{\lambda\to 0^+}\frac{\sum_{n=1}^{\infty} a_n\left[\lambda(1+2\beta x)e^{-2\beta x}\right]^n}{\sum_{n=1}^{\infty} a_n\lambda^n}$$

$$\lim_{\lambda\to 0^+} F(x)=1-\lim_{\lambda\to 0^+}\frac{a_1\lambda(1+2\beta x)e^{-2\beta x}+\sum_{n=2}^{\infty} a_n n\lambda^{n-1}\left[(1+2\beta x)e^{-2\beta x}\right]^n}{a_1\lambda+\sum_{n=2}^{\infty} a_n n\lambda^{n-1}}$$

$$=1-(1+2\beta x)e^{-2\beta x}$$

which is the distribution function of Ailamujia distribution.

PROPOSITION 2. The densities of $\text{APSD}(X;\beta,\lambda)$ can be expressed as an infinite linear combination of densities of 1st order statistics of Ailamujia distribution

PROOF. Since we have, $f(\mathrm{X};\beta,\lambda)=4\lambda\,\beta^2 xe^{-2\beta x}\,\dfrac{C'\left[\lambda(1+2\beta x)e^{-2\beta x}\right]}{C(\lambda)}$

It is known that

$$C'(\lambda)=\sum_{n=1}^{\infty} n a_n \lambda^{n-1}$$

Therefore it follows that

$$f(\mathrm{X};\beta,\lambda)=4\lambda\,\beta^2 xe^{-2\beta x}\sum_{n=1}^{\infty} n a_n\,\frac{\left[\lambda(1+2\beta x)e^{-2\beta x}\right]^{n-1}}{C(\lambda)}$$

$$f(\mathrm{X};\beta,\lambda)=\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{C(\lambda)}4n\beta^2 xe^{-2\beta x}\left[(1+2\beta x)e^{-2\beta x}\right]^{n-1}$$

$$f(\mathrm{X};\beta,\lambda)=\sum_{n=1}^{\infty} P(N=n)g_1(x,n) \qquad (4)$$

Where $g_1(x,n)=4n\beta^2 xe^{-2\beta x}\left[(1+2\beta x)e^{-2\beta x}\right]^{n-1}$ is the Ist order statistics of Ailamujia distribution. Therefore the densities of proposed distribution can be expressed as an infinite linear combination of the Ist order statistics of Ailamujia distribution. Hence it is obvious that properties of $\text{APSD}(X;\beta,\lambda)$ can be obtained from the Ist order statistics of Ailamujia distribution.

4. MOMENT GENERATING FUNCTION

In view of the proposition 2, the mgf APSD$(X;\beta,\lambda)$ can be obtained as

$$M_X(t)=\sum_{n=1}^{\infty}P(N=n)M_{X_{(1)}}(t)$$

where $M_{X_{(1)}}(t)$ is the moment generating function of I[st] order statistics of Ailamujia distribution

$$M_{X_{(1)}}(t)=\int_0^{\infty}e^{tx}4n\beta^2xe^{-2\beta x}\left[\left(1+2\beta x\right)e^{-2\beta x}\right]^{n-1}\,dx$$

$$M_{X_{(1)}}(t)=4n\beta^2\int_0^{\infty}e^{-(2n\beta-t)x}\left(1+2\beta x\right)^{n-1}\,dx$$

$$M_{X_{(1)}}(t)=n\left(2\beta\right)^{n-j+1}\sum_{j=1}^{n-1}\binom{n-1}{j}\int_0^{\infty}e^{-(2n\beta-t)x}x^{n-j-1}\,dx$$

$$M_{X_{(1)}}(t)=n\left(2\beta\right)^{n-j+1}\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{\Gamma(n-j)}{(2n\beta-t)^{\mathrm{n-j}}}$$

Hence we get

$$M_X(t)=\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{C(\lambda)}n\left(2\beta\right)^{n-j+1}\sum_{j=1}^{n-1}\binom{n-1}{j}\frac{\Gamma(n-j)}{(2n\beta-t)^{\mathrm{n-j}}}$$

In order to obtain the moments of proposed distribution we again use proposition 2

$$E(X^r)=\sum_{n=1}^{\infty}P(N=n)\int_0^{\infty}x^rg_1(x)\,dx$$

$$=\sum_{n=1}^{\infty}P(N=n)E\left(X_{(1)}{}^{r}\right)$$

Now consider

$$E\left(X_{(1)}{}^{r}\right)=\int_0^{\infty}x^rg_1(x)\,dx=4n\beta^2\int_0^{\infty}x^{r+1}\,e^{-2\beta x}\left[\left(1+2\beta x\right)e^{-2\beta x}\right]^{n-1}dx$$

$$=4n\beta^2\sum_{j=0}^{n-1}\binom{n-1}{j}(2\beta)^{\mathrm{n-1-j}}\int_0^{\infty}x^{r+n-j}\,e^{-2n\beta x}dx$$

$$=4n\beta^2\sum_{j=0}^{n-1}\binom{n-1}{j}(2\beta)^{\mathrm{n-1-j}}\,\frac{\Gamma(r+n-j+1)}{\left(2n\beta\right)^{r+n-j+1}}$$

$$=(2\beta)^{-r}\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{\Gamma(r+n-j+1)}{n^{r+n-j}}$$

Thus we have

$$E(X^r) = (2\beta)^{-r} \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\Gamma(r+n-j+1)}{n^{r+n-j}} \tag{5}$$

5. ORDER STATISTICS AND THEIR MOMENTS

We have developed a new lifetime distribution that can be used to model the lifetime data where order statistics plays a vital role. In this section we derive expressions for pdf and CDF of i^{th} order statistics of proposed distribution.

Let $X_1, X_2, ..., X_n$ be a random sample from APSD and $X_{1:n}, X_{2:n}, ..., X_{1:n}$ denote the corresponding order statistics. The pdf of i^{th} order statistics say $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\left[F(x)\right]^{i-1}\left[1-F(x)\right]^{n-i} f(x) \tag{6}$$

$$= \frac{n!\, f(x)}{(n-i)!(i-1)!}\left[1 - \frac{C\left[\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]}{C(\lambda)}\right]^{i-1}\left[\frac{C\left[\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]}{C(\lambda)}\right]^{n-i}$$

Expression (6) can be equivalently written as

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k f(x)\left[F(x)\right]^{k+i-1}$$

or

$$f_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k f(x)\left[1-F(x)\right]^{k+n-i}$$

In view of the fact

$$f(x)\left[F(x)\right]^{k+i-1} = \left(\frac{1}{k+i}\right)\frac{d}{dx}\left[F(x)\right]^{k+i}$$

The associated CDF of $f_{i:n}(x)$ denoted by $F_{i::n}(x)$ becomes

$$F_{i:n}(x) = \frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\left[1 - \frac{C\left[\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]}{C(\lambda)}\right]^{k+i} \tag{7}$$

The expression for r^{th} moment of i^{th} order statistics $X_{1:n,...,}X_{n:n}$ with CDF (7) can be obtained by exploiting a well-known result due to Barakat and Abdelkadir [7] as follows

$$E\left(X_{i:n}^{r}\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_{0}^{\infty}x^{r-1}S(x)^{k}\,dx$$

where $S(x)$ is the survival function of Ailamujia distribution. Therefore we have

$$E\left(X_{i:n}^{r}\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_{0}^{\infty}x^{r-1}\left(\frac{C\left[\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]}{C(\lambda)}\right)^{k}dx$$

where $r=1,2,3...$ and $i=1,2,...,n$.

6. PARAMETER ESTIMATIONS

The log-likelihood function of the proposed model with unknown parameter vector $\Theta=(\beta,\lambda)^{T}$ is given by

$$l_n=l_n(x,\Theta)=n\log 4+2n\log\beta+n\log\lambda-2\beta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\log\left\{C'\left[\lambda(1+2\beta x)e^{-2\beta x}\right]\right\}-n\log C(\lambda)$$

The corresponding score functions are

$$\frac{\partial l_n}{\partial\beta}=\frac{2n}{\beta}-\sum_{i=1}^{n}x_i-4\lambda\beta\sum_{i=1}^{n}\frac{C''\left[\lambda\left(1+2\beta x_i\right)e^{-2\beta x_i}\right]}{C'\left[\lambda\left(1+2\beta x_i\right)e^{-2\beta x_i}\right]}x_i^{2}e^{-2\beta x_i}$$

$$\frac{\partial l_n}{\partial\lambda}=\frac{n}{\lambda}-\frac{nC'(\lambda)}{C(\lambda)}+\sum_{i=1}^{n}\frac{C''\left[\lambda\left(1+2\beta x_i\right)e^{-2\beta x_i}\right]}{C'\left[\lambda\left(1+2\beta x_i\right)e^{-2\beta x_i}\right]}\left(1+2\beta x_i\right)e^{-2\beta x_i}$$

The maximum likelihood estimate of Θ say $\hat{\Theta}$ is obtained by solving the non-linear system of equations $U_n(\Theta)=\left(\frac{\partial l_n}{\partial\beta},\frac{\partial l_n}{\partial\lambda}\right)^{T}=0$. The solution of this non-linear system of equations can be found numerically by using software such as R.

7. CONSEQUENCES OF PROPOSED MODEL

In this section we will study some important consequences of proposed model in the form of some special cases. The graphical behavior these sub models will also be discussed to show the flexibility in terms of hazard and density function.

7.1. Ailamujia Poisson Distribution (APD)

Here, we frequently exploit the use of table 1, in which it is clear that classical distributions are embodied in PSD on specific function setting. For instance, Poisson distribution is a special case of power series distribution for $C(\lambda)=e^{\lambda}-1$ and $C'(\lambda)=e^{\lambda}$. Therefore, cdf and pdf of a compound of APD is $F(\text{x})=1-\dfrac{e^{\lambda(1+2\beta x)e^{-2\beta x}}-1}{e^{\lambda}-1}$. The associated pdf, survival and hazard rate function is

$$f(x)=\frac{4\lambda\beta^{2}}{e^{\lambda}-1}xe^{\lambda(1+2\beta x)e^{-2\beta x}-2\beta x}$$

$$S(x)=\frac{e^{\lambda(1+2\beta x)e^{-2\beta x}}-1}{e^{\lambda}-1} \text{ and } h(x)=\frac{4\lambda\beta^{2}xe^{\lambda(1+2\beta x)e^{-2\beta x}-2\beta x}}{e^{\lambda(1+2\beta x)e^{-2\beta x}}-1}$$

for $x,\beta>0$, $0<\lambda<\infty$ respectively.

7.2. Ailamujia Logarithmic Distribution (ALD):

Again from table 1, Logarithmic distribution is a special case of PSD when $C(\lambda)=-\log(1-\lambda)$ and $C'(\lambda)=(1-\lambda)^{-1}$. The cdf of ALD is

$$F(x)=1-\frac{\log\left[1-\lambda(1+2\beta x)e^{-2\beta x}\right]}{\log(1-\lambda)},\quad x>0$$

The associated pdf, survival and hazard rate function is

$$f(x)=\frac{4\lambda\beta^{2}e^{-2\beta x}}{\left[\lambda(1+2\beta x)e^{-2\beta x}-1\right]l\,\text{og}(1-\lambda)}$$

$$S(x)=\frac{\log\left[1-\lambda(1+2\beta x)e^{-2\beta x}\right]}{\log(1-\lambda)}$$

$$h(x)=\frac{4\lambda\beta^{2}e^{-2\beta x}}{\left[\lambda(1+2\beta x)e^{-2\beta x}-1\right]\log\left[1-\lambda(1+2\beta x)e^{-2\beta x}\right]}$$

for $x,\lambda>0$ and $0<\beta<1$ respectively.

7.3. Ailamujia Geometric Distribution (AGD)

We observe from the table 1 that Geometric distribution is a particular case of PSD when $C(\lambda)=\lambda(1-\lambda)^{-1}$ and $C'(\lambda)=(1-\lambda)^{-2}$. Therefore, cdf of AGD is

$$F(x)=\frac{1-(1+2\beta x)e^{-2\beta x}}{1-\lambda(1+2\beta x)e^{-2\beta x}},x>0$$

The associated pdf, survival and hazard rate function is

$$f(x)=\frac{4\beta^2 x e^{-2\beta x}(1-\lambda)}{\left[1-\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]^2}$$

$$S(x)=\frac{\left(1+2\beta x\right)e^{-2\beta x}(1-\lambda)}{1-\lambda\left(1+2\beta x\right)e^{-2\beta x}}$$

$$h(x)=\frac{4\beta^2 x e^{-2\beta x}(1-\lambda)}{\left[1-\lambda\left(1+2\beta x\right)e^{-2\beta x}\right]\left(1+2\beta x\right)e^{-2\beta x}},$$

for $x, \beta, 0<\lambda<1$ respectively.

7.4. Ailamujia Binomial Distribution (ABD)

Binomial distribution is a particular case of PSD for $C(\lambda)=(\lambda+1)^m-1$ and a compound of Ailamujia binomial distribution (ABD) is followed from (2) by using $C(\lambda)=(\lambda+1)^m-1$. This may be noted here that these sub models are new lifetime distributions that have been obtained on specific parameter setting in ALPSD.

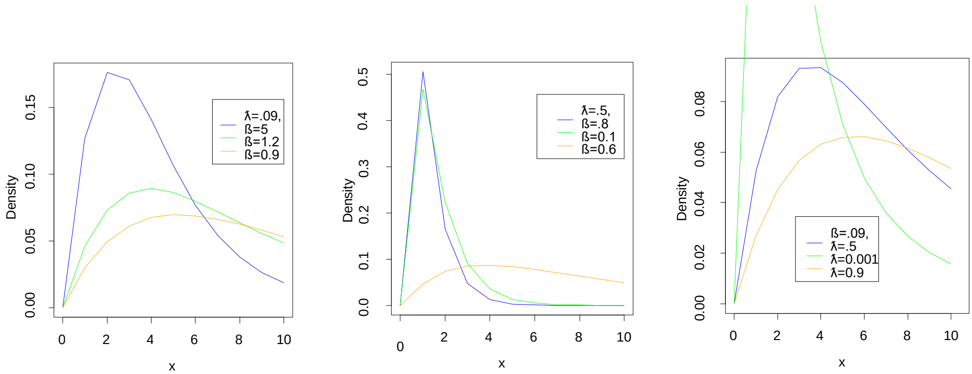

Fig. 1-3. These graphs show the flexibility of APD, ALD and AGD models for randomly selected values of parameters

8. APPLICATION

In this section, we will expose and compare the potentiality of proposed model on a real life data set based on Lifetime of fatigue fracture of Kevlar 373/epoxy [8], that are subject to constant pressure at the 90% stress level until all had failed. The data set is

0.0251	0.0886	0.0891	0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696	0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.912	0.9836
1.0483	1.0596	1.0773	1.1733	1.257	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.488	1.5728	1.5733	1.7083	1.7263	1.746	1.763	1.7746	1.8275	1.8375	1.8503
1.8808	1.8878	1.8881	1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.133	2.21
2.246	2.2878	2.3203	2.347	2.3513	2.4951	2.526	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455	3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.096	

Table 2: Analysis of model fitting

MODEL	*MLE*	*AIC*	*BIC*
LG	$\hat{\beta}=0.41,\ \hat{\lambda}=0.38$	249.52	252.58
LP	$\hat{\beta}=0.25,\ \hat{\lambda}=3.58$	248.68	251.73
LL	$\hat{\beta}=\ 0.44,\ \hat{\lambda}=0.51$	249.70	252.75

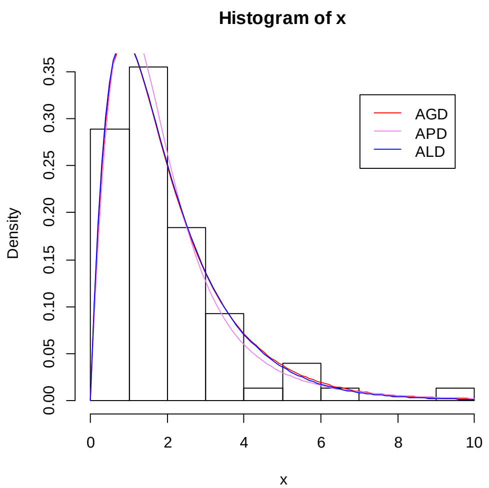

Fig 4. Fitting of AGD, APD, ALD to the fatigue lifetime data

9. CONCLUSION

We have developed a new class of compound lifetime distributions that has been named as Ailamujia power series distribution. Furthermore, we also discussed some special cases of this class of distributions that are very flexible in terms of density and hazard rate functions. Mathematical properties such as moments, order statistics and parameter estimation through MLE of the proposed class has also been discussed. Finally the potentiality of proposed model has been explored in lifetime data analysis. It is very clear from statistical analysis that proposed model performs well which is also corroborated by graphical analysis. So, we strongly recommend practitioners to use one of our models in order to get effective results when it comes to fit lifetime data.

The future course of work will be on a generalized version of Ailamujia power series distribution.

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