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Original Paper

Trends in the number of foreign students at Slovak universities and Slovak students at foreign universities

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ABSTRACT

Just as students from abroad come to Slovakia to study, Slovak students go to study at foreign colleges and universities. In our article we analyze trends in the number of foreign students studying at Slovak universities and Slovak students leaving mainly for the Czech Republic in 10 years from 2010 to 2019. In Slovakia, but also in the Czech Republic, we observe a decrease in the total number of students studying at universities. We are also following this trend among Slovaks studying in the Czech Republic. In contrast, in Slovakia we are seeing an increase in the number of foreign students studying at our colleges and universities. Czechs and Ukrainians study the most in our country, and the ratio between them has been changing since 2017 in favor of Ukrainian students. The share of Slovak students among all foreign students at Czech colleges and universities was almost 64% in 2010, in 2019 it is only less than 45%.

KEYWORDS: trends, Slovak students, foreign students

JEL CLASSIFICATION: C02, C11, I210

INTRODUCTION

As of 31 December 2021, there are 41 universities in the Slovak Republic, of which 20 are public, 3 state, 10 private and 8 foreign universities. A total of 133,558 students study at them, of which 18,243 are foreigners (MINEDU, 2022), which represents 13.66%. As of the same date, there are 79 universities in the Czech Republic, of which 26 are public, 2 state, 33 private and 18 foreign universities. 304,054 students study at these schools, of which 52,109 are foreigners (MEYS, 2022), which is 17.14%.

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In both republics, we are seeing a decline in the total number of students studying at universities. In contrast to these trends, the numbers of foreign students in both republics are gradually increasing. Therefore, we decided to examine trends in the number of students from 2010 to 2019, whether Slovaks and foreign students studying at Slovak universities, but also Czechs and foreign students, with an emphasis on Slovaks studying in the Czech Republic. Antalíková (2020) and Blanár (2020) dealt with similar topics. We also stated these trends in the contribution of Pechočiak and Drábeková (2020), although since 2018 we have observed that the decline of Slovak students studying in the Slovak Republic has stopped or increased slightly.

One of the most important factors in achieving a country´s prosperous socioeconomic environment is education. As Országhová et al. states (2018) investments in human capital, including education, skills upgrading, the development of education and science, are nowadays a significant prerequisite for the further development of society and its economic growth. Tholen (2014) in his work says that Policymakers has argued as well as assumed that the share of degree holders within the workforce is a direct measure of national competitiveness and economic strength within the global era. However, if the structure of graduates does not meet the expectations and needs of the labor market the growth of the number of university graduates won't produce the desired effect.

MATERIAL AND METHODS

In this article, we follow the trends in the number of Slovak and foreign students studying at Slovak and foreign universities for 10 years from 2010 to 2019. In 2020 and 2021, a pandemic of covid virus spread to the world, so we assumed that it would affect the numbers of these students. Therefore, we did not include them in the examined sample. We obtained data from databases of Slovak Center of Scientific and Technical Information (SCSTI), Statistical Office of the Slovak Republic (SOSR), The Ministry of Education, Science, Research and Sport of the Slovak Republic (MINEDU) and Ministry of Education, Youth and sports in Czech (MEYS).

We used methods of mathematical descriptive statistics to evaluate them. In Excel, we created spreadsheets in which we calculated Percentages. From this data, we drew graphs, which we transferred with tables to Word in our article.

Table 1 shows the numbers of enrolled and in Table 2 the numbers of enrolled Slovak and foreign students at Slovak universities in the years 2010 to 2019. We see that the differences between enrolled and enrolled students are many times large. For Slovak students it is two to three times, for foreign students almost double. Therefore, we took the data on enrolled students as more representative. We assume that some students applied to more universities, or some did not get to the university.

Table 1 Number of registered Slovak and foreign students at Slovak universities in the years 2010 to 2019

Year	Slovak students	foreign students
2010	144 038	3 472
2011	135 739	3 735
2012	117 806	3 625
2013	101 345	4 497
2014	87 109	3 747
2015	79 114	4 418
2016	72 156	5 184
2017	62 766	6 408
2018	65 643	6 910
2019	66 531	8 822

Source: SCSTI, SOSR

Table 2 Number of enrolled Slovak and foreign students at Slovak universities in the years 2010 to 2019

Year	Slovak students	foreign students	Share in percentage
2010	49 758	2 199	4,23
2011	49 074	2 230	4,35
2012	44 994	2 317	4,90
2013	41 894	3 013	6,71
2014	38 190	2 404	5,92
2015	35 601	2 447	6,43
2016	32 512	2 865	8,10
2017	30 158	3 418	10,18
2018	31 320	3 645	10,42
2019	31 639	4 788	13,14

Source: SCSTI, SOSR, own

If we notice from which countries most students come to us, then we find that they are mainly 2 countries, the Czech Republic and Ukraine. Data on the number of students from these countries, the total number of foreign students and the share in% between the sums of the numbers of Czechs and Ukrainians to the total number of foreign students studying in our country are given in Table 3.

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Table 3 Number of enrolled foreign students from the Czech Republic, Ukraine and from all countries together at Slovak universities in the years 2010 to 2019

Year	Czech Republic	Ukraine	all countries together	Share in percentage
2010	1548	44	2 199	72.40
2011	1314	55	2 230	61.39
2012	1195	54	2 317	53.91
2013	1277	604	3 013	62.43
2014	1048	206	2 404	52.16
2015	838	358	2 447	48.88
2016	764	608	2 865	47.89
2017	826	941	3 418	51.70
2018	613	1385	3 645	54.81
2019	822	1830	4 788	55.39

Source: SCSTI, SOSR, own

In terms of Slovak students studying abroad, we find that most of them study in the Czech Republic, the United Kingdom, Hungary, Austria and Germany. At the same time, up to 70% of these students study in the Czech Republic. That is why we focused our work on Slovaks studying at Czech universities. Table 4 shows the total numbers of students and Slovaks studying at universities in the Czech Republic.

Table 4 Total number of students and Slovaks studying at Czech universities in the years 2010 to 2019

Year	All students	Slovaks
2010	395982	23 930
2011	392039	24 086
2012	380893	23 565
2013	367772	22 937
2014	346811	22 166
2015	326437	21 713
2016	311060	21 614
2017	298682	20 955
2018	289680	20 766
2019	288644	20 642

Source: MEYS

Table 5 shows the total numbers of foreign students and Slovaks studying at Czech universities and the ratio of Slovaks to the total number of foreign students. This table, as well as Figure 4, shows that the total number of foreign students at Czech universities is growing and the number of Slovaks is declining, which is also reflected in the ratio of the number of Slovaks to all foreign students. While in 2010 this ratio was 63.82%, in 2019 it is only 44.53%.

Table 5 Numbers of all foreign and Slovak students at Czech universities

Year	All foreign students	Slovak students	Share in percentage
2010	37498	23 930	63.82
2011	38707	24 086	62.23
2012	39430	23 565	59.76
2013	40330	22 937	56.87
2014	40916	22 166	54.17
2015	42027	21 713	51.66
2016	43450	21 614	49.74
2017	43671	20 955	47.98
2018	44685	20 766	46.47
2019	46359	20 642	44.53

Source: MEYS, own

RESULTS AND DISCUSSION

In Table 2, in addition to the numbers of enrolled Slovak and foreign students studying at Slovak universities, we also stated the percentage share between the numbers of foreign students and all students studying at Slovak universities. While in 2010 this share was 4.23 %, in 2019 it was already 13.14 %. As we can also see in Figure 1, the trend in the number of Slovak students over the period under review is declining compared to the number of foreign students, which is why this ratio is growing.

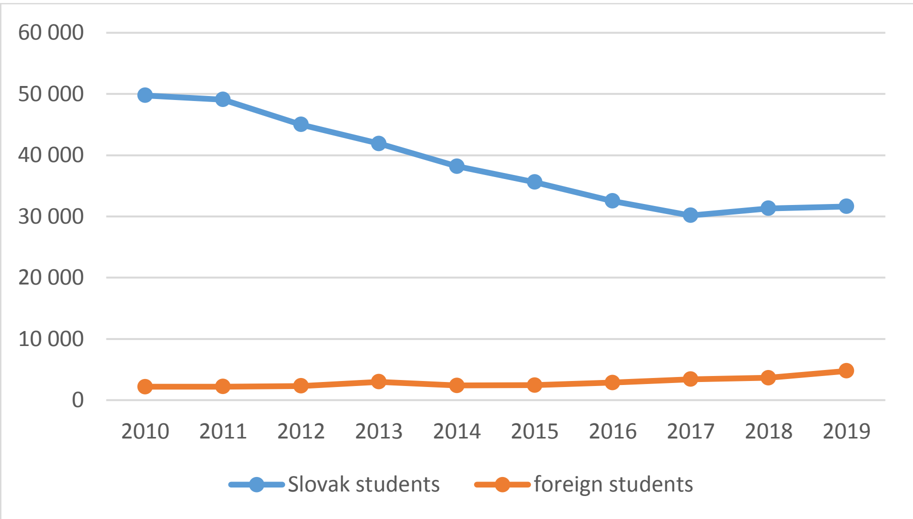

Figure 1 Number of enrolled Slovak and foreign students at Slovak universities in the years 2010 to 2019

Source: own

Most foreign students at our universities come from the Czech Republic and Ukraine. Data on the number of students from these countries are given in Table 3. The table shows that in 2010 1548 Czechs and only 44 Ukrainians studied here. Gradually, these numbers are "turning" in favor of Ukrainians, and we see that in 2019, 822 Czechs and 1,830 Ukrainians studied in our country, and the trend in the sum of these numbers is growing. In Table 3, we also stated the share in% between the sums of the numbers of Czechs and Ukrainians to the total number of foreign students studying in our country. This ratio decreased from 72.4 % in 2010 to 55.39 % in 2019. Trends in the numbers of Czechs, Ukrainians and the total number of foreign students can be seen in Figure 2. Trends in the total numbers of foreign students and Ukrainians are almost identical, growing, we observe in the Czechs decline.

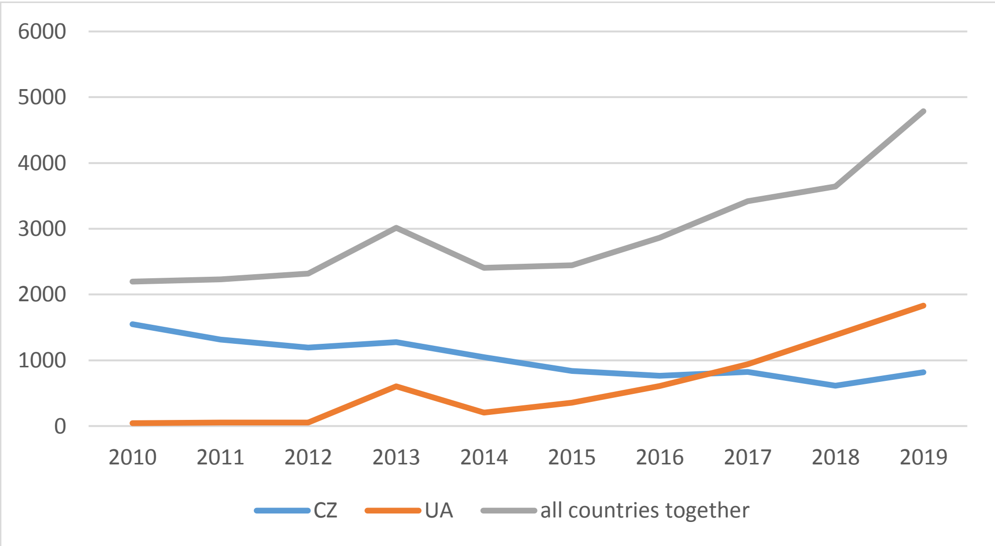

Figure 2 Number of enrolled foreign students from the Czech Republic, UA and from all countries together at Slovak universities in the years 2010 to 2019
Source: own

As we presented in Table 4 and we see this in Figure 3, which shows the trends in the numbers of these students, the total number of students and the number of Slovaks studying at universities in the Czech Republic has a declining trend, as we observe in Slovakia. In the total number, this decrease is 1.37-fold, for Slovaks I observe "only" 1.16-fold decrease.

Table 5 shows the total numbers of foreign students and Slovaks studying at Czech universities and the ratio of Slovaks to the total number of foreign students. This table, as well as Figure 4, shows that the total number of foreign students at Czech universities is growing and the number of Slovaks is declining, which is also reflected in the ratio of the number of Slovaks to all foreign students. While in 2010 this ratio was 63.82%, in 2019 it is only 44.53%.

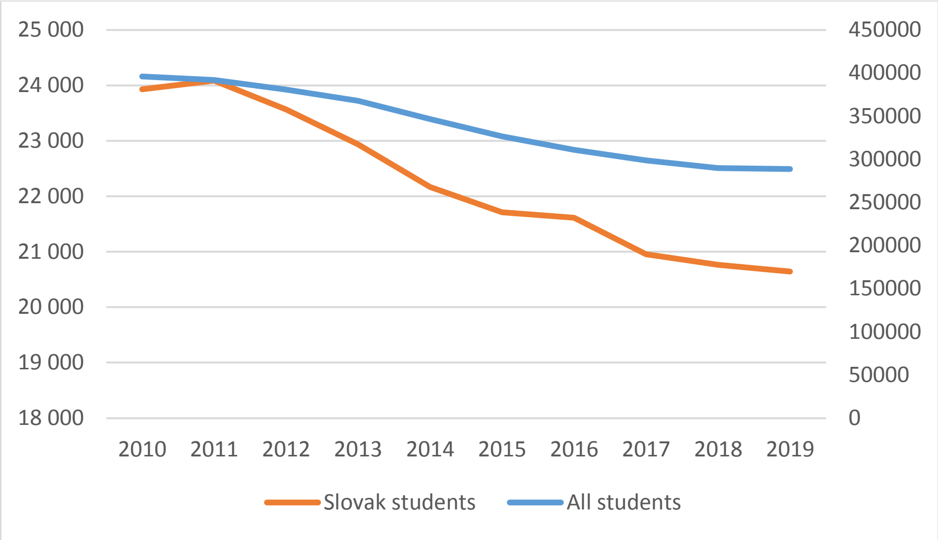

Figure 3 Total number of students and Slovaks studying at Czech universities in the years 2010 to 2019
Source: own

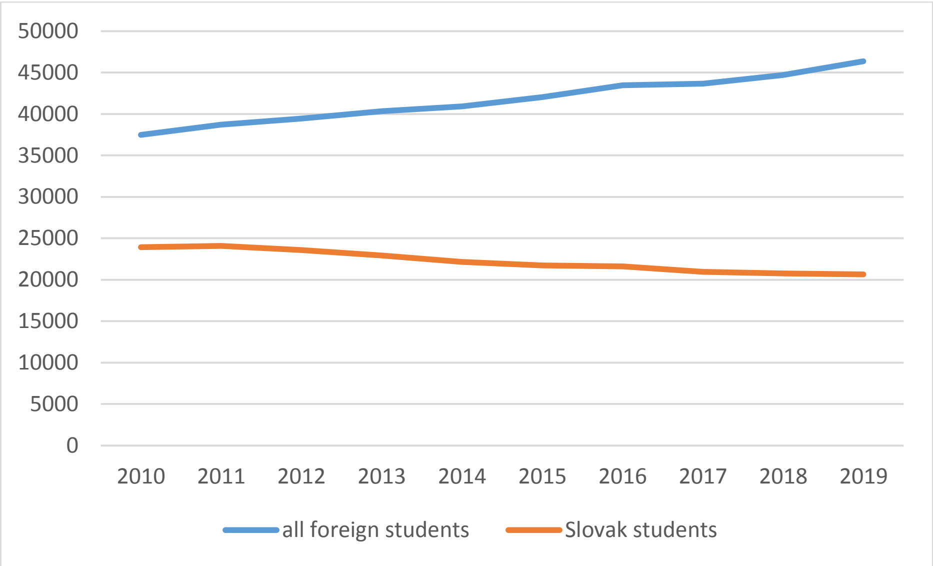

Figure 4 Numbers of all foreign and Slovak students at Czech universities in the years 2010 to 2019
Source: own

CONCLUSIONS

Hundreds of students choose to study abroad every year, as they can obtain quality study in a foreign language and on favorable terms. This applies not only to foreign students studying in Slovakia, but also to Slovaks studying at foreign universities. By studying abroad, students often improve their prospects for their future profession, have the opportunity to complete an internship after or during their studies, improve their language skills, get to know a new culture and people, and learn independence. The new way of education will not only broaden their horizons, but will also support their personal and intellectual development.

It is no coincidence that Slovak students decide to study in the neighboring Czech Republic. We are united by a common history and culture, it is close, so it is very accessible and it is very important that students can study there for free. The greatest interest is in the study of technical fields and informatics, economics, management, medicine and pharmacy.

Although we observe growing trends in the number of foreign students studying in Slovakia, but also in the Czech Republic, the number of Slovaks studying in our country and in the Czech Republic is declining. One of the factors is the decline of the population in a given age range, but certainly also the realization that skillful students can find employment without graduating from college.

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Original Paper

Statistical indicators of unemployment in the Slovak Republic

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ABSTRACT

One indicator of the economy is the unemployment rate. It is monitored in all countries of the world, especially in periods when it increases significantly in a short period of time. It is one of the highly debated problems of society. Countries around the world attempt to solve at least part of the problem through economic policy. The aim of this paper is to highlight the problem of unemployment in the Slovak Republic in all districts of the Slovak Republic. The research sample consisted of the unemployed in the individual districts of the Slovak Republic between 2019 and 2021. When we compared the years 2020 and 2021 with 2021, we found a significant difference in the number of unemployed, but when we compared the years 2020 and 2021, there was no significant difference.

KEYWORDS: unemployment rate, problem of unemployment, districts of the Slovak Republic, number of unemployed

JEL CLASSIFICATION: E010, E66

INTRODUCTION

The problem of unemployment has existed in the past, but it is now a major problem for our society. All the countries of the world are affected by this problem and are trying to tackle it in different ways. Obviously, there is no simple formula for dealing with unemployment. Unemployment depends on the current state of the economy, politics and other factors that affect it.

Slovakia has undergone significant political and economic changes since 1989. With the transition from a centrally planned economy to a market economy, the first data on the number of unemployed started to appear. By 1989 there was full employment, so there was almost no experience of unemployment. The first registration of jobseekers in the Slovak Republic appeared in February 1990, when 1949 jobseekers were registered at the labour

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offices and the unemployment rate reached 0.07% (Rievajová, 1996). At the end of 1990 there were 30,000 unemployed registered in Slovakia, the extreme growth came the following year and the number of unemployed exceeded 300,000 at an unemployment rate of 11.8% (Košta and Juríčková, 1995). Mass unemployment is usually considered when the unemployment rate exceeds 10%. The development of unemployment is influenced by both economic and social problems, such as poor mental and physical health, family breakdown, crime, etc. Essentially, unemployment results from a decline in economic growth and, as a consequence, a decline in the national economy.

Morrish (2022) et al. compare loneliness and unemployment in the working age. The impact of loneliness on unemployment is deepened further in the context of physical health. Particular attention should therefore be paid to loneliness with additional support from employers and the government to improve health and well-being. Similarly, Ikar et al. (2022) link unemployment to health. They compared the frequency of contact with GPs and patients' experiences of GPs among unemployed and employed people by using data from two surveys. They found out that unemployed people saw their GPs more often (65.9%) compared to employed people (42.9%).

Another problem is the transition between employment and unemployment. Liina et al. (2022) discussed the transition from unemployment to employment. They evaluated the association of various physical and psychiatric conditions with job search and job stability. They observed the Finnish population aged 30-60 with a period of unemployment between 2009-2018 within two years of the onset of unemployment. Poor health will reduce the unemployed's chances of finding a job, especially if they have alcohol or psychiatric problems. Park and Cho (2022) examined the economic impact of unemployment in the context of the pandemic and what greatest negative impacts were experienced by young people. Young people were significantly more likely to be economically inactive or discouraged job seekers after the COVID-19 outbreak compared to other age groups. In addition, through an analysis of employ support unitiesnity, young people reacted negatively to the possibility of future employment compared to middle-aged people after COVID-19. Galindo (2022) examined differences between firms created by unemployed people compared to firms created by employed people. He found that unemployed people are more likely to set up small firms but are not successful. He says that business assistance among the unemployed has little impact on job creation and shows low productivity.

The authors also addressed social issues in their works. Raimi, Fardeen and Sule (2022) dealt with social problems in the context of unemployment. They asked the question: "Are social enterprises in developed and developing countries characterized by the same social problems, social goals, social outcomes, and social impact objectives?" They reviewed 50 scholarly papers on social enterprises that provided an in-depth look at the topic. The first finding revealed that social enterprises in developed countries focused on "secondary-level social problems" such as education, health, environmental problems and property inequality. The second finding indicated that social enterprises in developing countries focused on 'primary level social problems' such as illiteracy, poor school enrolment, unemployment, poverty, social exclusion, gender inequality, and weak health care systems. Sengupta (2009) analysed the social bases of the various levels of unemployment available in the literature and listed measures to eliminate unemployment.

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Unemployment is also a long-term problem in Slovakia; unemployment affects the economic situation and brings problems to individuals in various forms. Unemployment, as a concept, is a much mentioned concept in ascertaining the economic health and stability of a country. Fad'os and Bohdalova (2018) describe unemployment in four regions of Slovakia in 1999-2016. The highest unemployment rate was in eastern Slovakia, while the lowest unemployment rate was in the Bratislava region. They compared the differences in unemployment between men and women. They found that since 2016, the unemployment rate for women increased by 22.73% in the Bratislava region, 86.11% in Western Slovakia, 29.41% in Central Slovakia and 2.80% in Eastern Slovakia. Bajzik (2020) focused on the current problems related to unemployment in the Slovak Republic. He dealt with a brief definition of unemployment and its types, an overview and analysis of unemployment in Slovakia based on selected demographic factors. Caposova (2020) said that unemployment is now a worldwide problem, high unemployment rate affects the overall social situation and worsens the economic situation and development of regions in Slovakia. The assessment focused on economic indicators such as GDP per capita and unemployment in different regions. The evaluation was carried out on data from individual regions. Similar issues were also observed by Svecova and Rajcakova (2010), who examined regional differences in unemployment in Slovakia. Stolicna and Grozak (2018) specified on the problem of unemployment and the automotive industry.

MATERIAL AND METHODS

The study sample consisted of all 79 districts of the Slovak Republic. We also evaluated unemployment in all regions of the Slovak Republic. The basic data for processing were obtained from the website of the Central Office of Labour, Social Affairs and Family (2022). We evaluated unemployment in all regions of the Slovak Republic. Table 1 shows the evaluation of unemployment in all regions of the Slovak Republic.

Table 1 The number of unemployed in each year of all regions of the Slovak Republic (%)

Region	2019	2020	2021	Difference 2020 and 2019	Difference 2021 and 2020	Difference 2021 and 2019
Banská Bystrica Region	6.69	9.57	9.01	2.88	-0.56	2.32
Bratislava Region	2.83	4.65	4.38	1.82	-0.27	1.55
Nitra Region	2.93	5.36	4.80	2.43	-0.56	1.87
Prešov Region	8.19	11.06	10.75	2.87	-0.31	2.56
Trnava Region	2.63	5.05	4.16	2.42	-0.89	1.53
Žilina Region	3.96	6.30	5.32	2.34	-0.98	1.36
Trenčín Region	3.20	5.27	4.28	2.07	-0.99	1.08
Košice Region	7.45	10.17	9.80	2.72	-0.37	2.35
Slovak Republic	4.92	7.38	6.76	2.46	-0.62	1.84

Source: data from the Central Office of Labour, Social Affairs and Family, author processing

The largest increase in unemployment was seen in the Košice and Prešov regions. Table 1 we see several information. There has been a lot of variation over time in unemployment rates across Slovakia. In years 2019 to 2021 was rather muted effect on unemployment in much of Slovakia, being felt fairly strongly in the eastern regions. We see the large, persistent differences in unemployment rates across regions. The west-lying regions was the unemployment rate much lower than any other region.

In Figure 1 we see the high unemployment rate in the above-mentioned districts.

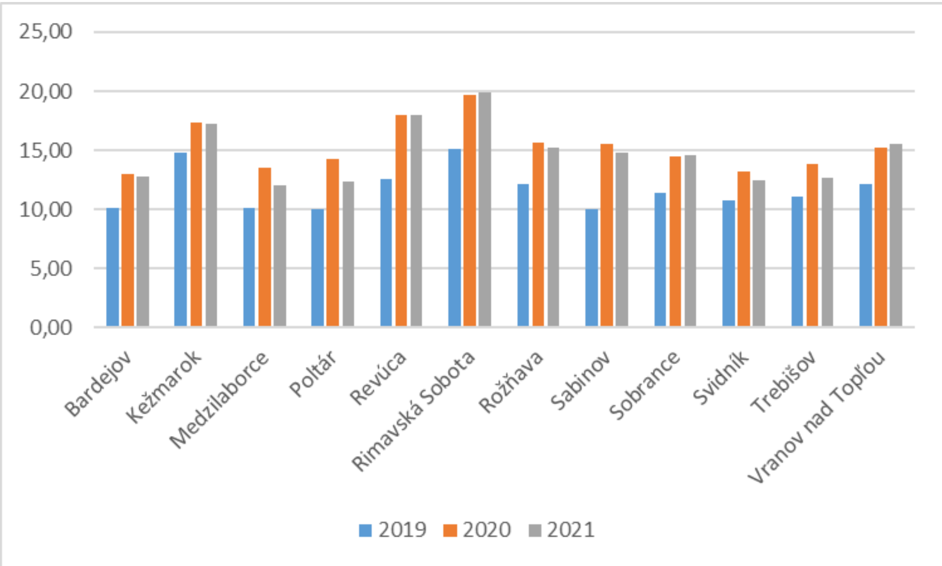

Figure 1 The high unemployment rate in the above-mentioned districts (%)
Source: Central Office of Labour, Social Affairs and Family, author processing

In all the years studied, the highest share of unemployed was in the districts in the east of Slovakia (Bardejov, Kežmarok, Medzilaborce, Poltár, Revúca, Rimavská Sobota, Rožňava Sabinov, Sobrance, Svidník, Trebišov and Vranov nad Topľou).

RESULTS AND DISCUSSION

We used basic methods of descriptive statistics and hypotheses testing. The values shown in Table 1 and Table 2 were applied as basic criteria to compare the number of unemployed in each year. Based on this, the objectives of the research were set:
- to determine whether the numbers of unemployed in each year are significantly different,
- to identify differences between districts.

We formulated the research hypotheses based on theoretical knowledge and economic practice. We want to test the hypothesis:
H0: The difference in the number of unemployed in 2019-2021 is not statistically significant.
H1: There is a significant difference in the number of unemployed in 2019 - 2021.

We use the z-test to evaluate the data. A z-test is a statistical test used to determine whether two population means are different when the variances are known and the sample size is large (Matejková et. al., 2013). Using z-test, we will test the null hypothesis. The z-test is a statistical test used to determine if the means of two populations are different when the variances are known and the sample size is greater than 30.

In Table 2, we see the number of unemployed in each year of all 79 districts.

Table 2 The number of unemployed in each year of all 79 districts of the Slovak Republic (%)

District	2019	2020	2021	District	2019	2020	2021	District	2019	2020	2021
Bánovce nad Bebravou	2.85	5.26	4.55	Košice IV	3.14	5.38	4.99	Rimavská Sobota	15.14	19.69	19.90
Banská Bystrica	3.30	4.72	4.57	Krupina	5.09	7.53	7.09	Rožňava	12.14	15.68	15.19
Banská Štiavnica	6.06	8.59	7.41	Kysucké Nové Mesto	4.74	7.23	6.36	Ružomberok	4.61	7.08	6.28
Bardejov	10.07	12.95	12.72	Levice	3.82	6.58	5.86	Sabinov	10.00	15.57	14.78
Bratislava I	2.77	3.98	3.62	Levoča	7.99	9.89	9.88	Senec	3.25	5.93	5.39
Bratislava II	3.22	4.82	4.36	Liptovský Mikuláš	4.50	6.94	6.13	Senica	5.17	7.17	6.76
Bratislava III	3.24	4.94	4.39	Lučenec	8.31	11.78	11.00	Skalica	3.35	4.81	3.98
Bratislava IV	2.97	4.52	4.45	Malacky	3.31	5.18	4.62	Snina	7.48	10.64	9.22
Bratislava V	2.00	3.67	3.63	Martin	3.07	5.31	4.47	Sobrance	11.35	14.52	14.61
Brezno	4.42	8.20	6.70	Medzilaborce	10.07	13.54	12.07	Spišská Nová Ves	6.46	9.83	8.89
Bytča	5.25	6.71	6.66	Michalovce	8.40	11.71	13.95	Stará Ľubovňa	6.07	8.68	7.95
Čadca	4.00	6.83	5.59	Myjava	2.87	4.40	3.86	Stropkov	8.86	11.88	11.25
Detva	4.90	7.30	6.22	Námestovo	3.57	6.46	4.87	Svidník	10.78	13.21	12.50
Dolný Kubín	4.67	7.56	5.74	Nitra	1.97	4.06	3.21	Šaľa	2.11	4.59	4.20
Dunajská Streda	2.15	6.22	5.00	Nové Mesto nad Váhom	2.84	4.55	3.58	Topoľčany	2.90	5.19	4.52
Galanta	2.21	4.26	3.76	Nové Zámky	2.73	5.30	5.53	Trebišov	11.02	13.81	12.71
Gelnica	8.66	11.77	11.19	Partizánske	3.28	5.55	3.91	Trenčín	1.93	4.09	3.12
Hlohovec	2.06	3.67	3.15	Pezinok	2.41	4.87	5.13	Trnava	2.40	4.58	3.32
Humenné	5.81	8.24	7.93	Piešťany	2.07	4.18	3.25	Turčianske Teplice	4.56	7.09	6.22
Ilava	2.22	3.97	3.20	Poltár	10.01	14.23	12.29	Tvrdošín	4.23	6.06	4.80
Kežmarok	14.79	17.36	17.24	Poprad	4.74	6.95	6.39	Veľký Krtíš	6.40	10.10	9.11
Komárno	4.26	7.04	6.18	Považská Bystrica	4.09	6.46	5.39	Vranov nad Topľou	12.13	15.20	15.55
Košice - okolie	9.33	12.10	11.06	Prešov	5.60	8.46	8.44	Zlaté Moravce	2.95	4.75	3.69
Košice I	4.01	5.96	5.57	Prievidza	4.70	6.93	5.76	Zvolen	3.48	5.30	4.69
Košice II	4.51	6.64	5.52	Púchov	3.01	4.68	4.09	Žarnovica	6.45	8.71	8.09
Košice III	2.88	4.50	4.08	Revúca	12.58	18.00	18.00	Žiar nad Hronom	4.58	6.26	5.59
								Žilina	3.46	5.40	4.59

Source: data from the Central Office of Labour, Social Affairs and Family, author processing

In the table 3 we test the null hypothesis which states that there is a significant difference in the number of unemployed between 2019 and 2021.

Comparing 2020 and 2021 (Table 3), the z-test value is negative and the critical value is 1.96 at the chosen significance level. Since the value of the z-test is less than the critical value, we do not reject the null hypothesis which states that the difference in the number of unemployed is not statistically significant. In Table 3, we see that the z-test value is 11.67 for 2019 and 2021, and the critical value is 1.96 at the significance level. For 2019 and 2020, the z-test value is 15.83, the critical value is 1.96 at the significance level. Since the z-test value is greater than the critical value in these two cases, we reject the null hypothesis. That is, we accept the alternative hypothesis that there is a significant difference in the number of unemployed.

Table 3 Results of z-test

	2019 and 2021		2019 and 2020		2020 and 2021	
Mean	7.23	5.38	5.38	7.90	7.90	7.23
Observations	79	79	79	79	79	79
z Stat	11.67		15.83		-4.15	
z Critical one-tail	1.96		1.96		1.96	

Source: data from the Central Office of Labour, Social Affairs and Family, author processing

CONCLUSIONS

The aim of this paper was to highlight a topical issue that is often discussed and related to the quality of employment services provided by public and private sector organizations. There is a significant difference in the number of unemployed between 2019 and 2021, according to our research. The study research results indicate an increase in the unemployment rate in Slovakia during 2020 by two to three percentages compared to the trend of its development, which would have occurred without a pandemic. This difference can be described as the impact of the COVID-19 pandemic. The main causes of the high unemployment rate in the above-mentioned districts are the overall stagnation of agriculture and industry, the closing down of enterprises and the inability to create new job positions. An important variable entering into the research analysis of unemployment trend analysis is the migration of labor force abroad.

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Original Paper

Evaluation of specific integrals by differentiation – part 3

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ABSTRACT

The technique of integration (anti-differentiation) represents one of the most important techniques in calculus. While its counterpart, differentiation, is a routine and relatively simple procedure, integration, in general, is a much more involving task. The inverse relationship between differentiation and anti-differentiation (evaluation of indefinite integrals) in some particular cases reveals the possibility to derive the form of the antiderivative and evaluate this antiderivative by differentiation and subsequent comparison of coefficients. This paper is a sequel to previous author's papers and deals with some other types of elementary functions whose indefinite integrals can be, at least partly, evaluated by differentiation and comparison of coefficients.

KEYWORDS: higher mathematics, differentiation, integration, undetermined coefficients

JEL CLASSIFICATION: I 20, C20

INTRODUCTION

In [2], we investigated integrals containing polynomials and various rational powers of a linear function. Namely, we discussed the antiderivatives of the following functions: $\frac{P_n(x)}{\sqrt{ax+b}}$, $\frac{P_m(x)}{\sqrt[n]{ax+b}}$, $\frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}$, where $P_n(x)$, $P_m(x)$, $P_r(x)$ denote polynomials of n-th, m-th and r-th degree, respectively. Let's recall the results from [2]. We showed that the indefinite integrals of these functions can be expressed as follows:

$$\int \frac{P_n(x)}{\sqrt{ax+b}}\,dx = Q_n(x)\sqrt{ax+b}$$

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$$\int \frac{P_m(x)}{\sqrt[n]{ax+b}}\,dx = Q_m(x)\sqrt[n]{(ax+b)^{n-1}}$$

$$\int \frac{P_r(x)}{\sqrt[n]{(ax+b)^m}}\,dx = Q_r(x)\sqrt[n]{(ax+b)^{n-m}}$$

For example

$$\int (x^2+1)\sqrt[4]{x-1}\,\mathrm{d}x = (Ax^2+Bx+C)\sqrt[4]{(x-1)^5}$$

Upon differentiation and comparison of coefficients we obtained the antiderivative without "classical" integration.

$$\int (x^2+1)\sqrt[4]{x-1}\,dx = \left(\frac{4}{13}x^2+\frac{32}{117}x+\frac{596}{585}\right)\sqrt[4]{(x-1)^5}+const$$

In this paper we investigate integrals containing polynomials and various rational powers of polynomials.

RESULTS AND DISCUSSION

Like in [1] and [2], we denote polynomials of degree *n, m, r* as $P_n(x), Q_m(x), \alpha_r(x)$ etc., respectively and their *k*-th derivatives as $P_{n-k}(x), Q_{m-k}(x), \alpha_{r-k}(x)$ etc., respectively, further *a, b, c* etc. are given (real) constants and *A, B, C,* α, β etc. are unknown coefficients. All the investigated integrals are considered on intervals where they are defined and in all cases and illustrative examples we set the integration constant equal to zero.

Let us consider the simplest case first.

1. $\displaystyle\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx.$

Since the square root of a quadratic polynomial is transformed by differentiation into its reciprocal multiplied by a linear term, we will consider the function $Q_n(x)\sqrt{ax^2+bx+c}$ and its derivative. In all discussions in this paper, except for the illustrative examples, we consider polynomials in general, hence we deliberately neglect the constant multiples of polynomials that arise during differentiation.

$$\left[Q_n(x)\sqrt{ax^2+bx+c}\right]' = Q_{n-1}(x)\sqrt{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}$$

$$Q_n(x)\sqrt{ax^2+bx+c} = \int Q_{n-1}(x)\sqrt{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx$$

$$Q_n(x)\sqrt{ax^2+bx+c}=\int\frac{Q_{n-1}(x)\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}+\frac{Q_n(x)(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx$$

Now let $P_{n+1}(x)=Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)$, and we can write

$$\int\frac{P_{n+1}(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_n(x)\sqrt{ax^2+bx+c}$$

But the polynomials $P_{n+1}(x), Q_n(x)$ differ in degree and $Q_n(x)$ contains fewer (unknown) coefficients than $P_{n+1}(x)$, so it is necessary to make up for the missing coefficient. This coefficient (in order to be "preserved" by differentiation) assumes the form

$\int\frac{\alpha}{\sqrt{ax^2+bx+c}}\,dx,$ hence for $n\geq 0$ we get

$$\int\frac{P_{n+1}(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_n(x)\sqrt{ax^2+bx+c}+\alpha\int\frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

Example 1: Evaluate $\int\frac{2x+4}{\sqrt{x^2+1}}\,dx$.

Solution:

$$\int\frac{2x+4}{\sqrt{x^2+1}}\,dx=A\sqrt{x^2+1}+\alpha\int\frac{1}{\sqrt{x^2+1}}\,dx$$

Now we take the derivatives of both sides

$$\frac{2x+4}{\sqrt{x^2+1}}=\frac{2Ax}{2\sqrt{x^2+1}}+\frac{\alpha}{\sqrt{x^2+1}}$$

$$2x+4=Ax+\alpha$$

We see that $A=2,\ \alpha=4$, hence

$$\int\frac{2x+4}{\sqrt{x^2+1}}\,dx=2\sqrt{x^2+1}+4\int\frac{1}{\sqrt{x^2+1}}\,dx\ .$$

Of course, the right hand side integral should further be evaluated. We do so only in this particular case. Since it is tabulated, we get

$$\int\frac{2x+4}{\sqrt{x^2+1}}\,dx=2\sqrt{x^2+1}+4\ln\left|x+\sqrt{x^2+1}\right|$$

Example 2: Evaluate $\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx$.

Solution:

$$\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx = \left(Ax^2+Bx+C\right)\sqrt{x^2+4x-5}+\alpha\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$$

$$\frac{2x^3+3x+1}{\sqrt{x^2+4x-5}} = (2Ax+B)\sqrt{x^2+4x-5}+\frac{\left(Ax^2+Bx+C\right)(2x+4)}{2\sqrt{x^2+4x-5}}+\frac{\alpha}{\sqrt{x^2+4x-5}}$$

$$2x^3+3x+1=(2Ax+B)\left(x^2+4x-5\right)+\left(Ax^2+Bx+C\right)(x+2)+\alpha$$

Now we distribute and compare the coefficients

$$2x^3+3x+1=2Ax^3+Bx^2+8Ax^2+4Bx-10Ax-5B+Ax^3+Bx^2+Cx+2Ax^2+2Bx+2C+\alpha$$

$$A=\frac{2}{3}, B=-\frac{10}{3}, C=\frac{89}{3}, D=-75$$

$$\int \frac{2x^3+3x+1}{\sqrt{x^2+4x-5}}\,dx = \left(\frac{2}{3}x^2-\frac{10}{3}x+\frac{89}{3}\right)\sqrt{x^2+4x-5}-75\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$$

Now we generalize this case to the *m*-th root of the polynomial in the denominator.

2. $\int \frac{P_n(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$ and $\int \frac{P_n(x)}{\sqrt[m]{ax^2+bx+c}}\,dx$.

We derive the form of the antiderivative for the first integral.

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right]' = Q_{n-1}(x)\sqrt[m]{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}$$

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right] = \int Q_{n-1}(x)\sqrt[m]{ax^2+bx+c}+\frac{Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

$$\left[Q_n(x)\sqrt[m]{ax^2+bx+c}\right] = \int \frac{Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

Let again $P_{n+1}(x)=Q_{n-1}(x)\left(ax^2+bx+c\right)+Q_n(x)(2ax+b)$

$\int \frac{P_{n+1}(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{ax^2+bx+c}$ and we again fix the missing coefficient, so

$$\int \frac{P_{n+1}(x)}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{ax^2+bx+c}+\alpha\int \frac{1}{\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}}\,dx$$

Analogously for $\int \frac{P_n(x)}{\sqrt[m]{ax^2+bx+c}}\,dx$ we would get

$$\int \frac{P_{n+1}(x)}{\sqrt[m]{ax^2+bx+c}}\,dx = Q_n(x)\sqrt[m]{\left(ax^2+bx+c\right)^{m-1}}+\alpha\int \frac{1}{\sqrt[m]{ax^2+bx+c}}\,dx.$$

Both these integrals are valid for $n \geq 0$ and $m \neq 0, 1$.

Example 3: Evaluate $\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx$.

Solution:

$$\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx = (Ax+B)\sqrt[5]{\left(x^2+x-1\right)^4}+\alpha\int \frac{1}{\sqrt[5]{x^2+x-1}}\,dx,$$

and differentiate

$$\frac{x^2-3x}{\sqrt[5]{x^2+x-1}} = A\sqrt[5]{\left(x^2+x-1\right)^4}+\frac{4}{5}\frac{(Ax+B)(2x+1)}{\sqrt[5]{x^2+x-1}}+\frac{\alpha}{\sqrt[5]{x^2+x-1}},$$

then a little algebra

$$\frac{x^2-3x}{\sqrt[5]{x^2+x-1}} = \frac{5A\left(x^2+x-1\right)+4(Ax+B)(2x+1)+5\alpha}{5\sqrt[5]{x^2+x-1}}$$

and finally we distribute and compare the coefficients

$$5x^2-15x = 5A\left(x^2+x-1\right)+4(Ax+B)(2x+1)+5\alpha$$

$$5x^2-15x = 5Ax^2+5Ax-5A+8Ax^2+8Bx+4Ax+4B+5\alpha$$

Upon evaluation of the coefficients we have

$$\int \frac{x^2-3x}{\sqrt[5]{x^2+x-1}}\,dx = \left(\frac{5}{13}x-\frac{30}{13}\right)\sqrt[5]{\left(x^2+x-1\right)^4}+\frac{29}{13}\int \frac{1}{\sqrt[5]{x^2+x-1}}\,dx$$

Next we generalize to the case of the *m*-th root of the *r*-th degree polynomial in the denominator.

3. $\int \frac{P_n(x)}{\sqrt[m]{(S_r(x))^{m-1}}}\,dx$ and $\int \frac{P_n(x)}{\sqrt[m]{S_r(x)}}\,dx$.

Again we derive the form of the antiderivative for the first integral.

$$\left[Q_n(x)\sqrt[m]{S_r(x)}\right]' = Q_{n-1}(x)\sqrt[m]{S_r(x)} + \frac{Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}$$

$$Q_n(x)\sqrt[m]{S_r(x)} = \int Q_{n-1}(x)\sqrt[m]{S_r(x)} + \frac{Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx$$

$$Q_n(x)\sqrt[m]{S_r(x)} = \int \frac{Q_{n-1}(x)\,S_r(x)+Q_n(x)\,S_{r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx$$

Let $P_{n+r-1}(x) = Q_{n-1}(x)\,S_r(x) + Q_n(x)S_{r-1}(x)$, then

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{S_r(x)}$$

Upon completion of the missing coefficients we get

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx = Q_n(x)\sqrt[m]{S_r(x)} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{\left(S_r(x)\right)^{m-1}}}\,dx$$

And for $\int \frac{P_n(x)}{\sqrt[m]{S_r(x)}}\,dx$ we have

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{S_r(x)}}\,dx = Q_n(x)\sqrt[m]{\left(S_r(x)\right)^{m-1}} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{S_r(x)}}\,dx$$

Note that $r \geq 2$, $n \geq 0$ and $m \neq 0, 1$.

Now let us replace *m* by –*m*, then

$$\int P_{n+r-1}(x)\sqrt[m]{S_r(x)}\,dx = Q_n(x)\sqrt[m]{\left(S_r(x)\right)^{m+1}} + \int \alpha_{r-2}(x)\sqrt[m]{S_r(x)}\,dx$$

Example 4: Evaluate $\int \frac{x^2+4x+1}{\sqrt[4]{x^3+x}}\,dx$.

Solution:

$$\int \frac{x^2+4x+1}{\sqrt[4]{x^3+x}}\,dx = A\sqrt[4]{\left(x^3+x\right)^3} + \int \frac{\alpha\, x+\beta}{\sqrt[4]{x^3+x}}\,dx$$

We differentiate again

$$\frac{x^2+4x+1}{\sqrt[4]{x^3+x}} = A\frac{3}{4}\frac{3x^2+1}{\sqrt[4]{x^3+x}} + \frac{\alpha\, x+\beta}{\sqrt[4]{x^3+x}}$$

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from which

$$x^2 + 4x + 1 = \frac{9}{4} Ax^2 + \frac{3}{4} A + \alpha\, x + \beta$$

$$\int \frac{x^2 + 4x + 1}{\sqrt[4]{x^3 + x}}\, dx = \frac{4}{9} \sqrt[4]{\left(x^3 + x\right)^3} + \int \frac{4\, x + \frac{2}{3}}{\sqrt[4]{x^3 + x}}\, dx\,.$$

And finally we consider the most general case, i.e. the general rational power of the r-th degree polynomial in the denominator.

4. $\int \frac{P_n(x)}{\sqrt[m]{(S_r(x))^{m-k}}}\, dx$ and $\int \frac{P_n(x)}{\sqrt[m]{(S_r(x))^{k}}}\, dx\,.$

Note again that $r \geq 2$, $n \geq 0$ and $m \neq 0$.

As in the previous cases we differentiate the function

$$\left[Q_n(x) \sqrt[m]{(S_r(x))^k} \right]' = Q_{n-1}(x) \sqrt[m]{(S_r(x))^k} + \frac{Q_n(x)\, S_{r-1}(x)}{\sqrt[m]{(S_r(x))^{m-k}}}$$

$$Q_n(x) \sqrt[m]{(S_r(x))^k} = \int \frac{Q_{n-1}(x)\, S_r(x) + Q_n(x)\, S_{r-1}(x)}{\sqrt[m]{(S_r(x))^{m-k}}}\, dx$$

Let $P_{n+r-1}(x) = Q_{n-1}(x)\, S_r(x) + Q_n(x) S_{r-1}(x)$, then

$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{(S_r(x))^{m-k}}}\, dx = Q_n(x) \sqrt[m]{(S_r(x))^k}$ and with the missing coefficients

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{(S_r(x))^{m-k}}}\, dx = Q_n(x) \sqrt[m]{(S_r(x))^k} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{(S_r(x))^{m-k}}}\, dx$$

It is easy to show that $\int \frac{P_n(x)}{\sqrt[m]{(S_r(x))^k}}\, dx$ assumes the form

$$\int \frac{P_{n+r-1}(x)}{\sqrt[m]{(S_r(x))^k}}\, dx = Q_n(x) \sqrt[m]{(S_r(x))^{m-k}} + \int \frac{\alpha_{r-2}(x)}{\sqrt[m]{(S_r(x))^k}}\, dx$$

Again let us replace m by $-m$, then

$$\int P_{n+r-1}(x) \sqrt[m]{(S_r(x))^k}\, dx = Q_n(x) \sqrt[m]{(S_r(x))^{m+k}} + \int \alpha_{r-2}(x) \sqrt[m]{(S_r(x))^k}\, dx$$

Example 5: Evaluate $\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{\left(x^2-4\right)^5}}\,dx$.

Solution:

$$\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{\left(x^2-4\right)^5}}\,dx = \left(Ax^2+Bx+C\right)\sqrt[3]{\left(x^2-4\right)^{-2}} + \int \frac{\alpha}{\sqrt[3]{\left(x^2-4\right)^5}}\,dx$$

$$\frac{x^3+2x^2-4x+5}{\sqrt[3]{\left(x^2-4\right)^5}} = \frac{\left(2Ax+B\right)}{\sqrt[3]{\left(x^2-4\right)^2}} - \frac{2}{3}\frac{\left(Ax^2+Bx+C\right)2x}{\sqrt[3]{\left(x^2-4\right)^5}} + \frac{\alpha}{\sqrt[3]{\left(x^2-4\right)^5}}$$

$$x^3+2x^2-4x+5 = \left(2Ax+B\right)\left(x^2-4\right) - \frac{2}{3}\left(Ax^2+Bx+C\right)2x+\alpha$$

$$x^3+2x^2-4x+5 = \frac{2}{3}Ax^3 - \frac{1}{3}Bx^2 - 8Ax - \frac{4}{3}Cx - 4B + \alpha$$

$$\int \frac{x^3+2x^2-4x+5}{\sqrt[3]{\left(x^2-4\right)^5}}\,dx = \left(\frac{3}{2}x^2-6x-6\right)\sqrt[3]{\left(x^2-4\right)^{-2}} - \int \frac{19}{\sqrt[3]{\left(x^2-4\right)^5}}\,dx$$

CONCLUSIONS

In the paper we investigated integrals containing polynomials and various rational powers of polynomials. The algebraic limitations of the method developed in the paper do not allow us to evaluate the “undone” integrals on the right hand side. These integrals are, in general, non-elementary and their evaluation requires, except for special cases, more sophisticated methods which are out of the scope of this paper.

However, the method presented in the paper simplifies the (reduces the degree) polynomial in the numerator and reveals the structure of the required antiderivative.

The use of the presented method is left to the reader in every particular case.

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Original Paper

The results of the university competence measurement in mathematics in the view of the tasks

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ABSTRACT
Experience shows that students applying to higher education have a highly differentiated knowledge of mathematics. They come from different types of secondary school, and it is not a general requirement to have an advanced level secondary school certificate in mathematics. However, adequate basic knowledge is essential for mastering the university mathematics material, so it is important to identify the shortcomings in time and, if necessary, to make interventions. This is the reason for measuring the mathematical competence of incoming students. In our article, we analyze the results of the mathematical competence measurement made in the 1st semester of the academic year of 2022/2023, among the 1st year BSc students of the University of Miskolc in the fields of IT, engineering and economics, highlighting the problematic areas and the parts of the academic material acquired at the appropriate level. 513 students completed the test, their average performance was 65% and only 5 students achieved the maximum point.

KEYWORDS: mathematical competence, measuring, university mathematics material, competence

JEL CLASSIFICATION: I21, C02

INTRODUCTION

The eternal questions of teaching mathematics are as follows: What should we teach? How should we teach? What level should we teach at? In terms of secondary education the question ‘What should we teach?’ can be answered on the basis of the National Basic Curriculum (NAT), the curriculum requirements, as well as the lesson plans, and in the case of universities the answer is given by the previously prepared syllabi for the given major. The answer to the question ‘How to teach?’ depends significantly on the age of the given students,

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as it is not the same thing whether we want to explain a particular subject to a young elementary school pupil or a university student, a student who already understands a lot of things using his mathematical knowledge. The third question asking at what level we should teach is not easy to answer either, but if we know the solution, we will get a partial answer to the other two questions as well [4].

Several years of experience show (the authors of the article have been teaching at the Institute of Mathematics for more than twenty years) that the mathematical knowledge of students applying to higher education in engineering, IT, and economic studies is highly differentiated. On the one hand, it is inherent to the fact that it is not compulsory to take an A-level exam in mathematics to get admission to university, and on the other hand, the type of secondary school the students come from (secondary technical school or secondary grammar school) is also crucial. It may happen that students coming from certain secondary technical schools have an advantage in technical drawing and mechanical drawing that they have already learnt in certain university subjects, such as Descriptive Geometry, however, in their case it is often observed that it is more difficult for them to acquire subjects based on mathematical knowledge, such as Analysis or Linear Algebra, since in their secondary school they studied mathematics in a lower number of hours.

In general, however, there are areas of mathematics that cause problems for many people, which makes it difficult for them to adapt to university studies. Furthermore, even for those students who are able to reproduce the knowledge they gained at an adequate level, applying it in new situations is problematic in some cases.

We tried to assess these shortcomings and strengths by the competence-based test to be completed by students at the very beginning of the year. We tried to compile the tasks in such a way that they should cover all topics and get a realistic picture of the students' adequate knowledge and their shortcomings. The purpose of our article is to explore the critical areas based on the results of the entrance competence test paper written by 513 people.

About competence in general

The changes taking place in the society and the entire world have fundamentally changed the expectations towards education. We can quite often hear that schools are demanded to provide useful knowledge to the students studying there. The term increasingly used to describe this demand is the development of competence [6].

In general, competence means a preparedness that enables us to act effectively in different situations [8]. We mean the preparedness that is based on knowledge and skills, but also on experience, values and attitudes. The role of the thinking ability is one of the most significant in terms of mathematical competence, but it relies on several abilities, such as systematization, combinativity, deductive and inductive approaches and reasoning. These properties must be components that also work in other areas, i.e. the ability to think mathematically must become an ability that can be used in other subjects as well, and vice versa, e.g. an analytical ability developed in literature or history classes must also be adaptable in mathematics classes. It can be read in several researches ([5], [7]) that it is advisable to develop the following abilities in order to improve mathematical competence.

Three main components of mathematical competence:

- Knowledge of mathematics as a subject
- Mathematics-specific skills and abilities
- Motives, attitudes

The following table shows the most important ability and skill components of mathematical competence [4].

Table 1 Ability and skill components

Skills	Thinking ability	Communication skills	Knowledge acquisition skills	Learning skills
• counting • calculation • quantitative inference • estimation • measurement • measurement unit conversion • solving word problems	• systematization • combinativity • deductive inference • induction • probabilistic inference • reasoning • proof	• mathematics vocabulary • reading comprehension • text interpretation • spatial vision, spatial conditions • representation • presentation	• problem sensitivity (questions) • problem representation • originality, creativity • problem solving • metacognition	• attention • part-whole perception • memory • task performance • speed of task solving

Topics of the school leaving exam

The school leaving exam system underwent a significant change in 2005, when the two-level exam was introduced. At intermediate level, the mathematical knowledge of a person with the ability to navigate and create must be required in today's society, which primarily means the knowledge and application of mathematical concepts and theorems in practical situations. The advanced level includes the requirements of the intermediate level, but among the requirements formulated in the same way, the advanced level is more difficult, also, tasks requiring more ideas are included. In addition, among the requirements of the advanced level, there are also special parts of the material, since the advanced level mainly prepares students who use and study mathematics in higher education [3].

In parallel with the introduction of the new school leaving system, the entrance exams required for admission to higher education were abolished. According to the original concept, the advanced-level school leaving exam would have been a condition for applying to higher education institutions, while the intermediate-level exam would have been intended for the completion of secondary studies. However, this idea did not come true [1]. Even though since 2020 the advanced level exam has been the admission requirement for all fields of study, it is not compulsory to complete it in mathematics.

The topics of the school leaving exam come from the following chapters of mathematics:

- Thinking methods, sets, logic, combinatorics, graphs
- Number theory, algebra
- Functions, elements of analysis
- Geometry, coordinate geometry, trigonometry

- Probability calculation, statistics

In the case of engineering and IT BSc courses, compulsory subjects usually include a course on analysis for two semesters, courses on linear algebra and discrete mathematics at least for one semester. In the case of BSc courses belonging to the field of economics, the topics of the two-semester mathematics subjects include the elements of analysis, probability calculation, applied linear algebra and operations research. For most students, these foundational mathematics courses are difficult to complete, as they have to account for the acquisition of a significant amount of new knowledge, and the good mathematical foundations acquired in high school are essential for successful graduation. In the absence of these, it is significantly more difficult for students to solve the tasks set in written examination papers and in exam papers. In order to successfully complete the courses, students arriving with insufficient mathematical knowledge must not only master the new material, but must also make up the corresponding chapters of the secondary school material. Therefore, it is justified to measure the entrance mathematical competence of freshers at the beginning of their university studies, so that intervention and bringing them up to the level can be carried out if necessary, thus avoiding later dropouts.

MATERIALS AND METHODS

The entrance competency test was written by a total of 697 first-year BSc students belonging to engineering, IT, and economics fields of study. 513 people participated in the first measurement. It took 60 minutes to write the test, and the set of tasks contained 20 multiple-choice questions, which were selected in connection with the above-mentioned secondary school exam topics of mathematics, namely:

- Thinking methods, sets, logic, combinatorics, graphs: 3 questions
- Number theory, algebra: 5 questions
- Functions, elements of analysis: 7 questions
- Geometry, coordinate geometry, trigonometry: 5 questions

The students completed the test online on the e-learning interface of the University of Miskolc. They could choose one out of five options, each correct answer was worth 4 points, an incorrect answer resulted in a deduction of 1 point, no points were given for the unanswered task. Among the answer we have hidden solutions that can be obtained using the typical incorrect methods.
Out of the maximum possible 80 points, the respondents achieved an average of 52 points, which means an average performance of 65%, but there were only 5 students who achieved the maximum point.

RESULTS AND DISCUSSION

The analysis of the most successful tasks

The following Figure 1 shows the number of correct answers to each task. It can be seen that most of the students managed to solve the 8th task in the test, a question related to percentage calculation.

The task was as follows:

The price of a television has been increased by 25%. By what percentage must the new price of the television be reduced to return to the original price?
Students could choose a solution out of five options:

- 91.62% of the students chose the solution of 20%
- the solution of 25% was chosen by 6.43% of the students
- the solution of 15% was chosen by 0% of the students
- 0.97% of the students chose 22.5%
- 0.58% of the students chose 30%
- 0.39% of students did not answer the question

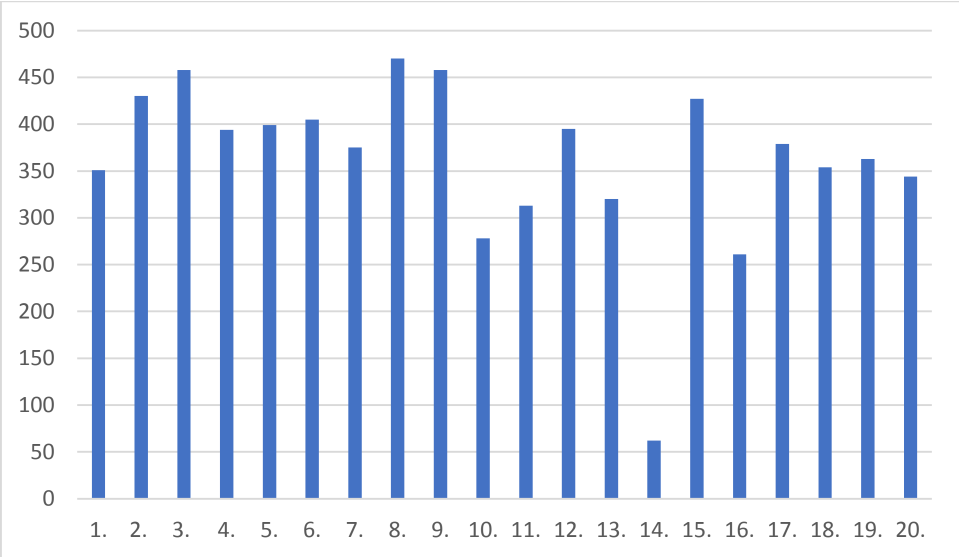

Figure 1 The number of correct answers

It is clear from the above results that almost everyone was able to do the basic percentage calculation task, with few exceptions. The correct solution of 20% was chosen by 470 out of 513 people. Based on this fact, we can expect that the solution of tasks related to percentage calculation occurring during university mathematics education will not cause difficulties either.

The other very successful task was task number 3, which belongs to the topic of the series:
The first element of a geometric series is 4, its quotient is 3. What is the sum of the first five elements of the series?
The students' five possible answers and their percentage distribution are as follows:

- 484 was chosen as the solution by 89.28% of the students
- 3.12% of students chose 324 as the solution
- 2.53% of students chose 16
- 476 was chosen as the solution by 1.17% of the students
- 1.36% of students did not choose a solution

Out of 513 students, 458 students answered this task correctly. Within the subject of analysis, in the topic of series, as well as in the material based on it, such as numerical series, it is hoped that most of the students will not have any problems when using and applying the knowledge about geometric series learned at secondary school. They were probably able to master this topic much better, and it is closer to them. This is good news for us university teachers, since we can rely on this knowledge and use these connections without repeating them in class assignments.

The third most successful task type was task 9 belonging to the area calculation topic:
One side of the rectangle is 120 cm, and the other side is a quarter of that. How many dm^2 is the area of the rectangle?

- 36 was chosen as the solution by 89.28% of the students
- 4.29% of students chose 3600 as the solution
- 3.9% of students chose 360
- 0.78% students chose 300
- 30 was chosen as the solution by 1.17% of the students
- 0.58% of students did not choose any of the solutions

Out of the 513 people who wrote the test, 458 people calculated this task correctly, i.e. solving this simple geometric task did not cause any problems for most of the students and they were able to give a good solution within the maximum 3 minutes allotted for one task. During their university mathematics studies, they will encounter many tasks based on geometry, such as area, surface, and volume calculations in the applications of integral calculus. Of course, we could list other examples, especially for students studying in the field of engineering training, not only mathematical tasks, but also tasks related to physics or descriptive geometry.

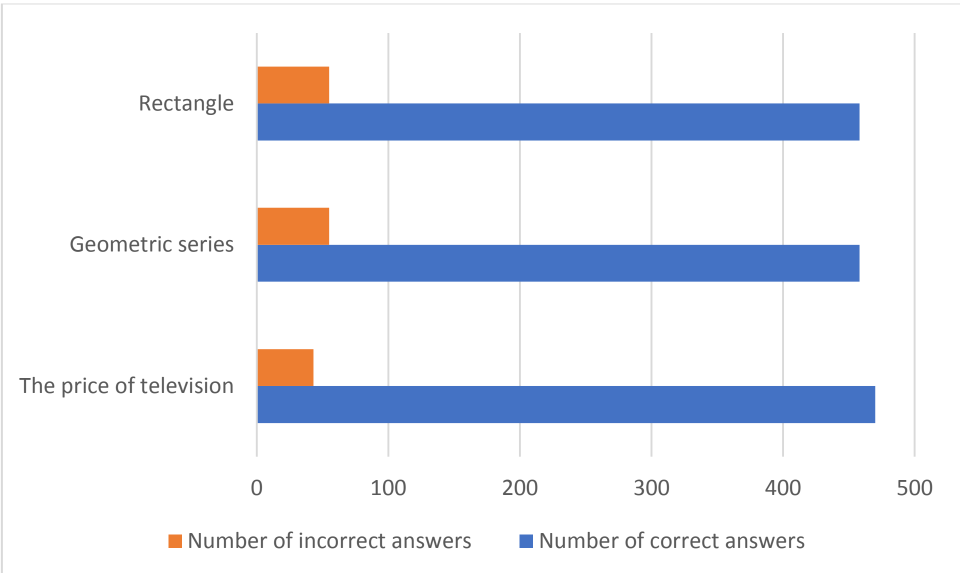

Figure 2 Number of correct and incorrect answers by the most successful tasks

Analysis of problematic tasks

A problem of combinatorics was the least successful among the tasks. Solving the task did not require the automatic application of basic knowledge, but it rather demanded logical thinking. The text of task 14 was as follows:
Bea makes bouquets. Each bouquet contains three types of flowers. For the bouquets, you can use 15 lilies, 25 gerberas, 25 roses and 35 tulips. How many bouquets will Bea make if there are as many as possible?

- 32 (the correct answer) was chosen by 12.09% of the students
- 25 was chosen as the solution by 57.31% of the students
- 23 was chosen as a solution by 1.17% of the students
- 33 was chosen as a solution by 20.66% of the students
- 35 was chosen as a solution by 4.48%% of the students
- 4.29% of students did not choose any of the solutions

Out of 513 respondents, only 62 solved the task correctly. It is instructive that most people chose 25, from which it can be concluded that they did not interpret the task correctly, they did not combine the four types of flowers, but only automatically used the three flowers with the highest number. As mentioned in the introduction, the majority of students find it difficult to apply knowledge in new situations. Our experience was confirmed by the answers to this task of the entrance competence measurement.

A little more than half of the students gave the correct answer to task 16; the question related to the following trigonometric equation:
How many solutions of the equation $cosx = \frac{1}{2}$ *are there at* $[0; 3\pi]$ *interval?*

- 3 (the correct answer) was chosen by 50.88% of the students
- 2 was chosen by 30.02% of the students as a solution
- 4 was chosen by 3.9% of the students as a solution
- 1 was chosen by 5.26% of the students as a solution
- 6 was chosen by 4.0% of the students as a solution
- 5.85% of the students, i.e. 30 people, did not choose any of the solutions

Most of the students' knowledge of trigonometric functions and identities is incomplete, which is a problem especially for students in the field of IT and technical education. Angular functions appear, for example, in the description of periodic phenomena, but in countless areas of technical life, knowledge of basic trigonometric relationships is also necessary for the successful completion of analysis subjects.

The question related to analytic geometry (task 10) was the third least successful task:
For which value of the parameter are the lines $3x + 2y = 4$ *and* $4x – ay = 17$ *perpendicular to each other?*

- 6 (the correct answer) was chosen by 54.19% of the students
- \- 6 was chosen by 18.32% of the students as a solution
- $\frac{8}{3}$ was chosen by 3.9% of the students
- $-\frac{8}{3}$ 4.87% of students chose it as a solution
- $\frac{17}{4}$ 3.31% of students chose it as a solution
- 15.4% of students did not choose any of the solution

Based on the answers, it can be seen that the second largest number of tips came for the opposite of the correct solution. The parameter $-\frac{8}{3}$, which would be the correct answer in the case of a parallel line was chosen by 25 people. It should be noted that 79 people did not answer at all. The results of the test have also confirmed that analytic geometry is a difficulty in many cases. One of the reasons for this may be that in order to be good at this subject, one must have solid geometric and algebraic foundations, since the task of analytic geometry is to solve geometric problems with algebraic tools [2].

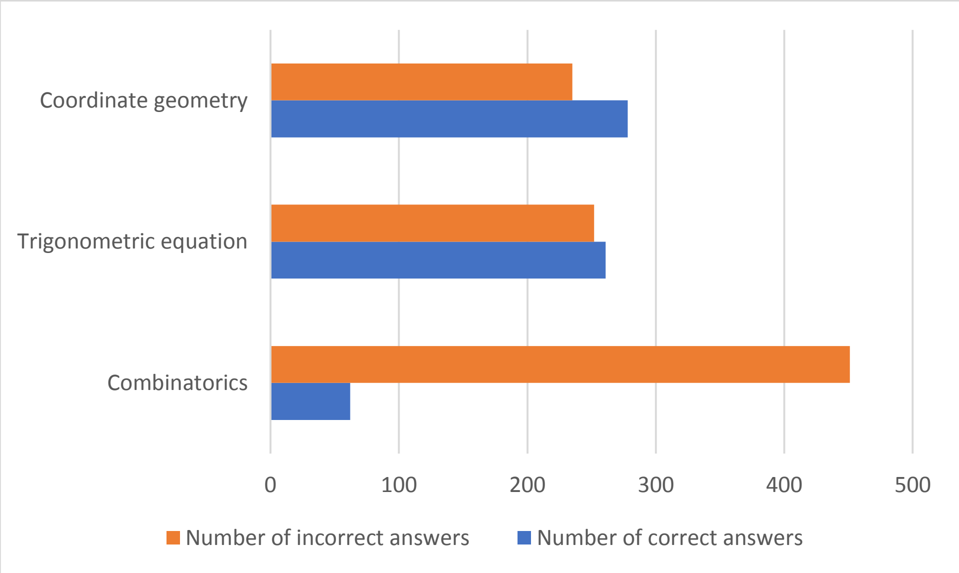

Figure 3 Number of correct and incorrect answers by the problematic tasks

CONCLUSIONS

The results of the competence measurement have confirmed our experience gained during university mathematics education. The newly recruited students proved to be less effective in the tasks that we had previously expected. Although as a result of the online implementation of the test, the tasks had to be solved in the form of multiple-choice questions, in the case of problematic topics, the offered options basically did not help the students either. Students who did not write the test at a sufficient level must participate in a competence development course. Within this framework, we can already focus on the shortcomings that need to be solved, thus helping those concerned to pass university mathematics subjects and to successfully acquire the professional knowledge relying on them.

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Original Paper

The sum of the series of reciprocals of the cubic polynomials with one zero and two different positive integer roots

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ABSTRACT

This contribution is a follow-up to five preceding author's papers and deals with the sum of the series of reciprocals of the cubic polynomials with one zero and two different positive integer roots. We derive the formula for the sum of these series and verify it by some examples using the basic programming language of the computer algebra system Maple 2020.

KEYWORDS: sum of the series, harmonic numbers, telescoping series, computer algebra system Maple 2020

JEL CLASSIFICATION: I30

INTRODUCTION

Let us recall some used basic terms concerning infinite series. We say that a series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

converges to a limit s if and only if the sequence of partial sums $s_n = a_1 + a_2 + \cdots + a_n$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$.
We than say that the series has a *sum* s and write

$$\sum_{k=1}^{\infty} a_k = s.$$

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions (see e.g. [9]). For example the sum of the reciprocals

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- of the square numbers (the *Basel problem*)

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

  is $\pi^2/6$, and equals approximately 1.644934,
- of the cubes

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$$

  is called *Apéry's constant* $\zeta(3)$, and equals approximately 1.202057,
- of the factorials

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

  is the transcendental number e $\doteq 2.718282$.

In contrast to these three convergent series, for example, the following two series of the reciprocals diverge:

- the series of the reciprocals of positive integers (the *harmonic series*)

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$$

- and the series of the reciprocals of all prime numbers *P*

$$\sum_{p \in P}^{\infty} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots .$$

Next, we will use *harmonic numbers*, where the nth *harmonic number* is the sum of the reciprocals of the first n positive integers:

$$H(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Basic and as well interesting information about harmonic numbers can be found in [1], [8]. The values of the harmonic numbers $H(n)$ for $n = 1,2,\ldots,10$ are stated in the following Table 1.

Table 1 First ten harmonic numbers

$\boldsymbol{n}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$\boldsymbol{H(n)}$	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$

Source: own modelling in Maple 2020

Let us note that there exists the Euler's constant $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$.

In addition to the harmonic numbers, we will also use telescoping series. The *telescoping series* is any series where nearly every term cancels with a preceding or following term, so its partial sums eventually only have a fixed number of terms after cancellation. Interesting facts about telescoping series, we can found in [2].

TELESCOPING SERIES EXAMPLE

Let us consider the following example, in which we determine the sum of the telescoping series formed by reciprocals of the cubic polynomial with one zero and two different positive integer roots.

Example 1 Using the nth partial sum calculate the sum $s(0,5,8)$ of the series

$$\sum_{\substack{k=1 \\ k\neq 5,8}}^{\infty} \frac{1}{k(k-5)(k-8)} \tag{1}$$

representing the telescoping series formed by reciprocals of the cubic polynomials with the integer roots $k_1 = 0$, $k_2 = 5$, and $k_3 = 8$.

By means of the result of the Maple 2020 command

```
> convert(1/(k*(k-5)*(k-8)),parfrac);
```

$$\frac{1}{24(k-8)} + \frac{1}{40k} - \frac{1}{15(k-5)}$$

we get the partial fraction decomposition of the kth term, where $k \neq 5$ and $k \neq 8$,

$$a_k = \frac{1}{k(k-5)(k-8)}$$

in the form

$$a_k = \frac{1}{40k} - \frac{1}{15(k-5)} + \frac{1}{24(k-8)} = \frac{1}{120}\left(\frac{3}{k} - \frac{8}{k-5} + \frac{5}{k-8}\right).$$

The sum in parentheses we express as the reciprocal of the integers:

$$a_k = \frac{1}{120}\left(\frac{3}{k} - \frac{5+3}{k-5} + \frac{5}{k-8}\right) = \frac{3}{120}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{5}{120}\left(\frac{1}{k-5} - \frac{1}{k-8}\right),$$

i.e.

$$a_k = \frac{1}{40}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{1}{24}\left(\frac{1}{k-5} - \frac{1}{k-8}\right).$$

The nth partial sum $s_n(0,5,8)$ of the series (1) is

$$s_n(0,5,8) = \frac{1}{40}\sum_{\substack{k=1 \\ k\neq 5,8}}^{n}\left(\frac{1}{k} - \frac{1}{k-5}\right) - \frac{1}{24}\sum_{\substack{k=1 \\ k\neq 5,8}}^{n}\left(\frac{1}{k-5} - \frac{1}{k-8}\right) = \frac{1}{40}s_n(0,5) - \frac{1}{24}s_n(5,8),$$

where

$$s_n(0,5) = \left(\frac{1}{1} + \frac{1}{4}\right) + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{1}\right) + \left(\frac{1}{6} - \frac{1}{1}\right) + \left(\frac{1}{7} - \frac{1}{2}\right) +$$

$$+\left(\frac{1}{9}-\frac{1}{4}\right)+\left(\frac{1}{10}-\frac{1}{5}\right)+\left(\frac{1}{11}-\frac{1}{6}\right)+\left(\frac{1}{12}-\frac{1}{7}\right)+\left(\frac{1}{13}-\frac{1}{8}\right)+\left(\frac{1}{14}-\frac{1}{9}\right)+\left(\frac{1}{15}-\frac{1}{10}\right)+\cdots$$
$$\cdots+\left(\frac{1}{n-6}-\frac{1}{n-11}\right)+\left(\frac{1}{n-5}-\frac{1}{n-10}\right)+\left(\frac{1}{n-4}-\frac{1}{n-9}\right)+$$
$$+\left(\frac{1}{n-3}-\frac{1}{n-8}\right)+\left(\frac{1}{n-2}-\frac{1}{n-7}\right)+\left(\frac{1}{n-1}-\frac{1}{n-6}\right)+\left(\frac{1}{n}-\frac{1}{n-5}\right)=$$
$$=2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+$$
$$+\left(\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)$$

and

$$s_n(5,8)=\left(-\frac{1}{4}+\frac{1}{7}\right)+\left(-\frac{1}{3}+\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{5}\right)+\left(-\frac{1}{1}+\frac{1}{4}\right)+\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{1}\right)+$$
$$+\left(\frac{1}{4}-\frac{1}{1}\right)+\left(\frac{1}{5}-\frac{1}{2}\right)+\left(\frac{1}{6}-\frac{1}{3}\right)+\left(\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{8}-\frac{1}{5}\right)+\left(\frac{1}{9}-\frac{1}{6}\right)+\cdots$$
$$\cdots+\left(\frac{1}{n-12}-\frac{1}{n-9}\right)+\left(\frac{1}{n-11}-\frac{1}{n-8}\right)+\left(\frac{1}{n-10}-\frac{1}{n-7}\right)+$$
$$+\left(\frac{1}{n-9}-\frac{1}{n-6}\right)+\left(\frac{1}{n-8}-\frac{1}{n-5}\right)=$$
$$=-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-$$
$$-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{n-7}+\frac{1}{n-6}+\frac{1}{n-5}\right).$$

Therefore, we have

$$s_n(0,5,8)=\frac{1}{40}\left[2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\right.$$
$$\left.+\left(\frac{1}{n-4}+\frac{1}{n-3}+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\right)\right]-$$
$$-\frac{1}{24}\left[-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)-\right.$$
$$\left.-\left(\frac{1}{n-7}+\frac{1}{n-6}+\frac{1}{n-5}\right)\right].$$

Since for arbitrary real c it holds

$$\lim_{n\to\infty}\frac{1}{n+c}=0$$

and because

$$s(0,5,8)=\lim_{n\to\infty}s_n(0,5,8)\,,$$

we have

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$$s(0,5,8) = \frac{1}{40}\left[2\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{7}\right)-\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)\right]-$$
$$-\frac{1}{24}\left[-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+2\left(\frac{1}{1}+\frac{1}{2}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)\right]=$$
$$=\frac{1}{40}[2H(4)+H(7)-H(5)-H(2)-H(8)+H(3)]-$$
$$-\frac{1}{24}[-H(4)+H(7)-H(3)+2H(2)-H(3)]=$$
$$=\frac{1}{40}\left(\frac{25}{6}+\frac{363}{140}-\frac{137}{60}-\frac{3}{2}-\frac{761}{280}+\frac{11}{6}\right)-\frac{1}{24}\left(-\frac{25}{12}+\frac{363}{140}-\frac{11}{3}+3\right)=$$
$$=\frac{1}{40}\cdot\frac{251}{120}-\frac{1}{24}\cdot\left(-\frac{11}{70}\right)=\frac{251}{4800}+\frac{11}{1680}=\frac{659}{11200}\doteq 0.058839\,.$$

The sum of the series (1) we can also compute by means of Maple 2020 this way:

```
> evalf(sum(1/(k*(k-5)*(k-8)),k=1..4)+ sum(1/(k*(k-5)*
(k-8)),k=6..7)+ sum(1/(k*(k-5)*(k-8)),k=9..infinity),10);
```

0.058839

THREE LEMMAS

This paper is a free follow-up to author's papers [3], [4], [5], [6], [7] dealing with the sum of the telescoping series formed by reciprocals of the cubic polynomials with some positive integer roots. Before we derive the main result of this paper, we present three following lemmas:

Lemma 1 Let $a < b$ be positive integers. Then a fraction

$$\frac{1}{k(k-a)(k-b)}$$

can be rewritten in the form

$$\frac{1}{ab(b-a)}\left(\frac{b-a}{k}-\frac{(b-a)+a}{k-a}+\frac{a}{k-b}\right).$$

This expression can also be rewritten as a difference

$$\frac{1}{ab}\left(\frac{1}{k}-\frac{1}{k-a}\right)-\frac{1}{b(b-a)}\left(\frac{1}{k-a}-\frac{1}{k-b}\right). \qquad (2)$$

Proof. Can be simply made in Maple using the simplify command applied to expression (2).

Lemma 2 Let $a < b$ be positive integers. Then it holds

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\left(\frac{1}{k}-\frac{1}{k-a}\right)=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}\,, \tag{3}$$

where $H(n)$ is the nth harmonic number.
Proof. The sum $s(0,a)$ of the infinite series in (3) is the limit of the sequence $\{s_n(0,a)\}_{n=1}^{\infty}$of the partial sums

$$s_n(0,a)=\sum_{k=1}^{a-1}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\sum_{k=a+1}^{b-1}\left(\frac{1}{k}-\frac{1}{k-a}\right)+\sum_{k=b+1}^{n}\left(\frac{1}{k}-\frac{1}{k-a}\right).$$

In fact

$$s_n(0,a)=\left(\frac{1}{1}-\frac{1}{1-a}\right)+\left(\frac{1}{2}-\frac{1}{2-a}\right)+\cdots+\left(\frac{1}{a-2}-\frac{1}{-2}\right)+\left(\frac{1}{a-1}-\frac{1}{-1}\right)+$$
$$+\left(\frac{1}{a+1}-\frac{1}{1}\right)+\left(\frac{1}{a+2}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{b-2}-\frac{1}{b-2-a}\right)+\left(\frac{1}{b-1}-\frac{1}{b-1-a}\right)+$$
$$+\left(\frac{1}{b+1}-\frac{1}{b+1-a}\right)+\left(\frac{1}{b+2}-\frac{1}{b+2-a}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n-1-a}\right)+\left(\frac{1}{n}-\frac{1}{n-a}\right)=$$
$$=2H(a-1)+H(b-1)-H(a)-H(b-1-a)+H(n)-H(b)-H(n-a)+H(b-a)=$$
$$=[2H(a-1)-H(a)]+[H(b-1)-H(b-1-a)]-[H(b)-H(b-a)]+[H(n)-H(n-a)]\,.$$

Since $a<b$, then

$$H(b)-H(b-a)=\frac{1}{b-a+1}+\frac{1}{b-a+2}+\cdots+\frac{1}{b-1}+\frac{1}{b}$$

and we have

$$s_n(0,a)=\left[H(a-1)-\frac{1}{a}\right]+\left[\frac{1}{b-1}+\frac{1}{b-2}\cdots+\frac{1}{b-a+1}+\frac{1}{b-a}\right]-$$
$$-\left[\frac{1}{b}+\frac{1}{b-1}+\cdots+\frac{1}{b-a+2}+\frac{1}{b-a+1}\right]+\left[\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right]=$$
$$=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}+\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}.$$

Hence, we have

$$s(0,a)=\lim_{n\to\infty}\left[H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}+\frac{1}{n-a+1}+\frac{1}{n-a+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right]=$$
$$=H(a-1)-\frac{1}{a}-\frac{1}{b}+\frac{1}{b-a}.$$

Lemma 3 Let $a<b$ be positive integers. Then it holds

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\left(\frac{1}{k-a}-\frac{1}{k-b}\right)=-H(a-1)+H(b-1)-\frac{2}{b-a}\,, \tag{4}$$

where $H(n)$ is the nth harmonic number.

Proof. The sum $s(a,b)$ of the infinite series in (4) is the limit of the sequence $\{s_n(a,b)\}_{n=1}^{\infty}$ of the partial sums

$$s_n(a,b) = \sum_{k=1}^{a-1}\left(\frac{1}{k-a}-\frac{1}{k-b}\right) + \sum_{k=a+1}^{b-1}\left(\frac{1}{k-a}-\frac{1}{k-b}\right) + \sum_{k=b+1}^{n}\left(\frac{1}{k-a}-\frac{1}{k-b}\right).$$

In fact

$$s_n(a,b) = \left(\frac{1}{1-a}-\frac{1}{1-b}\right)+\left(\frac{1}{2-a}-\frac{1}{2-b}\right)+\cdots+\left(\frac{1}{-2}-\frac{1}{a-2-b}\right)+\left(\frac{1}{-1}-\frac{1}{a-1-b}\right)+$$
$$+\left(\frac{1}{1}-\frac{1}{a+1-b}\right)+\left(\frac{1}{2}-\frac{1}{a+2-b}\right)+\cdots+\left(\frac{1}{b-2-a}-\frac{1}{-2}\right)+\left(\frac{1}{b-1-a}-\frac{1}{-1}\right)+$$
$$+\left(\frac{1}{b+1-a}-\frac{1}{1}\right)+\left(\frac{1}{b+2-a}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{n-1-a}-\frac{1}{n-1-b}\right)+\left(\frac{1}{n-a}-\frac{1}{n-b}\right)=$$
$$=-\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{a-1}\right)+\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{b-a-1}+\frac{1}{b-a+1}+\cdots+\frac{1}{b-1}\right)+$$
$$+\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{b-1-a}+\frac{1}{b+1-a}+\cdots+\frac{1}{n-a}\right)-\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n-b}\right)=$$
$$=-H(a-1)+H(b-1)-\frac{1}{b-a}+H(n-a)-\frac{1}{b-a}-H(n-b).$$

Since $a<b$, then

$$H(n-a)-H(n-b)=\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}$$

and we have

$$s_n(a,b) = -H(a-1)+H(b-1)-\frac{2}{b-a}+\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}.$$

Hence, we have

$$s(a,b)=\lim_{n\to\infty}\left[-H(a-1)+H(b-1)-\frac{2}{b-a}+\frac{1}{n-a}+\frac{1}{n-a-1}+\cdots+\frac{1}{n-b+1}\right]=$$
$$=-H(a-1)+H(b-1)-\frac{2}{b-a}.$$

THE MAIN RESULT

Now, let us consider the series formed by reciprocals of the cubic polynomial with one zero and two different positive integer roots $a<b$, i.e. the series

$$\sum_{\substack{k=1\\k\neq a,b}}^{\infty}\frac{1}{k(k-a)(k-b)}, \tag{5}$$

and let us determine its sum $s(0,a,b)$.

According to Lemma 1 we can the kth term a_k of the series (5) write in the form

$$a_k = \frac{1}{ab}\left(\frac{1}{k} - \frac{1}{k-a}\right) - \frac{1}{b(b-a)}\left(\frac{1}{k-a} - \frac{1}{k-b}\right),$$

so the sum $s(0,a,b)$ of the series (5) is

$$s(0,a,b) = \frac{1}{ab}\sum_{\substack{k=1\\ k\neq a,b}}^{n}\left(\frac{1}{k} - \frac{1}{k-a}\right) - \frac{1}{b(b-a)}\sum_{\substack{k=1\\ k\neq a,b}}^{n}\left(\frac{1}{k-a} - \frac{1}{k-b}\right).$$

According to the Lemmas 2 and 3 we have

$$s(0,a,b) = \frac{1}{ab}\left[H(a-1) - \frac{1}{a} - \frac{1}{b} + \frac{1}{b-a}\right] - \frac{1}{b(b-a)}\left[H(b-1) - H(a-1) - \frac{2}{b-a}\right] =$$

$$= \frac{1}{ab}\left[H(a-1) + \frac{a^2+ab-b^2}{ab(b-a)}\right] - \frac{1}{b(b-a)}\left[H(b-1) - H(a-1) - \frac{2}{b-a}\right] =$$

$$= \frac{H(a-1)}{ab} + \frac{a^2+ab-b^2}{a^2b^2(b-a)} - \frac{H(b-1) - H(a-1)}{b(b-a)} + \frac{2}{b(b-a)^2} =$$

$$= \frac{H(a-1)}{ab} - \frac{H(b-1) - H(a-1)}{b(b-a)} - \frac{a^3 - 2a^2b - 2ab^2 + b^3}{a^2b^2(b-a)^2}$$

$$= \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3 - 2a^2b - 2ab^2 + b^3}{a^2b^2(b-a)^2}.$$

We have derived the following statement:

Theorem 1 The series

$$\sum_{\substack{k=1\\ k\neq a,b}}^{\infty}\frac{1}{k(k-a)(k-b)},$$

where $a < b$ be positive integers, has the sum

$$s(0,a,b) = \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3 - 2a^2b - 2ab^2 + b^3}{a^2b^2(b-a)^2}, \tag{6}$$

where $H(n)$ is the nth harmonic number.

NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(0,a,b)$ for $a = 1,2,\dots,10$ and for $b = a+1, a+2, \dots, a+10$. We use on the one hand an approximate direct evaluation of the sum

$$s(0,a,b,t) = \sum_{\substack{k=1\\ k\neq a,b}}^{t}\frac{1}{k(k-a)(k-b)},$$

where $t = 10^7$, using the basic programming language of the computer algebra system Maple 2020, and on the other hand the formula (6) for evaluation the sum $s(0, a, b)$. We compare one hundred pairs of these ways obtained sums $s(0, a, b, t)$ and $s(0, a, b)$ to verify the formula (6). We use the following simple procedure `ts0abpos` and the following double repetition statement:

```
> ts0abpos:=proc(a,b,t)
      local k,s0ab,s0abt,s0abt1,s0abt2,s0abt3;
      s0abt1:=0; s0abt2:=0; s0abt3:=0;
      s0ab:=harmonic(a-1)/(a*(b-a))-harmonic(b-1)/(b*(b-a))
           -(a*a*a-2*a*a*b-2*a*b*b+b*b*b)/(a*a*b*b*(b-a)*(b-a));
      print("s(0",a,b,")=",evalf[12](s0ab));
      for k from 1 to a-1 do
          s0abt1:=s0abt1+1/(k*(k-a)*(k-b)); end do;
      for k from a+1 to b-1 do
          s0abt2:=s0abt2+1/(k*(k-a)*(k-b)); end do;
      for k from b+1 to t do
          s0abt3:=s0abt3+1/(k*(k-a)*(k-b)); end do;
      s0abt:=s0abt1+s0abt2+s0abt3;
      print("s(0,a,b,t)=",evalf[12](s0abt));
      print("diff=",evalf[12](abs(s0abt-s0ab)));
  end proc:
> for i from 1 to 10 do
      for j from i+1 to i+10 do ts0abpos(i,j,10000000); end do;
  end do;
```

The approximate values of the sums $s(0, a, b)$ rounded to 10 decimals obtained by this procedure are written into the following Table 2.

Table 2 Some approximate values of the sums $s(0, a, b)$ obtained by means of the formula (6)

$a \setminus b$	$b = a+1$	$b = a+2$	$b = a+3$	$b = a+4$	$b = a+5$
$a = 1$	0.2500000000	−0.3611111111	−0.3263888889	−0.2691666667	−0.2238888889
$a = 2$	0.6944444444	0.1145833333	0.0355555556	0.0159722222	0.0097959184
$a = 3$	0.5763888889	0.1394444444	0.0675925926	0.0445861678	0.0344146825
$a = 4$	0.4691666667	0.1256944444	0.0657312925	0.0452752976	0.0356834215
$a = 5$	0.3905555556	0.1092517007	0.0588392857	0.0412014991	0.0327539683
$a = 6$	0.3327097506	0.0950644841	0.0519106408	0.0366369048	0.0292575102
$a = 7$	0.2889668367	0.0834687579	0.0458824641	0.0325070042	0.0260235862
$a = 8$	0.2549548060	0.0740426587	0.0408154871	0.0289633688	0.0232150809
$a = 9$	0.2278527337	0.0663215853	0.0365804206	0.0259639103	0.0208192028
$a = 10$	0.2058005378	0.0599257456	0.0330266101	0.0234258995	0.0187807563
$a \setminus b$	$b = a+6$	$b = a+7$	$b = a+8$	$b = a+9$	$b = a+10$
$a = 1$	−0.1898526077	−0.1639668367	−0.1438436949	−0.1278527337	−0.1148914469
$a = 2$	0.0076140873	0.0068090199	0.0065128968	0.0064056875	0.0063621332
$a = 3$	0.0288874192	0.0254138322	0.0229892267	0.0211636871	0.0197123176
$a = 4$	0.0302116402	0.0266470362	0.0240984999	0.0221518629	0.0205925820
$a = 5$	0.0278576894	0.0246329795	0.0223113372	0.0223113372	0.0191006956
$a = 6$	0.0249554072	0.0221121369	0.0200611640	0.0184863696	0.0172211382
$a = 7$	0.0222384199	0.0197362445	0.0179319919	0.0165473594	0.0154353700
$a = 8$	0.0198616446	0.0176481026	0.0160545726	0.0148334195	0.0138537867

$a = 9$	0.0178236167	0.0158509962	0.0144342593	0.0133507928	0.0124829743
$a = 10$	0.0160828494	0.0143113853	0.0130426798	0.0120747685	0.0113009843

Source: own modelling in Maple 2020

Computation of 100 pairs of the sums $s(0,a,b)$ and $s(0,a,b,10^7)$ took over 34 minutes. The absolute errors, i.e. the differences $|s(0,a,b) - s(0,a,b,10^7)|$, are all only about $5 \cdot 10^{-15}$.

CONCLUSIONS

We dealt with the sum of the telescoping series formed by reciprocals of the cubic polynomials with one zero and two different positive integer roots $0 < a < b$. We derived that the sum $s(0,a,b)$ of this series is given by the formula

$$s(0,a,b) = \frac{H(a-1)}{a(b-a)} - \frac{H(b-1)}{b(b-a)} - \frac{a^3 - 2a^2b - 2ab^2 + b^3}{a^2b^2(b-a)^2},$$

where $H(n)$ is the nth harmonic number. This series so belong to special types of infinite series, such as geometric series, which sums are given analytically by means of a simple formula. This series so belong to special types of infinite series, such as geometric series, which sums are given analytically by means of a simple formula.

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Original Paper

Quantitative comparison of the amount of produced waste in selected Slovak municipalities: a case study

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ABSTRACT

The environment gives a clear challenge to the inhabitants of the planet that it is necessary to reduce the amount of generated waste. Disposal, separation and processing of waste, or landfilling of produced waste is connected to processes that require additional financial resources. The main goal of the contribution was to analyze the differences in the amount of total produced waste, municipal solid waste, and separated waste in selected Slovak municipalities. The data were obtained by the questionnaire method from 33 municipalities in the Nitra region, which provided data on the amount and type of waste in the period 2015 - 2017. Selected statistical methods, paired t-test, analysis of variance and Bonferroni test were used to analyze the obtained data. The results of the analysis show that there are significant differences in the volume of waste generated by municipalities together between individual years. Similarly, statistically significant differences were demonstrated between the amount of produced waste in relation to the size of the municipality according to the number of inhabitants. In the evaluated period, medium-sized municipalities showed the lowest amount of produced waste per inhabitant.

KEYWORDS: waste management, descriptive statistics, paired T-Test, analysis of variance, Bonferroni test

JEL CLASSIFICATION: C12, Q 53

INTRODUCTION

Preserving the environment without waste pollution requires significant adjustments in waste management, as well as changes in the consumer way of life. The processing and disposal of produced waste is one of the current challenges of every economy, which aims to minimize the amount of waste and obtain raw materials for recycling and reuse. Different ways of dealing with communal waste and its processing bring new possibilities for reducing the

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impact of waste on the water, soil, Earth's atmosphere, and economic benefits of recycling for society [4].

Municipal waste management in the Slovak Republic is governed by the Act No. 79/2015 Coll., Waste Act [1]. The processing and disposal of produced waste also requires financial costs, therefore residents of municipalities and companies pay local taxes and a local fee for municipal waste and small construction waste, which are established according to Act No. 582/2004 Coll. [3] and Act No. 335/2022 Coll. [2]. The goal of waste management is that waste does not have a negative impact on the health of residents or the environment. This goal is associated with defined priorities: prevent waste, reuse waste, recycle, recover waste and dispose of waste [10].

Municipal solid waste is mixed waste and separately collected waste from households, which is sorted according to the determined components of municipal waste. Small construction waste is waste from normal maintenance work, for which the local fee for municipal waste and small construction waste is paid [6]. Landfills are considered the last resort in the hierarchy of waste management, which significantly affect the climate change of the entire planet because they release methane. The disposal of biological waste from households is often the result of food waste, which contributes to the formation of harmful gases in the waste. Improvements in municipal waste management between 1995 and 2008 led to significantly lower greenhouse gas emissions [11]. In the study [7], it is proven that appeals to "reduce waste" had the highest effectiveness in changing the attitude of consumers towards food waste.

Effective disposal of municipal solid waste contributes to environmental protection. The material and energy recovery of municipal solid waste is the subject of many researches that address the complex cycle of recycling, recovery and reuse of this type of waste within the circular economy [5], [9].

The production of mixed municipal waste in Slovakia in 2019 reached almost 2.37 million tons. According to the new law on waste, valid from January 2021, Slovakia must achieve 90% sorting of municipal waste and 65% recycling by 2035 [8]. The analyst of WOOD & Company Eva Sádovská informed that in 2021 inhabitants of Slovakia created 2.7 million tons of municipal solid waste, which is an average of 497 kg of garbage per inhabitant. Approximately 41% of waste from households ends up in landfills, which makes Slovakia one of the landfill superpowers in Europe [12].

Charges for municipal waste have a growing tendency, as more and more types of waste are sorted, which require ever higher financial costs for processing and disposal. Current high energy prices are a prerequisite for increasing fees for municipal waste.

MATERIAL AND METHODS

The main research source was data from a questionnaire that was conducted in selected municipalities of the Nitra region in the period 2018-2019 (total n = 33). The municipalities provided data on the amount and types of waste in the period 2015-2017. The questionnaire contained following questions, which were used as sorting criteria: Name of the municipality, Number of inhabitants, and Area of the territory of the municipality.

In the paper, there are analyzed data obtained to these selected questions:

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1. Enter the amount of total produced waste (in tons) according to years 2017, 2016, 2015:
a) Of which, the amount of municipal solid waste (in tons),
b) Of which, the amount according to individual types of waste: Paper, Glass, Plastics, Iron waste, Electrical waste, Construction waste, Bulk waste, Hazardous waste, Other.
2. Enter the amount of total separated waste (in tons) according to years 2017, 2016, 2015:
a) Of this amount of municipal solid waste (in t),
b) Of which, the amount by individual type: Paper, Glass, Plastics, Iron waste, Electrical waste, Construction waste, Bulk waste, Hazardous waste, Other.

The research objectives were as follows:
- analyzing and determining the significance of differences in the amount of individual types of waste in municipalities within 2015-2017,
- comparison of the amount of produced waste in relation to the size of the municipality.

In analyses, the villages were divided into 3 groups according to the number of inhabitants:
- Small villages (n = 9): up to 1,000 inhabitants,
- Middle villages (n = 11): from 1,001 to 2,000 inhabitants,
- Big villages (n = 13): over 2,001 inhabitants.

The analysis of municipal waste data was done using methods of descriptive statistics, paired t-test, one-way analysis of variance and Bonferroni test of multiple comparisons.

RESULTS

The results of the analysis are presented according to the type of waste together for all municipalities, by individual years 2015, 2016, 2017 and by the size of the municipality. Following abbreviations were used for the types of waste: TPW – Total produced waste, MSW – Municipal solid waste, TSW – Total separated waste. For the respective year, the waste abbreviation is supplemented with numbers, e.g.: TPW15 – Total produced waste in year 2015.

In the first part, there are presented statistical characteristics for individual types of waste for all municipalities together: total produced waste, solid municipal waste and total separated waste. The data shows that between 2015 and 2016 there was an increase in produced and separated waste. Subsequently, between 2016 and 2017, there was a decrease in produced and separated waste. There was a decrease in the volume of municipal solid waste between 2015 and 2016, followed by an increase in 2017 (Table 1).

Table 1 Basic statistical characteristics of TPW, MSW and TSW

Year	$\bar{x}$	SD	SE $_{\bar{x}}$	C.V.	Asymmetry, A	Excess, E
Total produced waste, n = 33						
2015	508.85	311.65	54.252	61.246	0.992	0.822
2016	602.36	415.27	72.289	68.940	2.278	6.873
2017	582.54	305.03	53.099	52.362	0.884	0.789
Municipal solid waste, n = 33						
2015	319.89	244.35	42.536	76.387	1.024	-0.084
2016	298.95	260.12	45.281	87.012	2.622	8.358
2017	307.81	230.05	40.046	74.737	1.465	1.574
Total separated waste, n = 33						

2015	388.20	255.46	44.470	65.807	1.175	1.658
2016	500.18	400.31	69.685	80.034	2.873	10.635
2017	476.52	273.83	47.667	57.464	1.306	2.805

$A_{0.05}(30) = 0.916$, $A_{0.01}(30) = 1.236$, $E_{0.05}(30) = 1.832$, $E_{0.01}(30) = 2.472$

In Table 1, the values for asymmetry are positive for all types of waste, i.e. the data distribution is skewed to the left, statistically significant to highly significant (in the sample there are more small values and few larger ones in the set). In the case of excess, the values are also positive (except for one case: MSW 2015), so it is a leptokurtic distribution (most of the values in the sample are close to the average).

Using a paired t-test, the significance of the differences was proved between individual type of waste and between year. Based on the results of the paired t-test, statistically significant or highly significant differences were found in four analyzed cases: between TPW15 - TPW16, TPW15 - TPW17, TSW15 - TSW16, and TSW15 - TSW17 (Table 2).

Table 2 Paired t-tests of waste type between years

Waste type and year	*t*	Waste type and year	*t*	Waste type and year	*t*
TPW15 - TPW16	$-2{,}32^{*}$	MSW15 - MSW16	0.81	TSW15 - TSW16	-2.98^{*}
TPW15 - TPW17	4.76^{**}	MSW15 - MSW17	-0.78	TSW15 - TSW17	6.47^{**}
TPW16 - TPW17	0.62	MSW16 - MSW17	0.52	TSW16 - TSW17	-0.78

$t_{0.05}(32) = 2.037$, $t_{0.01}(32) = 2.738$

In the following section there are results of descriptive statistics for individual types of waste according to the defined size of the village: small, middle and big villages.

In small villages, an increase in total produced waste occurred every year. There was a decrease in the volume of total municipal waste in 2016 and then an increase in 2017. Total separated waste showed an increasing trend in individual years. Asymmetry for produced and separated waste is negative, so it is a right-skewed distribution (a higher number of larger values than smaller ones in the sample). Asymmetry for municipal waste is positive, the distribution is left-skewed (a higher number of smaller values and few larger ones). Excess is negative for all three types of waste, so the data distribution is flatter (platykurtic). In the sample there are many low and high values, and they are not close to the average (Table 3).

Table 3 Basic statistical characteristics of TPW, MSW and TSW for small villages

Year	$\bar{x}$	**SD**	**SE** $_{\bar{x}}$	**C.V.**	**Asymmetry, A**	**Excess, E**
Total produced waste, n = 9						
2015	229.05	113.21	37.737	49.427	-0.270	-1.539
2016	264.67	104.90	34.968	39.636	-0.867	-0.617
2017	283.43	119.16	39.718	42.041	-0.706	-0.894
Municipal solid waste, n = 9						
2015	113.35	62.730	20.910	55.340	0.297	-0.542
2016	106.91	53.777	17.926	50.303	0.103	-0.997
2017	113.66	62.380	20.793	54.881	0.042	-1.266
Total separated waste, n = 9						
2015	216.38	129.05	43.016	59.639	-0.3109	-1.576
2016	253.55	107.62	35.873	42.445	-0.5258	-1.112
2017	270.99	122.28	40.761	45.125	-0.3804	-1.296

$A_{0.05}(10) = 1.786$, $A_{0.01}(10) = 2.599$, $E_{0.05}(10) = 3.572$, $E_{0.01}(10) = 5.198$

Data on the amount of total produced waste per inhabitant enable comparison of developments within individual years. A graphic representation of the amount of TPW per inhabitant in small villages by year is in the Figure 1. It is a growing trend in the amount of total produced waste in the examined period in small villages.

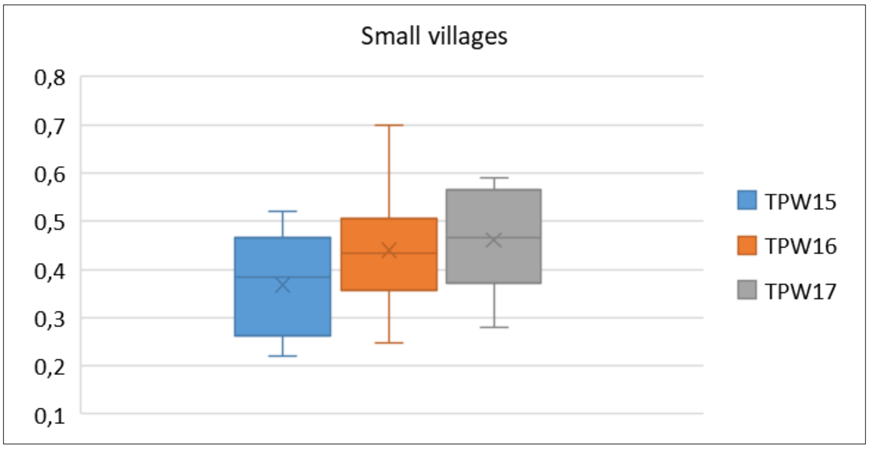

Figure 1 Total produced waste (TPW) per capita in small villages (2015 – 2016)

A growing trend in all types of waste was evaluated in medium-sized municipalities (Table 4). Figure 2 shows that the development of the amount of total produced waste (TPW) per inhabitant was slightly stabilized in 2017 after an increase in 2016.

Table 4 Basic statistical characteristics of TPW, MSW and TSW for middle villages

Year	$\bar{x}$	SD	SE $_{\bar{x}}$	C.V.	Asymmetry, A	Excess, E
Total produced waste, n = 11						
2015	361.94	82.032	24.734	22.665	-0.424	-1.188
2016	459.50	83.136	25.066	18.093	-0.082	-0.958
2017	467.82	74.356	22.419	15.894	0.177	-1.171
Municipal solid waste, n = 11						
2015	202.97	69.100	20.834	34.045	0.073	-0.906
2016	216.11	42.498	12.814	19.665	0.724	-0.591
2017	224.36	48.683	14.679	21.699	0.929	-0.369
Total separated waste, n = 11						
2015	346.58	114.10	34.403	32.923	-1.161	0.508
2016	432.38	118.86	35.838	27.490	-1.509	2.229
2017	450.72	111.61	33.651	24.762	-1.168	1.181

$A_{0.05}(10) = 1.786$, $A_{0.01}(10) = 2.599$, $E_{0.05}(10) = 3.572$, $E_{0.01}(10) = 5.198$

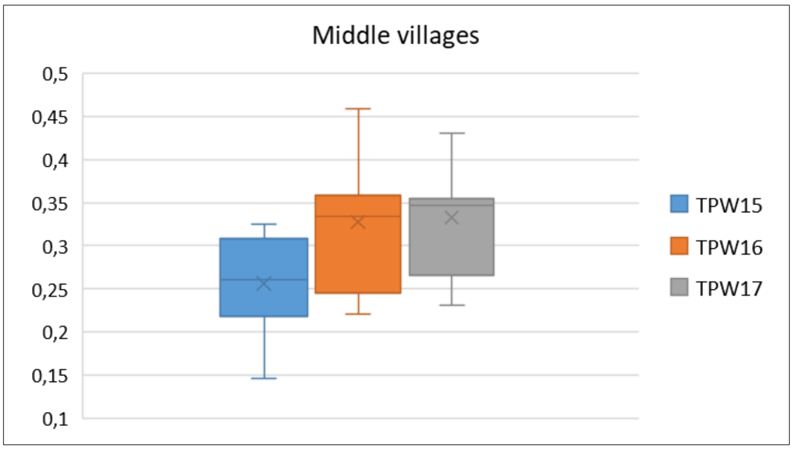

Figure 2 Total produced waste (TPW) per capita in middle villages (2015 – 2016)

In big municipalities the development of the amount of waste for each type has a different character. Asymmetry for all types of waste is positive, i.e. the distribution is left-skewed (the sample contains a larger number of smaller values and few larger ones) (Table 5).

Table 5 Basic statistical characteristics of TPW, MSW and TSW for big villages

Year	$\overline{x}$	**SD**	**SE** $_{\overline{x}}$	**C.V.**	**Asymmetry, A**	**Excess, E**
Total produced waste, n = 13						
2015	826.88	238.30	66.091	28.819	1.482	1.309
2016	957.03	454.16	125.96	47.455	2.252	4.178
2017	886.70	233.79	64.843	26.367	1.449	0.919
Municipal solid waste, n = 13						
2015	561.81	211.41	58.633	37.629	0.275	-1.637
2016	501.99	313.67	86.997	62.486	2.004	3.544
2017	512.83	237.42	65.849	46.296	0.837	-0.819
Total separated waste, n = 13						
2015	542.37	320.31	88.838	59.058	0.404	-0.505
2016	728.29	549.21	152.32	75.412	1.802	3.416
2017	640.64	343.70	95.325	53.650	0.582	0.455

$A_{0.05}(15) = 1.366$, $A_{0.01}(15) = 1.905$, $E_{0.05}(15) = 2.733$, $E_{0.01}(15) = 3.810$

The development of the amount of total produced waste (TPW) per inhabitant for big villages is shown in Figure 3. In 2016, several outliers can be seen. The amount of TPW per inhabitant increased in 2016 and subsequently decreased in 2017.

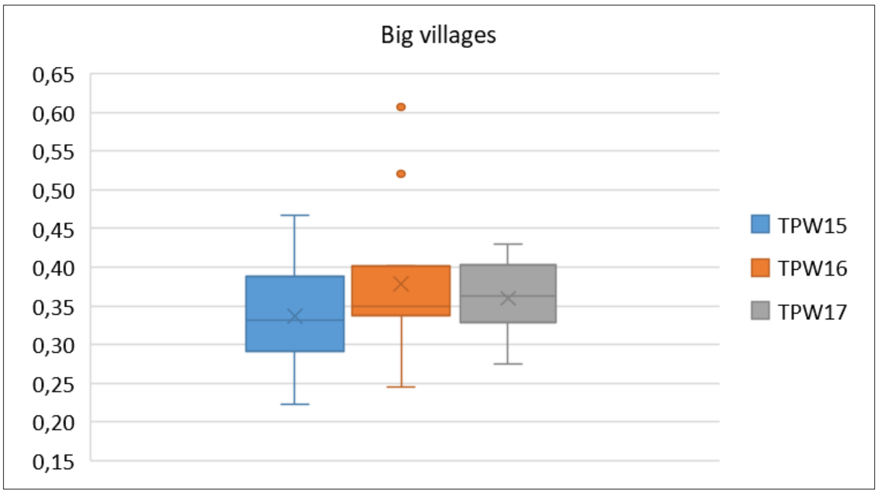

Figure 3 Total produced waste (TPW) per capita in big villages (2015 – 2016)

Following results were obtained by analyzing data on produced waste between municipalities in relation to the number of their inhabitants. Using the method of one-way analysis of variance for evaluated traits and via Bonferroni tests, we examined the significance of differences in the amount of waste within each year and the type of waste in relation to the size of the municipality. The results show that there are statistically significant differences between municipalities in the production of individual types of waste within a given year. The differences are significant at the chosen level of significance α = 0.01. Statistically insignificant differences at the 0.01 level were also insignificant at the 0.05 level (Table 6).

Table 6 One-way analyses of variance for evaluated traits via Bonferroni multiple comparison test between sizes of villages

	MS/F	**Group**	**Error**	**Means and Bonferroni test**		
		$f_G = 2$	$f_e = 30$	1 - small	2 - middle	3 - big
Total produced waste						
TPW15	**MS**	1128436	28375	229.05	361.94	826.88
	F	39.77**		3 : (1, 2)**		
TPW16	**MS**	1443028	87743	264.67	459.50	957.03
	F	16.45**		3 : (1, 2)**		
TPW17	**MS**	1076326	27493	283.43	467.82	886.70
	F	39.15**		3 : (1, 2)**		
Municipal solid waste						
MSW15	**MS**	647562	20518	113.35	202.97	561.81
	F	31.56**		3 : (1, 2)**		
MSW16	**MS**	471660	40729	106.91	216.12	501.99
	F	11.58**		3 : (1, 2)**		
MSW17	**MS**	481132	24375	113.66	224.36	512.83
	F	19.74**		3 : (1, 2)**		
Total separated waste						
TSW15	**MS**	296863	49820	216.38	346.58	542.37
	F	5.96**		3 : 1**		
TSW16	**MS**	637216	128452	253.55	432.38	728.29
	F	4.96**		3 : 1**		
TSW17	**MS**	368827	55392	270.99	450.72	640.64
	F	6.66**		3 : 1**		

$F_{0.05}(2, 30) = 3.316$, $F_{0.01}(2, 30) = 5.390$

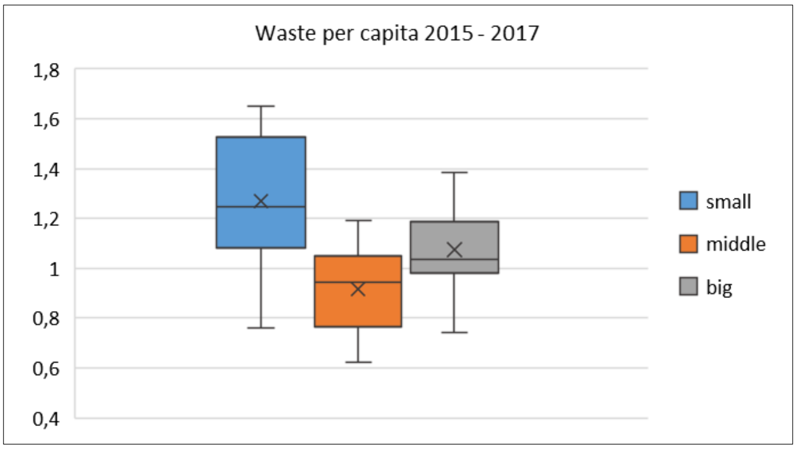

Figure 4 TPW per inhabitant in 3 years according to the size of the municipality (in tons)

To compare the amount of total produced waste between municipalities, we used data on the amount of TPW per capita in each year and determine the sum for three years 2015-2017 (in tons). It is evident from the graph (Figure 4) that individual groups of municipalities achieved different results in waste production per inhabitant in 3 years. The best values were shown by medium-sized municipalities, which have the analyzed parameter in the range 0.62 - 1.19 (t). In the second place there are big municipalities, which have the amount of produced waste per inhabitant in the sum of 3 years in the range of 0.74 - 1.38 (t). The worst results achieved small municipalities with analyzed parameter in the range of 0.76 - 1.65 (t).

To compare all municipalities with each other it was again applied mentioned indicator: the amount of total produced waste per inhabitant together for 3 years. The names of municipalities were anonymized and denoted as follows: small villages from S1 to S9, middle villages from M1 to M11, and big villages from B1 to B13. Then municipalities were ranked according to the amount of TPW per inhabitant for three observed years together (Figure 5). Overall, in the research sample the TPW range is from 0.62 t (medium village M10) to 1.65 t (small village S3). In the first part of the graph, there are medium-sized municipalities (they produce less waste per inhabitant). Small municipalities are in the second part of the list, even to the bottom of it (they produce more waste per inhabitant).

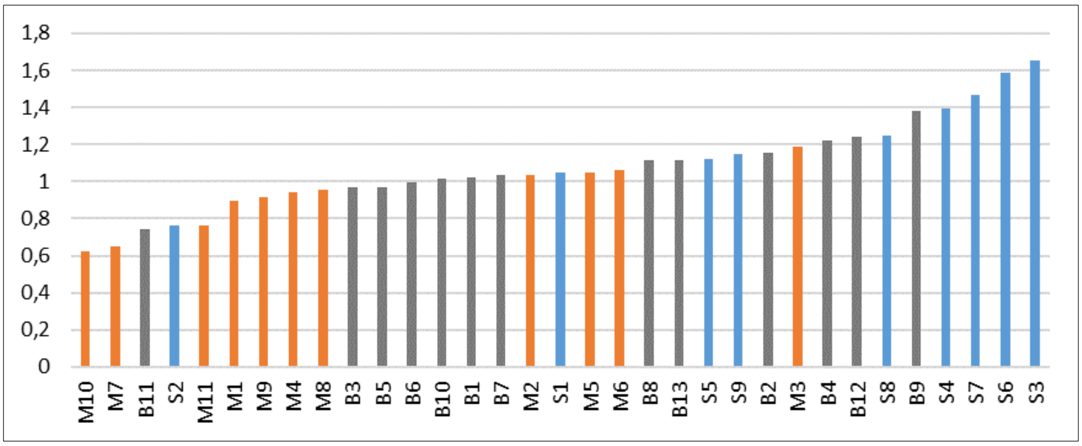

Figure 5 Villages ranked by TPW per capita (total over 3 years)

The percentage comparison within all municipalities shows:
- Middle-sized municipalities (63% of them) are in the first third of the list with lower waste production,
- Big municipalities (54% of them) are in the first half of the list with lower waste production,
- Small municipalities (78% of them) are in the second half of the list (higher waste production).

CONCLUSION

The generated waste in municipalities is associated with many processes: sorting, processing, recycling, evaluation, disposal, or storage. Residents of municipalities pay fees for waste disposal, which are determined based on the type of waste and the method of its processing or disposal.

In this contribution municipalities' data on produced waste were analyzed according to the type of waste in relation to the particular years 2015-2017. The results of the paired t-test confirmed statistically significant differences in the amount of waste according to the type of waste and years in the four analyzed cases.

Data on produced waste in the period 2015-2017 were analyzed in relation to the size of the village, which was defined in accordance with the number of inhabitants: small, middle and big village. Using the Bonferroni multiple comparison test, statistically significant differences were confirmed in waste production between individual municipalities in relation to their size. The best results in waste production were shown by medium-sized municipalities, which had the lowest waste production per inhabitant. The worst results were achieved by small municipalities, which have the highest waste production per inhabitant. Medium-sized municipalities could communicate their experience about wastes with other municipalities.

To preserve the quality of life of people, flora and fauna, it is necessary to change people's approach to the environment so that the generated waste does not damage it. The logical solution is not to create waste, or to create it in a minimal amount. It is important to separate consistently the generated waste into components and then evaluate them. Separating as much waste as possible is also a motivation for residents to reduce local waste fees.

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