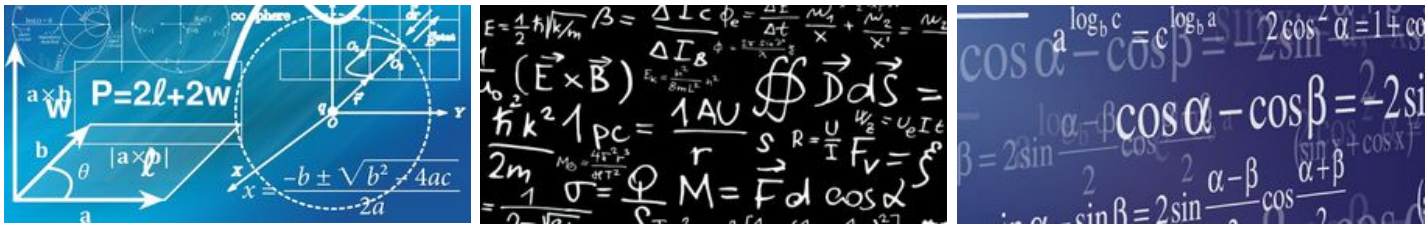

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Original Paper

Modelling of the efficient system of the daylighting in a stable for dairy cows

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ABSTRACT

In the article, we focused on the modelling of the uniformity of daylighting in a stable for dairy cows, depending on the opening structures in the roof of the building. We verified size effect of lighting openings on the daylighting in the stable. To the assessment of daylight was designed stable in which we have changed the size of the openings in the ceiling - roof construction. Ground plan dimensions and the size of the side openings remained the same. By using the program WDLS 5.0 we calculated values of daylight factor in the five possible solutions. In the first solution daylight was coming only from the sides. In the second to fifth solutions were also the upper lighting, which was ensured through the upper windows and skylight. In these solutions were changing the sizes of the upper openings. Researched points were evenly distributed over the entire stable area. From the results, we have found that for large widths are not only the side opening constructions but also the upper opening constructions. The best lighting conditions are achieved at an equally spaced upper windows and skylights.

KEYWORDS: daylight, daylight factor in stable, opening structures, WDLS program

JEL CLASSIFICATION: C 63

INTRODUCTION

The quality environment in the stables is important for health as well as the usefulness of dairy cows. The buildings for livestock breeding therefore must be made so, as to ensure a healthy indoor environment and do not jeopardize the stabling and breeding [6]. It is therefore necessary to monitor in the stable microclimatic characteristics of the environment as well as air-containing gases, dust and microorganisms, which are by-product of the decomposition animal excrements often due to imperfect metabolism of nutrients [4]. One

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important factor of the quality of the environment is also daylighting. In the building is lighting beside the heating and ventilation third component of technical security of internal microclimate. It creates not only favourable conditions for working people but also suitable photobiological conditions for the animals [2]. According to Chastain [3], proper lighting is an environmental factor that is often overlooked or given little attention during the planning, construction and maintenance of livestock facilities. However, it is just as important to the efficient operation of a livestock operation as ventilation, heating, or cooling. Daylighting can be obtained from the side openings and the roof structure by means of roof windows and skylights. The disadvantage is that, ridge skylights lighting improved, but in the summer poses a risk of increased thermal load of animals [5].

MATERIAL AND METHODS

To assessing of daylighting in the stable for different types of upper lighting we simulated a stable. Interior dimensions of the stable were 66.16 m x 28.05 m. Dimensions of gates and side openings were in all cases the same. They were changed only openings in the roof ceiling structure. In the front walls of the stable were six gates, on both sides the same. Two sets of gates had dimensions 2.7 x 3.0 m; two sets 3.0 x 3.0 m and two sets 2.7 x 2.9 m. On the one side wall were 11 openings, one of them had dimensions 4,744 x 1,427 mm, nine openings had dimensions 5,828 x 1,427 mm and one opening had dimension 5,244 x 1,727 mm. Lying opposite side wall had 12 openings, where one had dimension 994 x 1,427 mm, eight had dimensions 5,828 x 1,427 mm, other dimensions were 1,264 x 1,427 mm, 1,764 x 1,427 mm 5,244 x 1,427 mm. In the middle of the stables were the columns with circular section, with diameter of 150 mm. In the roof ceiling structure, we considered with five possible solutions.

In the first solution (I) was no opening in the roof ceiling structure.

In the second solution (II) was in the roof placed skylight, which had dimensions 2.4 x 54 m. It was in the middle of the roof located.

The third solution (III) was a stable with roof windows. In this case had skylight, which was placed in the middle of stable dimensions 65,760 mm x 1,000 mm. In addition, there were 28 equally spaced roof windows with dimensions 1,200 x 1,800 mm.

The fourth solution (IV) – stable with roof windows 1 had the same dimension as the skylight in the third solution 65,760 x 1,000 mm. There were 28 roof windows too, their dimensions were 1,400 x 2,000 mm.

The fifth solution (V) – stable with roof windows 2 had dimensions of skylight as in the second solution 2.4 x 54 m. There were 28 roof windows too and their dimensions were 1,600 x 2,000 mm.

In the Figure 1 we can see an example of spacing of openings in the roof of building. Daylight factor is calculated according to equation (1):

$$D = \frac{\overline{E}}{\overline{E}_h} x100 \ (\%) \qquad (1)$$

where: D – daylight factor (%),
$\bar{E}$ – average illuminance in the point of the given plane of indoor premises (lx),
$\bar{E}_h$ – average value of outside comparative illuminance (lx).

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The assessment points, where values of daylight factor were calculated using software WDLS 5.0 (WDLS 5.0 is software from programmers at ASTRA MS Software, s.r.o. for the calculation of daylighting, daylight factor and a mixed lighting), were equally spaced all over area of stable amounting to 0.5 m above the floor. This height is the height of measurement of physiological and working lighting for object with beef-cattle. Number of rated points was 198. Points were arranged in nine rows and twenty-two profiles (Figure 2).

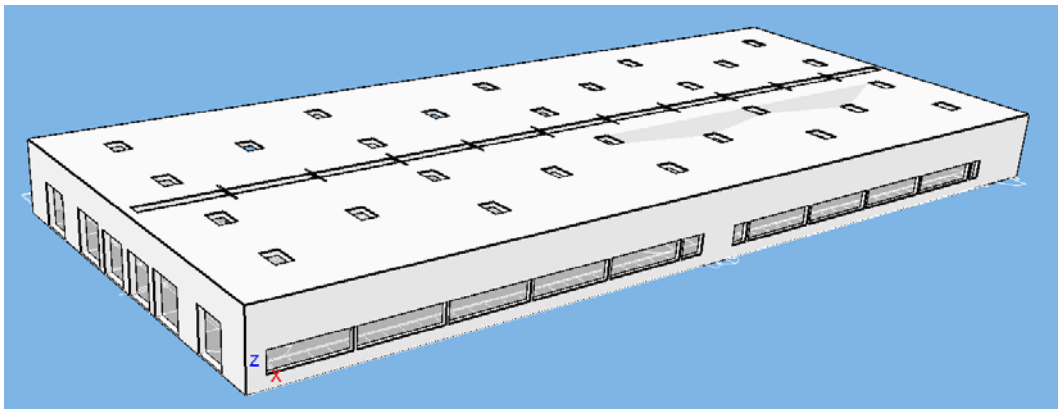
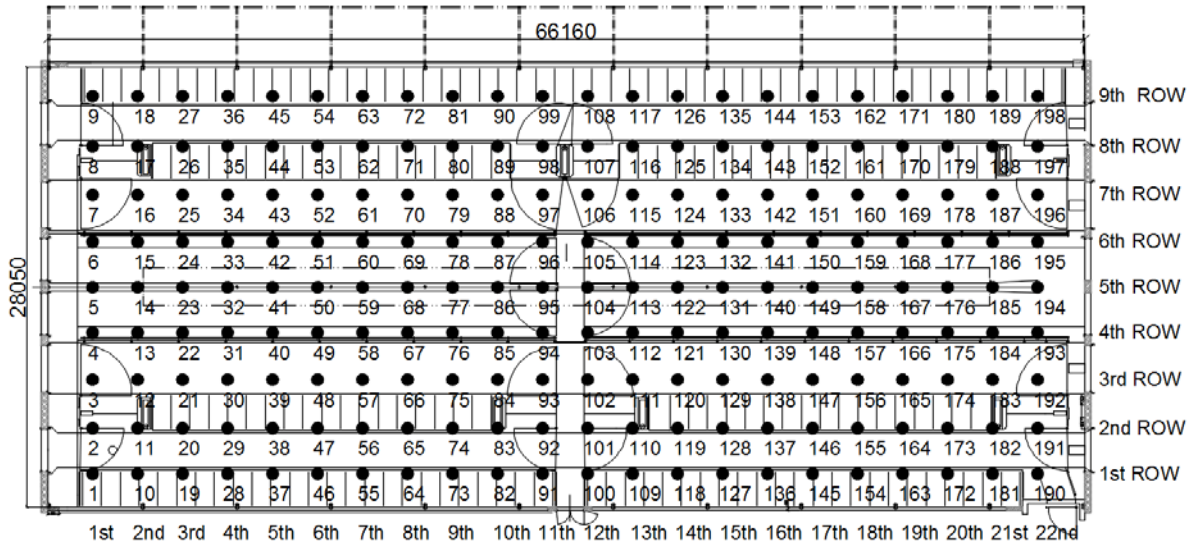


Figure 1 Location of openings in the roof of the building. Source: authors



1st – 22nd: numbers of profile
● 1 – ●198: numbers of measuring points

Figure 2 Plan of the assessed object and measuring points. Source: authors

Uniformity of daylight (U_o) is an important parameter in monitoring the stable building. For indoor premises with upper or combined lighting, the following relationship applies:

$$U_0 = \frac{D_{\min}}{D_m}, (-) \qquad (2)$$

where: D_{min} – lowest value of daylight factor (%),
D_m – average value of daylight factor (%).

According to STN 36 00 88, the minimum value of daylight factor (D) in the stable for free stall for dairy cows should be 1.0 %. The uniformity of technological lighting for animals (U_o) in spaces for dairy cows should be 0.2 [2].

RESULTS AND DISCUSSION

In the Tables 1 and 2 there are the results of the demonstrative examples. The selected assessed points were in first and 10th profiles and in second and 5th row for all solutions of stables. The first profile is close to the open barn-doors (Figure 2), tenth profile is located approximately in the middle of stable. The second row is formed in cubicles, which are in the longitudinal wall of the stable located. The fifth row is routed through the feed passage, which is in the middle of the stable. From these tables, have been created summary graphs (Figures 3 - 6). Table 3 shows the minimum, the maximum and average values of daylight factor and daylight uniformity values for individual solutions. All the values were calculated using the software WDLS 5.0.

Table 1 Values of daylight factor on various types of roof construction in 2nd and 5th row

The serial number of point	Solut. (I) D, % 2nd row	Solut. (I) D, % 5th row	Solut. (II) D, % 2nd row	Solut. (II) D, % 5th row	Solut. (III) D, % 2nd row	Solut. (III) D, % 5th row	Solut. (IV) D, % 2nd row	Solut. (IV) D, % 5th row	Solut. (V) D, % 2nd row	Solut. (V) D, % 5th row
1	3.8	4.9	4	4.6	7.3	12.2	7.7	13.4	7.8	9.8
2	1.7	1.5	2	4	4.6	7.5	5.4	9.6	5.8	7.8
3	1.3	0.8	1.7	7.4	2.9	5.6	3.9	8.1	4.1	11.5
4	1.1	0.6	1.6	9.1	3.2	5.1	3.9	7.3	4.4	13.2
5	1	0.5	1.6	9.4	2.8	5	3.5	7.1	4	13.7
6	1	0.4	1.6	9.4	3.2	5.6	3.9	7.9	4.5	14.6
7	1	0.4	1.6	9.4	3.5	6.1	4.3	7.5	5	15.1
8	0.9	0.4	1.6	8.7	2.6	4.9	3.2	7.5	3.8	12.5
9	0.9	0.3	1.5	9.2	2.9	6.1	3.5	7.1	4.1	13.4
10	0.9	0.3	1.3	8.8	2.5	6.2	3	7.3	3.5	12.9
11	0.8	0.3	1.2	8.9	3	6.5	3.6	8.2	4.2	14.3
12	0.8	0.3	1.2	8.9	2.9	6.6	3.6	8.2	4.3	14.5
13	0.9	0.3	1.3	8.8	2.4	6.2	2.9	7.2	3.5	12.9
14	0.9	0.3	1.4	9.2	2.9	6.1	3.5	7.1	4.1	13.4
15	0.9	0.3	1.5	8.7	2.6	4.9	3.2	7.2	3.8	12.4
16	1	0.4	1.6	9.4	3.5	6.1	4.2	8.1	4.9	14.9
17	1	0.4	1.6	9.4	3.2	5.6	4	8	4.7	14.8
18	1	0.5	1.6	9.4	2.8	4.9	3.4	7.4	3.9	13.7
19	1.1	0.6	1.5	9	3.1	5.1	3.8	7.3	4.3	13.1
20	1.1	0.8	1.5	7.4	2.6	5.5	3.6	8.1	3.8	11.2
21	1.4	1.5	1.7	4	4.2	7.4	4.9	9.6	5.2	7.7
22	3.5	4.9	3.7	5.5	6.6	12.1	7.2	13.5	7.5	9.9

Source: authors

Table 2 Values of daylight factor on various types of roof construction in 1st and 10th profile

The serial number of point	Solut. (I) D, % 1st prof.	Solut. (I) D, % 10th prof.	Solut. (II) D, % 1st prof.	Solut. (II) D, % 10th prof.	Solut. (III) D, % 1st prof.	Solut. (III) D, % 10th prof.	Solut. (IV) D, % 1st prof.	Solut. (IV) D, % 10th prof.	Solut. (V) D, % 1st prof.	Solut. (V) D, % 10th prof.
1	5.2	3.1	5.3	3.4	8.9	5.4	9.1	5.5	9.3	5.8
2	3.8	0.9	4	1.3	7.3	2.5	7.7	3	7.8	3.5
3	4.8	0.5	5.2	1.6	9	2.5	9.5	3	9.3	4.2
4	4.8	0.4	5.9	5.2	10.8	4.1	11.5	5.1	9.9	8.3
5	4.9	0.3	4.6	8.8	12.2	6.2	13.4	7.3	9.8	12.9
6	4.8	0.4	5.2	5.2	10.6	4.1	11.2	4.8	9.8	8.3
7	4.8	0.5	5	1.6	8.9	2.5	9.3	3.1	9.2	4.3
8	3.8	0.9	3.9	1.3	7.1	2.5	7.4	3	7.7	3.6
9	4.8	3.4	4.6	3.1	8.2	4.9	8.4	5.2	8.6	5.5

Source: authors

Table 3 Values D_{min}, D_{max}, D_{av} and U_o for different types of solutions. Source: authors

Solution	D_{min}, %	D_{max}, %	D_{av}, %	U_o, -
(I)	0.3	5.2	1.6	0.06
(II)	1.2	9.4	3.6	0.13
(III)	2.4	12.2	4.8	0.20
(IV)	2.8	13.5	5.5	0.21
(V)	3.5	15.1	6.9	0.23

D_{min} – minimum value of daylight factor
D_{max} – maximum value of daylight factor
D_{av} – average value of daylight factor
U_o – uniformity of daylight
Source: authors

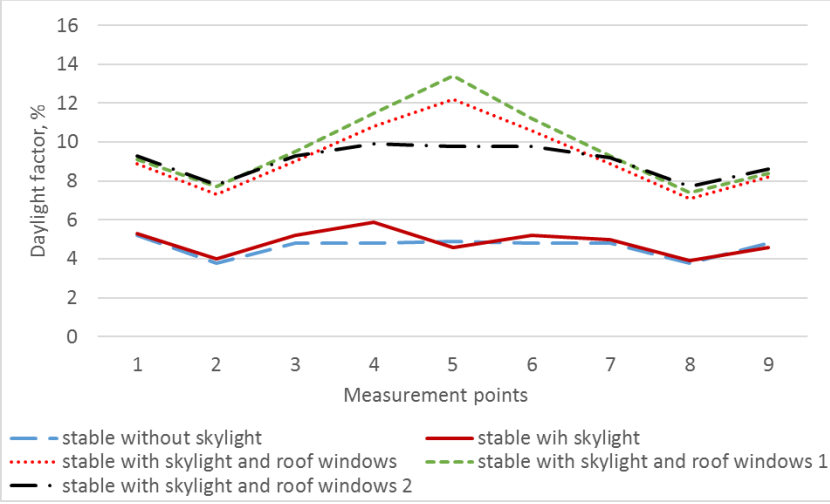


Figure 3 Values of daylight factor in the 1st profile. Source: authors

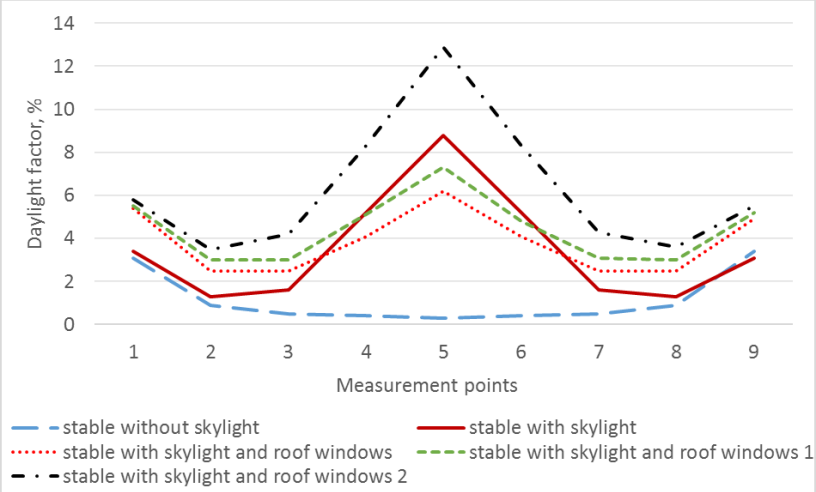


Figure 4 Values of daylight factor in the 10th profile. Source: authors

Figure 3 shows course of lighting in individual solutions in the first profile, which is located near the open barn-doors. In this part impact on except for side and roof windows and skylights also open barn-doors. Here in the first and second solution are values approximately the same, because this part does not affect skylight. In this section are the best values obtained in the fourth solution, which are large enough roof windows and skylight is also over this part of stables. In the tenth profile (Figure 4) did not affect already opened barn-doors on the lighting. Here is clearly see the difference in the values of daylight factor for individual solutions. While in the first solution calculated values did not reach desired values, in the fifth solution these values were exceeded many times over.

Figure 5 shows course of daylight factor values in the second row, which was in the cubicles by the longitudinal wall of stable. In this solution, the best lighting in all cases was near open barn-doors. Other assessed points were near the open side walls. In this section was the effect of skylight minimal. Here the lighting has been affected by skylights. Here we can see, that the best results were in the fifth solution, where were sufficiently large skylight and at the same time sufficiently large roof windows. In the Figure 6 is shown course of the lighting in the fifth row, which is in the part, where is the feed passage. Due to the fact, that above this part is skylight (except solution I) here is the best to see difference in daylighting in the individual solutions. In the solution without skylight here resulting values did not meet standard conditions [8]. In the other cases, this condition significantly exceeded.

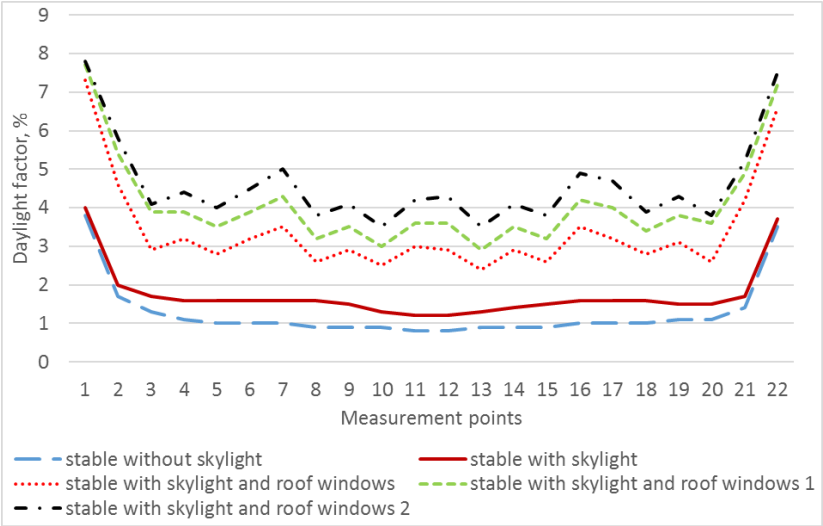


Figure 5 Values of daylight factor in the second row. Source: authors

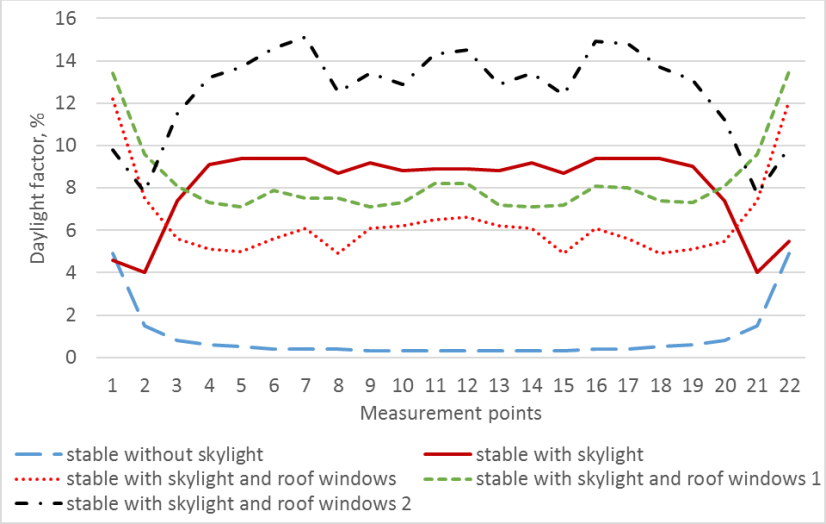


Figure 6 Values of daylight factor in the 5th row. Source: authors

From the above results, we found that the worst daylighting is in the stable, where opening constructions only in the side walls of stable are. In the cubicles located in the side walls of the stables it is sufficient lighting. Towards the middle part the lighting conditions worsen and fall short of the desired values already. It is equally low value of uniformity of daylighting. In the second solution, where was in the middle part of the stable skylight, lighting conditions significantly improved. They satisfy the conditions for housing animals. However, the light in the stable was unevenly distributed. Under the skylight were significantly higher values. In the following three solutions, we have tried to improve the uniformity of daylighting by evenly spaced roof windows. The best results were at the fifth solution. Influence of skylights on a daylighting in the stable we can see in the specific example where we measured values of illuminance [1]. By the longitudinal walls and under the skylight there were high values of daylight factor, however inside the stable they did not fulfil the conditions for housing animals.

In the Figure 7 is the stable with the worst daylighting (solution I) and in the Figure 8 is a view into this stable. In the Figure 9 is the stable with the best daylighting (solution V) and in the Figure 10 is a view into this stable.

Improvement of lighting conditions can worsen thermal conditions in the stable. The problem occurs mainly during the summer period. In this case, it would be necessary to address, what material is the best used for glazing. Temperature increase in the interior due to the opening constructions is affected by many factors. It is kind of glazing, glass thickness, the location of the openings with respect to the points of the compass, the size of the openings and so the like. Vanhoutteghem at al. [9] dealt with the effect of windows for daylighting and thermal comfort in buildings. He focused on the relationship between the size, orientation and characteristics of glazing. For glazing opening structures is appropriate to use materials, that transmits light but reflect sunlight. Samant and Sharples [7] investigated the effect of surface reflectivity to average daylight factor in the atrium of the building, in which they discovered the importance of reflective properties of surfaces atrium and changes in the distribution of the reflectivity of the daylight in the atrium of the building.

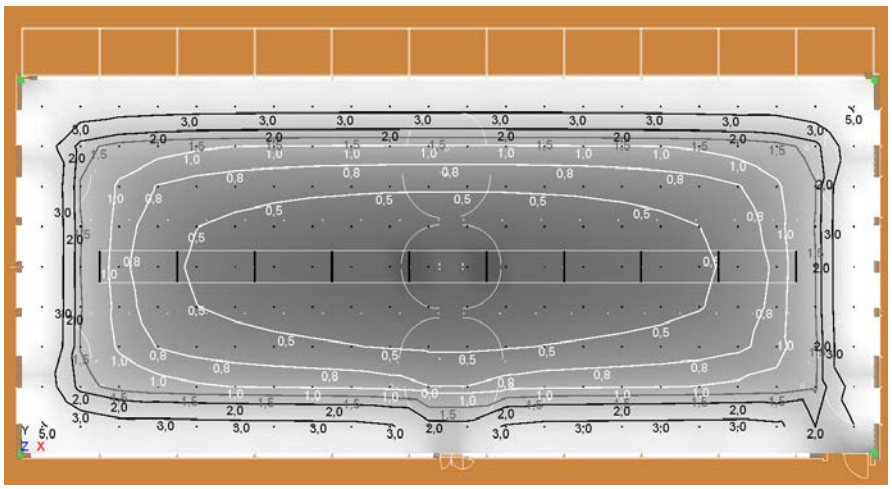


Figure 7 Stable without skylight – solution I. Source: authors

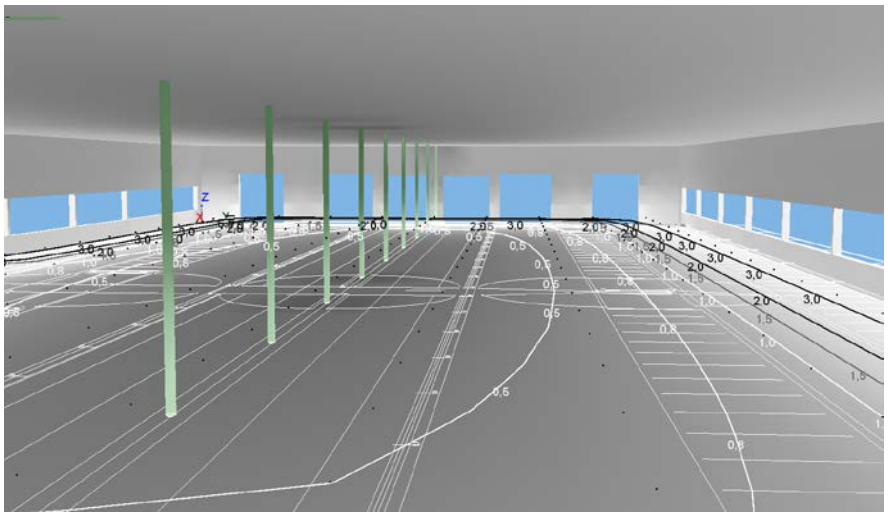


Figure 8 View into the stable with the worst daylighting - solution I. Source: authors

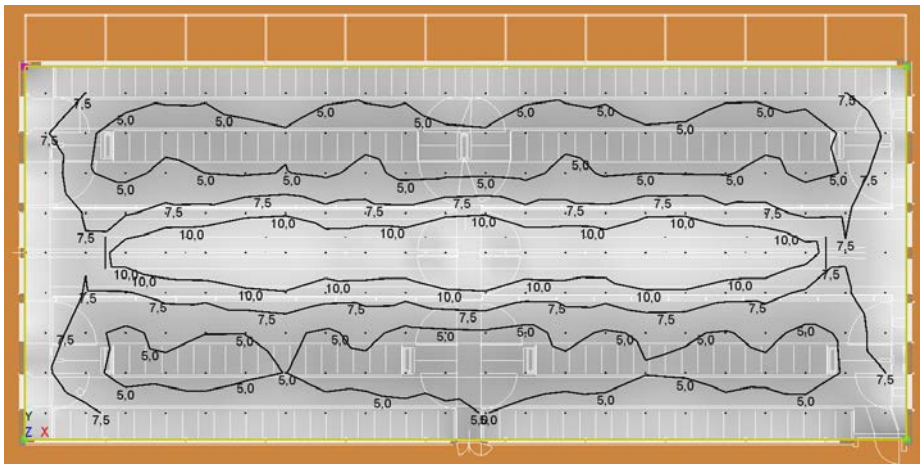


Figure 9 Stable with roof windows 2 – solution V. Source: authors

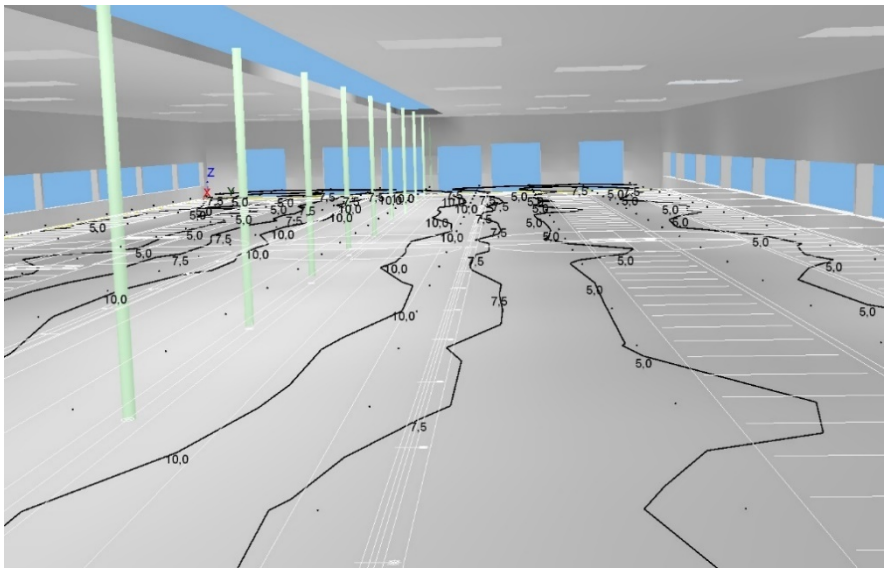


Figure 10 View into the stable with the best daylighting - solution V. Source: authors

CONCLUSIONS

The lighting in the stables for dairy cows is one of the basic parameters of environmental engineering. Many authors have focused on the influence of lighting on the health and cow production efficiency. It has been shown that well-designed lighting has a positive effect on dairy cows. The best lighting is from a health and economic aspect just natural lighting. That is just it important already during the drafting of the new respectively reconstruction of the original stables also focus on the fact make them well designed light openings.

In the article it was proposed five possible options of daylighting. Ground plan dimensions of stable as well as side openings and barn-doors were identical in all cases. It has been changed only roof ceiling structure. From the results, we can see that right plan lighting elements in the ceiling structure are the best solution. In proposals of daylighting it is also important that there was light in the building evenly distributed. That some parts were not too light-filled and on

the other hand, some parts dark. It is best to see the differences in the proposals I and V. In the first solution, we have in the roof structure no opening and part of the stable is, despite the large openings dark. In the fifth solution the light is most evenly deployed. For the draft of roof windows and skylights we must bear in mind what material suggest glazing openings. Here we must see to it that the stable especially in the summer does not overheat.

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Original Paper

B-spline surface for distribution of the agriculture soil moisture

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ABSTRACT

In the agricultural locality Pohranice were measured soil moisture values for the selected control points. Over these points we put the B-spline surface, which relatively detailed description we supply. On a simple example of 4 x 4 points we debugged program for generation of base functions. Furthermore, we verified process to generate knots of B-spline surface, which we subsequently used for the programming of the surface itself. In this manner formed algorithms we applied to 11 x 11 measured values of soil moisture. Discrete values of measurements were by clamped B-spline surface smoothed and its corner values constitute the edge measurement points. Evenly spaced points at a distance of 1 m in both directions slightly simplified the procedure, which can deal with irregular layout of points.

KEYWORDS: basis functions, B-spline knots, surface fitting.

JEL CLASSIFICATION: M55, N55

INTRODUCTION

Geometry modelling in existing computer applications prefers for objects definition instead of complex mathematical expressions enter the coordinates of several points in the plane or in the space that are easy to handle. These points then very efficiently approximate arbitrarily complex shapes and to edit in this way formed objects is often enough to edit positions of the control points [9]. Typical examples of this approach are Bézier curve [5], [8], B-spline curve [6] and NURBS curve [10]. Likewise are modelled also surfaces as Bézier surface [7], or B-spline surface, which is covered in detail in this article. In addition to description of the B-spline surface we practically apply it to the values of soil moisture. Methods of measurement of the soil moisture as well as processing of the measured data are described in [3].

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MATERIAL AND METHODS

B-spline surface

For unambiguous identification of the B-spline surface we need several information. First, the matrix of (m + 1) rows and (n + 1) columns of control points $P_{i,j}$, where i is from 0 to m and j is in the range from 0 to n. Next, there are required two vectors of knots, one in the direction of u with (h+1) knots $U = \{u_0, u_1, ... , u_h\}$ and the other in the direction of v with (k+1) knots $V = \{v_0, v_1, ... , v_k\}$. Finally, it is necessary to define the degree of the function p in the direction u and the degree of the function q in the direction v. For the mentioned parameters the equation of B-spline surface takes form

$$S(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n} N_{i,p}(u).N_{j,q}(v).P_{i,j}\,, \tag{1}$$

where $N_{i,p}(u)$ and $N_{j,q}(v)$ represent B-spline basis functions of degree p, respectively q. The basic equation of the B-spline function that we described for the curve [6] has to match in both directions, thus h = m + p + 1 and k = n + q + 1. Like Bézier surface is also the B-spline surface the surface of tensor product. The set of control points $P_{i,j}$ generates control net and the parameters u and v take values in the range from 0 to 1. B-spline surface thus transforms the unit square to rectangular surface.

B-spline curves can be open, closed, or clamped and therefore the B-spline surface may have the same characteristics in each direction. It may be required that in the direction u is the surface clamped and in the direction v is the same surface closed. If the B-spline surface is clamped in both direction then passes through the control points $P_{0,0}$, $P_{m,0}$, $P_{0,n}$, $P_{m,n}$ and the end flowlines of the control points constitute eight tangents. We deal in the contribution with just such a clamped surface in both directions. If the B-spline surface is closed in one direction, then also all the isoparametric curves are closed in this direction. In the case of open B-spline surface in both directions this does not pass through any of the control points $P_{0,0}$, $P_{m,0}$, $P_{0,n}$, $P_{m,n}$.

Basis functions

Let $U = \{u_0, u_1, u_2, ... , u_h\}$ is non-decreasing sequence of (h+1) real numbers, thus $u_0 \le u_1 \le u_2 \le ... \le u_i \le u_{i+1} \le ... \le u_h$. These numbers are called knots, set U is named knot vector and semi-open interval $\langle u_i, u_{i+1})$ is designated as the i-th knot span. Since knots u_i can be identical, some knot spans do not exist and are therefore zero. If knot u_i occurs s-times, thus $u_i = u_{i+1} = ... = u_{i+s-1}$, where $s > 1$, u_i is called s-multiple knot of multiplicity s, what is written $u_i(s)$. Otherwise, if u_i appears only one, it is simple knot. If the knots are equidistant, i.e. $(u_{i+1} - u_i)$ = constant, for any $0 \le i \le (h-1)$, the knot vector, or the knot sequence is called uniform, otherwise is non-uniform.

The knots can be considered as dividing points that divide the interval $\langle u_0, u_h\rangle$ to the knot spans. The interval $\langle u_0, u_h\rangle$ represents definition domain of the B-spline basis function.
Basis function of B-spline surface, shown in the formula (1) as $N_{i,p}(u)$ a $N_{j,q}(v)$ are defined by the Cox de Boor recursive formula

$$\begin{aligned} N_{i,0}(u) &= 1, \ u_i \le u \le u_{i+1} \\ N_{i,0}(u) &= 0, \ \text{for other u} \end{aligned} \tag{2}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \quad (3)$$

where p indicates function degree.

Basis functions of the B-spline surface (1) are the coefficients of the control points referred to (2) and (3). The definition (1) shows that the two-dimensional basis function is a product of two one-dimensional basis functions.

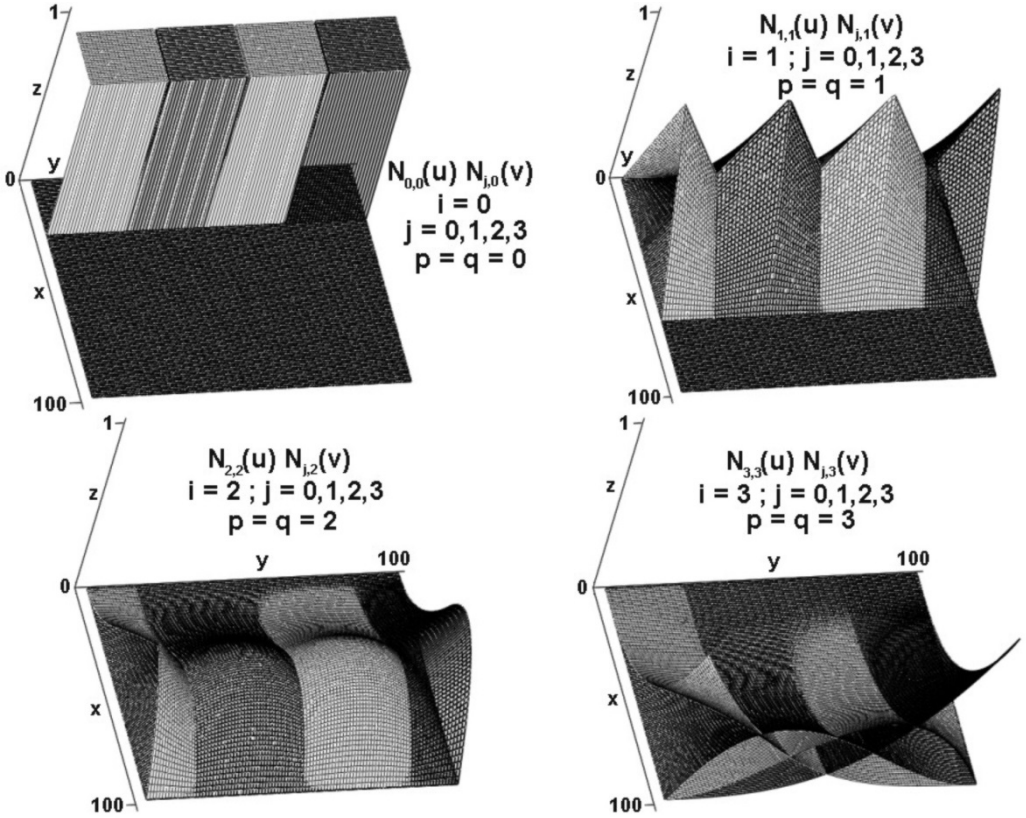

Fig. 1 Basis functions of the B-spline surface.

The Fig. 1 illustrates the basis functions for m = 3 and n = 3. We show them for different degrees of surfaces, p = q = 0, 1, 2, 3. On each of the four parts of the Fig. 1 are drawn four functions for p = q, and with the same i changing gradually along the x axis. The functions in the same part of the Fig. 1 differ by the different values of j along the y axis. Every part stated precisely its settings. Basis functions m = n = 3, correspond to a training example, which we solve by using p = q = 2. This example has total sixteen basis functions, each match to a single control point $P_{i,j}$. It must be noted that the last part below right, that assignment p = q = 3, is the property of B-spline surface its limit state, Bézier surface.

Realization of the experiment

The measurement was carried out in the village Pohranice on agricultural site with GPS N48.33213° E18.16568° at an altitude of 223 m. The terrain was slightly undulate with a declining trend to the northwest. On the field was growing wheat and was performed agricultural operation autumn tillage.

To measure soil moisture field, where the vehicle moved during execution of working manoeuvers, we used Moisture Meter HH2. Moisture meter from Delta-T [1] is a universal reader unit, which allows comfortable display format and stores measurement data of the connected sensor. The device reads outputs of probes Profile Probe type PR1 (depth probe to measure soil moisture profile), of the probes Theta Probe (types ML1, ML2 and ML2x) and of isometric tensiometer (types EQ1 and EQ2). Moisture meter size is 150 x 80 x 40 mm and weight 450 g. As mentioned by producer, the range of humidity is measured from zero to saturation, 0 - 1.5 V on the voltage scale. The measuring accuracy is specified at ± 0.13% readings in mV +1.0 mV and the resolution is 1 mV.

RESULTS AND DISCUSSION

Algorithm creation

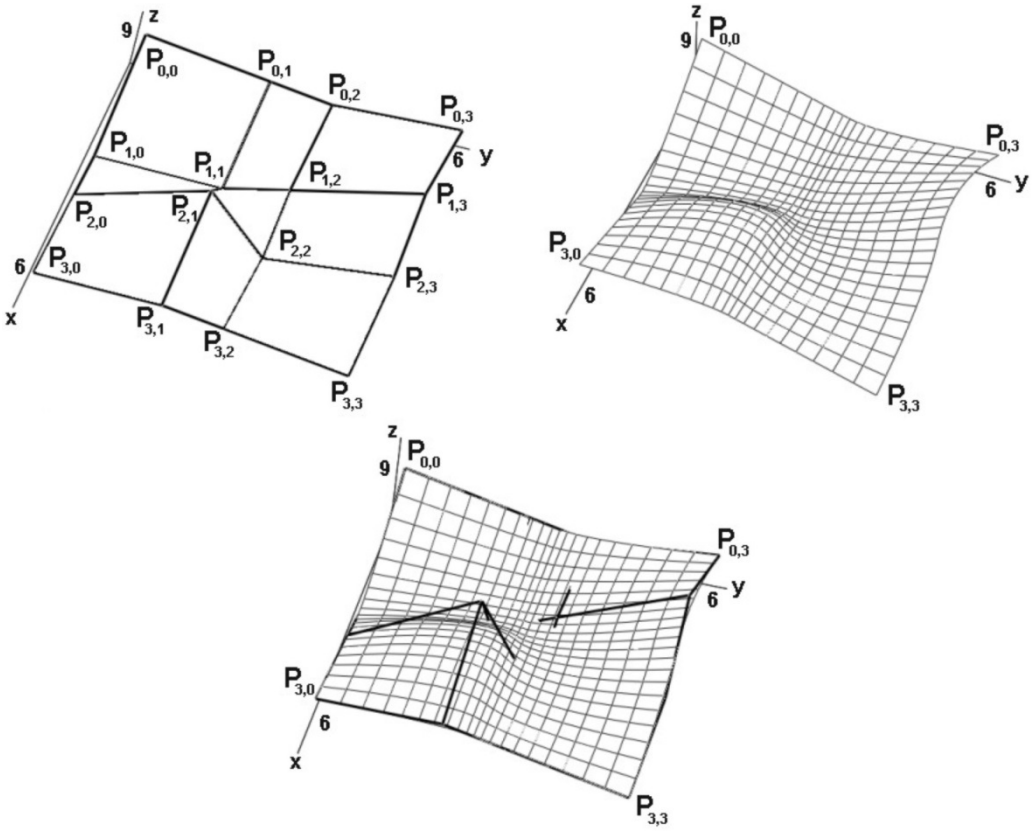

Fig. 2 Control points, B-spline surface of order (2, 2) and their comparison.

In the Fig. 2 is plotted the net of control points and through them is by the algorithm fitted B-spline surface. The surface in both directions has the same degree p = q = 2. Corner points $P_{0,0}$, $P_{3,0}$, $P_{0,3}$ and $P_{3,3}$ of surface are identical to the control points as shown in the Fig. 2. The Fig. 2 allows simultaneously a comparison of given control points with formed surface and shows smoothing of polygon surface defined by discrete points. Like this measured quantity modelled by B-spline surface remains quite smoothly without restrictions arising by specification only a finite number of measured values.

To create B-spline surfaces, we used the algorithm according to [4]. Debugged algorithm we examined on a training example 4 x 4 points, whose coordinates are presented by (4).

$$x=\begin{pmatrix}1&3&4&6\\1&3&4&6\\1&3&4&6\\1&3&4&6\end{pmatrix}\quad y=\begin{pmatrix}1&1&1&1\\3&3&3&3\\4&4&4&4\\6&6&6&6\end{pmatrix}\quad z=\begin{pmatrix}7&3&4&6\\5&3&9&6\\4&5&2&5\\5&9&4&3\end{pmatrix} \tag{4}$$

Application of the B-spline surface for the distribution of moisture

Soil moisture denotes actual water content in the soil expressed in relative units. It is often measured by volume percentage. Its changes depend on weather conditions as well as on the water intake to plant roots. We conducted our measurements by Moisture Meter HH2, in the area Pohranice, on November 12, 2015.

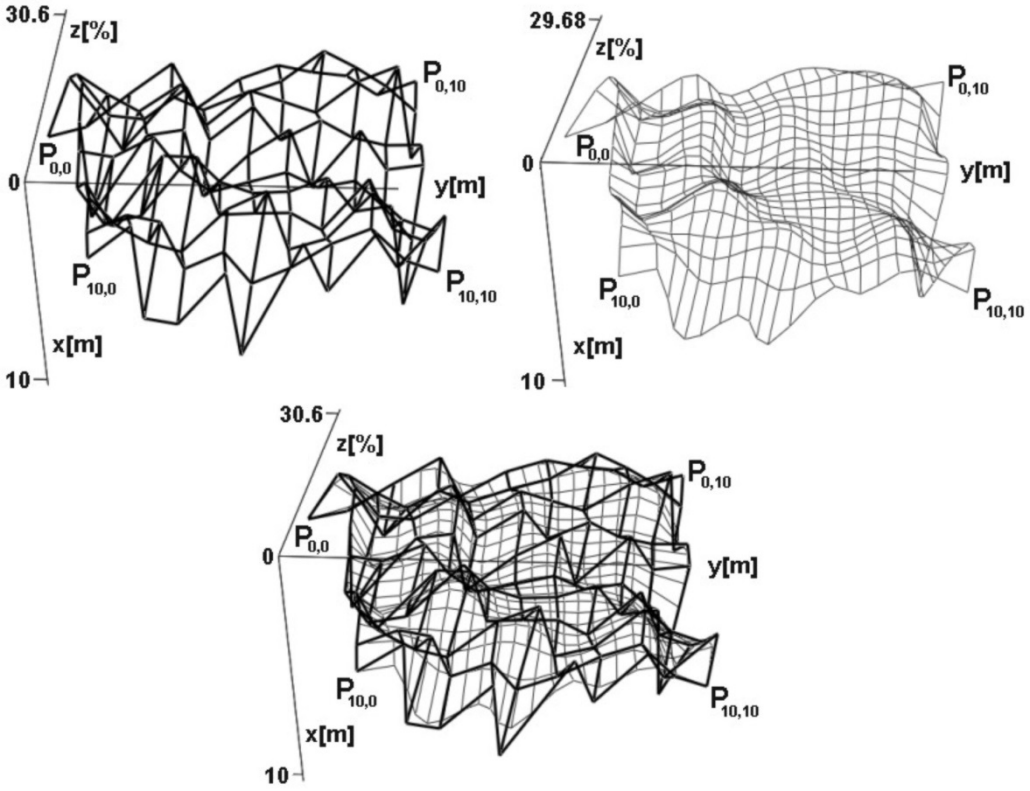

Fig. 3 Control points, B-spline surface of order (2, 2) and their comparison.

Measurements we performed on an area of 10 x 10 meters, each measuring point had from the neighbouring point distant 1 m in both directions. So we created a regular grid of measuring points, which we then transformed to the control points $P_{i,j}$, used in the modelling of B-spline surface with indices i = 0,1,...,10 a j = 0,1,...10.

The measured values in the form of polygon surface are drawn in the Fig. 3. Using the algorithm mentioned above we made a B-spline surface and compared it with measurements in the Fig. 3. Control points $P_{0,0}$, $P_{10,0}$, $P_{0,10}$ and $P_{10,10}$ are identical to the measured values. Obviously, the B-spline surface smoothed original polygon and for creation of the surface have been used all values that completely define surface.

CONCLUSIONS

B-spline surface stands out for its great adaptability in application to a set of spatially placed points. Number of parameters to be defined for it could give the impression of its complexity. As shown in the Fig. 3 for applied clamped B-spline surface, its corner points are identical to the endpoints of the specified spatial polygon and other specified points the surface smooths and makes continuous discrete course of points for the polygon.

We compiled the algorithm for creation of basis functions of B-spline surface and verified the procedure for obtaining the knot points and for making surface itself. The net of control points exhibited regular arrangement, although the application of the algorithm is also admissible on an irregular layout. The use of B-spline surfaces on the distribution of soil moisture shows how it is possible to approximate by this surface also other similar technical problems.

Measurement of soil moisture is important for farmers, due to proper setup and use of irrigation systems [2]. When we know the exact terms of the origin and amount of the moisture we can save on the consumed water. At the same time significantly improves the process of plants irrigation and set to the important stages of plant growth.

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Original Paper

The sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots

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ABSTRACT

This contribution, which is a follow-up to author's papers dealing with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, different negative integer roots, and one negative and one positive integer root, deals with the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots. We derive the formula for the sum of these series and verify it by some examples evaluated using the basic programming language of the computer algebra system Maple 16. This contribution can be an inspiration for teachers of mathematics who are teaching the topic Infinite series or as a subject matter for work with talented students.

KEYWORDS: sum of the series, harmonic number, purely imaginary conjugate roots, hyperbolic cotangent, computer algebra system Maple

JEL CLASSIFICATION: I30

INTRODUCTION AND BASIC NOTIONS

In the papers [6], [5] and [4] author dealt with the sums of the series of reciprocals of quadratic polynomials with different positive integer roots, different negative integer roots, and one negative and one positive integer root. This contribution is focused on the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots.

Let us recall the basic terms. For any sequence $\{a_k\}$ of numbers the associated *series* is defined as the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots \ .$$

The *sequence of partial sums* $\{s_n\}$ associated to a series $\sum_{k=1}^{\infty} a_k$ is defined for each n as the sum

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$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The series $\sum_{k=1}^{\infty} a_k$ *converges* to a limit s if and only if the sequence $\{s_n\}$ converges to s, i.e. $\lim_{n\to\infty} s_n = s$. We say that the series $\sum_{k=1}^{\infty} a_k$ has a sum s.

The *sum of the reciprocals* of some positive integers is generally the sum of unit fractions. For example the sum of the reciprocals of the square numbers (the *Basel problem*) is $\frac{\pi^2}{6}$:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \doteq 1.644934.$$

The *nth harmonic number* H_n is the sum of the reciprocals of the first n natural numbers:

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The *hyperbolic cotangent* is defined as a ratio of hyperbolic cosine and hyperbolic sine

$$\coth x = \frac{\cosh x}{\sinh x}, \quad x \neq 0.$$

Because (see [7]) hyperbolic cosine and hyperbolic sine can be defined in terms of the exponential function

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} + 1}{2\mathrm{e}^x}, \qquad \sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \frac{\mathrm{e}^{2x} - 1}{2\mathrm{e}^x},$$

we get

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{\mathrm{e}^x - \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1}, \qquad x \neq 0. \qquad (1)$$

By means of the *gamma function*, which is defined via a convergent improper integral

$$\Gamma(z) = \int_0^{\infty} x^{z-1}\mathrm{e}^{-x}\mathrm{d}x,$$

is defined so called *digamma function* (see [1])

$$\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln(\Gamma(z)) = \frac{\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(z)}{\Gamma(z)}.$$

MATERIAL AND METHODS

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH REAL ROOTS

As regards the sum of the series of reciprocals of the quadratic polynomials with different positive integer roots a and,, i.e. with the series

$$\sum_{\substack{k=1 \\ k\neq a,b}}^{\infty} \frac{1}{(k-a)(k-b)},$$

in the paper [6] it was derived that the sum $s(a,b)$ is given by the following formula using the nth harmonic numbers H_n

$$s(a,b) = \frac{1}{b-a}(H_{a-1} - H_{b-1} + 2H_{b-a} - 2H_{b-a-1})\,. \qquad (2)$$

In the paper [5] it was shown that the sum of the series of reciprocals of the quadratic polynomials with different negative integer roots a and,, i.e. with the series

$$\sum_{k=1}^{\infty} \frac{1}{(k-a)(k-b)},$$

is given by the simple formula

$$s(a,b) = \frac{1}{b-a}(H_{-a} - H_{-b})\,. \qquad (3)$$

The sum of the series

$$\sum_{\substack{k=1 \\ k\neq b}}^{\infty} \frac{1}{(k-a)(k-b)}$$

of reciprocals of the quadratic polynomials with integer roots, $b > 0$ was derived in the paper [4]. This sum is given by the formula

$$s(a,b) = \frac{(b-a)(H_{-a} - H_{b-1}) + 1}{(b-a)^2}\,. \qquad (4)$$

THE ASSIGNMENT OF THE SOLVED PROBLEM

Now, we deal with the problem to determine the sum $s(b)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + b^2},$$

where, i.e. if the quadratic polynomial in the denominator has conjugate purely imaginary roots $k_{1,2} = \pm bi$, where is the imaginary unit. So,

we can write

$$s(b) = \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2}. \tag{5}$$

For example, we want to determine the sum, corresponding with the complex conjugates roots $k_{1,2} = \pm 4i$, of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + 16} = \frac{1}{17} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{41} + \cdots .$$

THE SUM OF THE SERIES OF RECIPROCALS OF THE QUADRATIC POLYNOMIALS WITH PURELY IMAGINARY CONJUGATE ROOTS

Through the Weierstrass product (see [3]) for the hyperbolic sine function

$$\frac{\sinh z}{z} = \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 k^2}\right),$$

where $z \in C - \{0\}$, and by the logarithmic derivation of both its sides

$$\ln \frac{\sinh z}{z} = \ln \sinh z - \ln z \quad \overset{'}{\rightarrow} \quad \frac{1}{\sinh z} \cdot \cosh z - \frac{1}{z} = \coth z - \frac{1}{z},$$

$$\ln \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{\pi^2 k^2}\right) = \sum_{k=1}^{\infty} \ln \frac{\pi^2 k^2 + z^2}{\pi^2 k^2} \quad \overset{'}{\rightarrow} \quad \sum_{k=1}^{\infty} \frac{\pi^2 k^2}{\pi^2 k^2 + z^2} \cdot \frac{2z}{\pi^2 k^2} = \sum_{k=1}^{\infty} \frac{2z}{\pi^2 k^2 + z^2}$$

we get the equality (see web pages [1] and [2])

$$\coth z - \frac{1}{z} = \sum_{k=1}^{\infty} \frac{2z}{z^2 + \pi^2 k^2}.$$

For $z = \pi \cdot b, b > 0$, we have

$$\coth \pi b - \frac{1}{\pi b} = \sum_{k=1}^{\infty} \frac{2\pi b}{\pi^2 b^2 + \pi^2 k^2} = \frac{2b}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^2 + b^2},$$

and after multiplication by the fraction $\pi/2b$ we get the equality

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + b^2} = \frac{\pi}{2b} \coth \pi b - \frac{1}{2b^2}. \tag{6}$$

By the relation (1) we get

Theorem 1 The series

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + b^2},$$

where $b > 0$, has the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^2+b^2} = \frac{\pi}{2b} \cdot \frac{e^{2\pi b}+1}{e^{2\pi b}-1} - \frac{1}{2b^2}. \tag{7}$$

Example 1 Evaluate the sum $s(4)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+4^2} = \frac{1}{17} + \frac{1}{20} + \frac{1}{25} + \frac{1}{32} + \frac{1}{41} + \cdots$$

i) by formula (7),
ii) in the CAS Maple 16 by means of the partial sum using the first one million terms. Compare the obtained results.

Solution

i) The series has by formula (7) from Theorem 1, where, the sum

$$s(4) = \frac{\pi}{2\cdot 4} \cdot \frac{e^{2\pi\cdot 4}+1}{e^{2\pi\cdot 4}-1} - \frac{1}{2\cdot 4^2} = \frac{\pi}{8} \cdot \frac{e^{8\pi}+1}{e^{8\pi}-1} - \frac{1}{32} \doteq 0.3614490817.$$

ii) By using the CAS Maple 16 and evaluation the approximate value $s_{10^6}(4)$ of the sum $s(4)$ we get the following sequence of commands and results:

```
> sum(1/(k^2+16),k=1..1000000);
```

$$-\frac{1}{8} I\Psi(1000001-4I) + \frac{1}{8} I\Psi(1000001+4I) + \frac{1}{8} I\Psi(1-4I) - \frac{1}{8} I\Psi(1+4I)$$

```
> evalf(%,10);
```

0.3614480818 + 0.I

So we can state that both these obtained results, using ten first decimals, are almost the same and they differ from each other only about 10^{-6}. Let us note that in the Maple notation the symbol means the imaginary unit and the function Ψ denotes the digamma function.

RESULTS AND DISCUSSION - NUMERICAL VERIFICATION

We solve the problem to determine the values of the sum $s(b)$ of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2+b^2}$$

for. We use on the one hand an approximative direct evaluation of the sum

$$s(b,t) = \sum_{k=1}^{t} \frac{1}{k^2+b^2},$$

where $t = 5\cdot 10^5$, using the basic programming language of the CAS Maple 16, and on the other hand formula (7) for evaluation the sum $s(b)$.

We compare 10 pairs of these two ways obtained sums $s(b,5\cdot10^5)$ and $s(b)$ to verify formula (7). Let us note that the evaluation of the sums $s(b,10^6)$ in this way would be too time consuming and would take dozens of hours. For computation the sums $s(b,5\cdot10^5)$ and $s(b)$ we use following simple procedure `sumb` and one **for** statement:

```
> sumb:=proc(b,t)
      local f,k,r,s;
      s:=0;
      r:=0;
      for k from 1 to t do
          r:=1/(k^2+b^2);
          s:=s+r;
      end do;
      print("s(",b,")=",evalf[10](s));
      f:=Pi/(2*b)*(exp(2*Pi*b)+1)/(exp(2*Pi*b)-1)-1/(2*b^2);
      print("f(",b,")=",evalf[10](f));
      print("abserr=",evalf[10](abs(f-s)));
      print("relerr=",evalf[10](abs((f-s)/f)));
  end proc:

> for b from 1 to 10 do
      sumb(500000,b);
  end do;
```

This computation of the sums $s(b,5\cdot10^5)$ and $s(b)$ took almost 17 hours while the following direct computation of the sums $s(b,10^6)$ without using the programming language took only under 1 second:

```
> for b from 1 to 10 do
     print("s(",b,")=",evalf[10](sum(1/(k^2+b^2),k=1..1000000)));
  end do;
```

The approximative values of the sums, $s(b,5\cdot10^5)$, and obtained by formula (7), by the procedure `sumb` and by direct computation and rounded to 9 decimals, are written into the following table 1.

Tab.1: Some approximative values of the sums, $s(b,5\cdot10^5)$

b	1	2	3	4	5
$s(b)$	1.076674048	0.660403642	0.468043227	0.361449082	0.294159265
$s(b,5\cdot10^5)$	1.076672047	0.660401641	0.468041227	0.361447082	0.294157265
$s(b,10^6)$	1.076673047	0.660402642	0.468042227	0.361448082	0.294158265
b	6	7	8	9	10
$s(b)$	0.247910499	0.214195394	0.188537041	0.168360086	0.152079633
$s(b,5\cdot10^5)$	0.247908499	0.214193394	0.188535041	0.168358086	0.152077633
$s(b,10^6)$	0.247909499	0.214194394	0.188536041	0.168359086	0.152078633

The relative errors, i.e. the ratios $\left|\frac{s(b)-s(b,5\cdot10^5)}{s(b)}\right|$, range between $2\cdot10^{-6}$ for $b=1$ and 10^{-5} for.

CONCLUSIONS

We dealt with the sum of the series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots $\pm bi$, i.e. with the series $\sum_{k=1}^{\infty}\frac{1}{k^2+b^2}$.

By means of the Weierstrass product and logarithmic derivation we derived that the sum $s(b)$ of this series is given by the formula

$$s(b)=\frac{\pi}{2b}\coth\pi b-\frac{1}{2b^2},$$

$$s(b)=\frac{\pi}{2b}\cdot\frac{e^{2\pi b}+1}{e^{2\pi b}-1}-\frac{1}{2b^2}.$$

We verified this main result by computing 10 sums by using the CAS Maple 16.

The series of reciprocals of the quadratic polynomials with purely imaginary conjugate roots so belong to special types of infinite series, such as geometric and telescoping series, which sums are given analytically by means of a formula which can be expressed in closed form.

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Original Paper

Assessment of competences acquired via CLIL in selected subjects at the Slovak University of Agriculture in Nitra

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ABSTRACT

The main objective of this paper follows from the education of students at the Slovak University of Agriculture in Nitra who will work as experts in different areas. The educational system of experts and professionals cannot operate in the isolation but in the context of changes in the professional field and society. In the paper we focused on the analysis of students' mathematics and language competences acquired via method CLIL (Content and Language Integrated Learning). The learners can realize the impact of their acquired knowledge to the study of other specialized courses and to better conditions for study stays. In the research part it is described the survey through which we verified the efficiency of CLIL methodology in the education of chosen topics of two study subjects: Mathematics and English. The results of pedagogical experiment were analyzed using statistical methods. The experiment confirmed that the above-mentioned methodology of mathematics teaching was effective. Based on the obtained results we can conclude that the students have the abilities to use mathematical apparatus and language skills, which is an important prerequisite for their professional advancement.

KEYWORDS: education, CLIL methodology, mathematics, foreign language education, communication skills, statistical hypotheses testing

JEL CLASSIFICATION: D40, C50, M10

INTRODUCTION

At the Slovak University of Agriculture in Nitra undergraduates are educated for successful careers in different areas such as: agriculture, food, biotechnology, economy and technical industry. The contemporary professional and scientific practice emphasizes the requirements for the students to be able to communicate in a foreign language. Apart from the common

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social communication it is important that the students could acquire new information and present the professional knowledge in a foreign language. The ability to communicate in a foreign language is becoming the significant advantage for any person looking for the appropriate job at the labor market. Therefore, teaching subjects in a foreign language means the excellent opportunity for the connection of a language and professional knowledge in the particular subject [11].

Moravcová and Maďarová [8] claim that "the degree of language mastery is based on the needs and interests, in our case, of the agricultural learners. It also depends on the amount of vocabulary, knowledge of grammar, as well as the correct ways of the verbal interaction. It is also important to have knowledge in social communication, culture, and language variations".

A teacher of a professional foreign language is required to have profound linguistic knowledge, to be familiar with the terminology of a particular science and also to be able to enhance teaching process via the information and communication technologies [5].

The "Faculty of European Studies and Regional Development" of the Slovak University of Agriculture in Nitra provides the students with the opportunity to choose the study program in English entitled "Regional Development and Policies of the EU". They learn all subjects in the given language, e.g. Macroeconomics, Microeconomics, Mathematics, English Language, Economic Geography, Computer Data Processing, Statistics and Sustainable Development. The learners have to combine their knowledge effectively in thinking, problem solving, reasoning and case studies. The cross-curricular educational process enables to transfer learners´ knowledge from one context into another and from one subject to others. This course means the embodiment of CLIL (Content and Language Integrated Learning) in practice.

CLIL is a methodology for studying content through a foreign language, thus acquiring both the subject and the language. This term was used for the first time by David Marsh from the University of Jyväskylä, Finland in 1994. This author claimed: "CLIL refers to situations where subjects, or parts of subjects, are taught through a foreign language with dual-focused aims, namely the learning of content and the simultaneous learning of a foreign language" [9].

The European Commission has been looking into the state of bilingualism and language education since the 1990s, and has a clear vision of a multilingual Europe in which people can function in two or three languages. Languages will play a key role in curricula across Europe. Attention needs to be given to the training of teachers and the development of frameworks and methods which will improve the quality of language education [1].

In the period of globalization the exchange mobility of university students and teachers are becoming common. English language is being used as lingua franca. The foreign students coming to study to the Slovak University of Agriculture usually choose English as the language to acquire knowledge in subjects and also for communication with peers and teachers.

The principal objective of CLIL methodology is to enable students the achievement of a high level of communicative competence, i.e. accuracy and fluency, and at the same time to present them content of particular subjects with the emphasis on specialized vocabulary in English.

The expression ´4Cs´ (Communication, Cognition, Culture, Content) is also being used for the definition of CLIL methodology. Content matter does not mean only acquiring knowledge

and skills; it also involves the learners creating their own knowledge and understanding and developing skills (personalized learning). Cognition comprises the content related to learning and thinking. If the learners want to achieve the ability to interpret the content, they have to analyze it for its linguistic demands; thinking processes (cognition) need to be analyzed in terms of their linguistic demands. Language needs to be learned and practiced through communication, reconstructing the content and its related cognitive processes. The interaction in the learning context is essential to learning. The relationship between culture and languages is evident. The acquisition of a foreign language always implies the intercultural awareness. The CLIL methodology develops certain aspects of language competence, predominantly listening and reading, vocabulary, grammar, in particular morphology, fluency of the spoken language, creativity, critical thinking and reasoning. This methodology also motivates students to concentrate on both a foreign language as well as content of a subject matter. Still another aspect is the increased learners´ competence. On the contrary, CLIL requires more preparation, special skills and competences from teachers [3].

CLIL increases students` motivation towards learning foreign languages. It also leads to the cognitional development, i.e. knowledge acquired in one context can be transferred and utilized in another context. CLIL groups can achieve a higher level of foreign language competence.

One of the important tasks of education should be the increase in students' motivation to learn. Based on the obtained results it can be concluded that learning objectives are not clear to students, who probably assume that the study subject is not needed for their further education and professional application [4]. Pokrivčáková [12] states that there is no question about CLIL and the fact that nowadays it represents the most effective preparation of graduates to meet the requirement of the multilingual European labor market as well the international study opportunities. According to Kováčiková [6] it seems that ESP (English for Specific Purposes) can work under CLIL methodology, using first of all contents of some other subjects via methods reinforcing language communicative competence, such as projects, presentations, reports, etc. Then, the students can acquire the specialized vocabulary needed for their branch of study as well as the language competence in everyday situations or professional talks.

MATERIAL AND METHODS

In the academic year 2015/2016 we carried out the survey at the Faculty of European Studies and Regional Development. The survey was targeted at the comparison of acquired knowledge of the selected tasks at Mathematics and the English language in the first year of the bachelor degree.

We obtained the data about students' knowledge from the teaching of the obligatory subject `Mathematics` (taught in the 1st study year of bachelor degree). The range of hours in one semester is: 1 hour of lecture per week and 3 hours of practical seminar per week. We summed up the results from the students' partial tests during semester and final exam tests. We focused on the basic topics of this mathematics course, which were put into the following pairs of the representative tasks for each student:

- Task 1: Function with one real variable; Task 2: Derivative of a function,
- Task 3: Matrices; Task 4: Systems of linear equations.

The next data were obtained from the partial and final tests in English, particularly in the subjects `Communication in English` and `English for Specific Purposes` (ESP). Given test tasks were in accordance with the study content of these subjects. The range is two hours of practical seminar per week. We focused on the basic topics of these language courses, which were set into the following pairs of tasks for each student:

- Communication in English: testing vocabulary by multiple choice (Task 1 – agricultural terminology; Task 2 – economic terminology),
- ESP: testing specialist vocabulary by matching (Task 3 – Food and Agriculture Organization; Task 4 – accounting).

The existence of dependences and statistical significance among data will be examined by statistical testing methods: F-test and parametric two-sample t-test. The detailed description of these methods can be found in [7]. We will test the null hypothesis providing that the level of students´ knowledge is the same, when testing the representative tasks at Mathematics and English. In order to test the null hypothesis we will use the two-sample test of equality means. At first we verify by F-test if both random choices have the same variability and then we apply t-test. The calculations and the graphic interpretation are executed via the table processor MS Excel 2010 which has the given testing functions installed.

RESULTS AND DISCUSSION

Teaching a subject in a foreign language requires further study of teachers not only in their own subject area but the language improvement is also essential. Teaching course in a foreign language constitutes a certain specialization which contributes to the increase of quality of the university education. Still another opportunity to utilize the acquired competences in this area is the possibility to participate in the students` and teachers` mobility projects abroad.

Table 1 Results of paired two sample *t*-test for means in Mathematics, n = 61

Variables	**Task 1**	**Task 2**	**Task 3**	**Task 4**
Mean	4.11	5.51	6.88	7.10
Variance	7.00	6.89	5.14	5.02
Pearson Correlation	0.911**		0.941**	
Hypothesized Mean Difference	0		0	
Df	60		60	
t – test	- 9.763		- 2.142	
P(T ≤) two-tail	5.257E-14		0.036	
t Critical value	2.000		2.000	

Source: authors' calculations

The correlations between Task 1 and Task 2 and also between Task 3 and Task 4 were statistically highly significant. The differences between studied variables are significant at the significance level $\alpha = 0.05$.

In the Figure 1a and Figure 1b we present box-plots of the achieved points in math tasks.

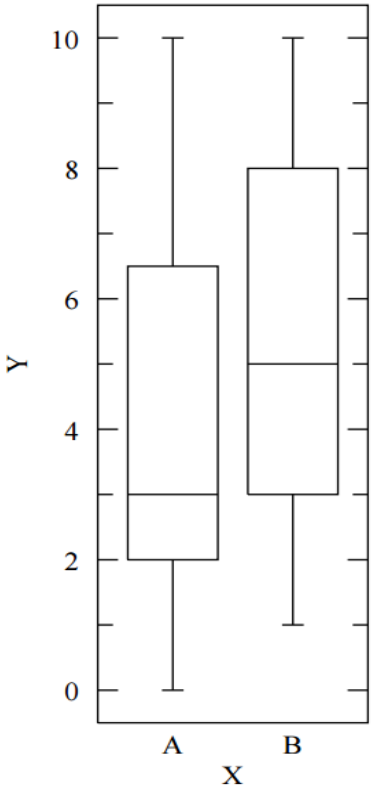

Figure 1a Box-plots of the achieved points in tasks Functions – Derivatives

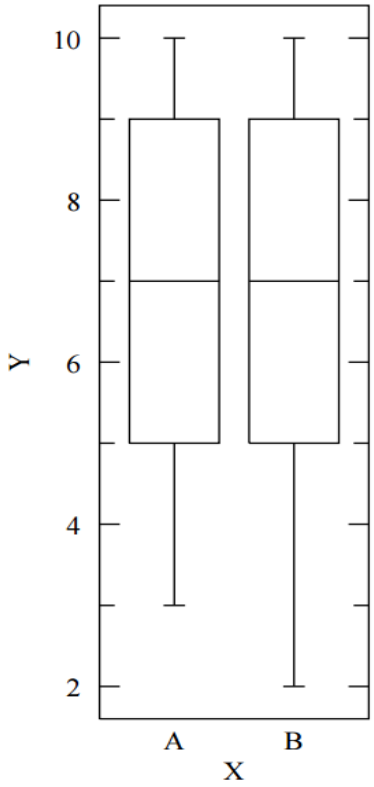

Figure 1b Box-plots of the achieved points in tasks Matrices – Systems of equations

Table 2 Results of paired two sample *t*-test for means in the subject Communication in English and in the subject English for Specific Purposes, n = 41

	Communication in English		**English for Specific Purposes**	
Variables	**Task 1**	**Task 2**	**Task 3**	**Task 4**
Mean	8.00	8.02	8.07	7.76
Variance	7.00	6.62	5.12	4.24
Pearson Correlation	- 0.154		0.347*	
Hypothesized Mean Difference	0		0	
Df	40		40	
t test	- 0.039		0.820	
P(T ≤ t) two-tail	0.969		0.417	
t Critical value	2.021		2.021	

Source: authors' calculations

The correlation between Task 3 and 4 was statistically significant. The differences between studied tasks are not significant at the significance level $\alpha = 0.05$.

In the Figure 2a and Figure 2b we present box-plots of the achieved points in tasks from the language tests.

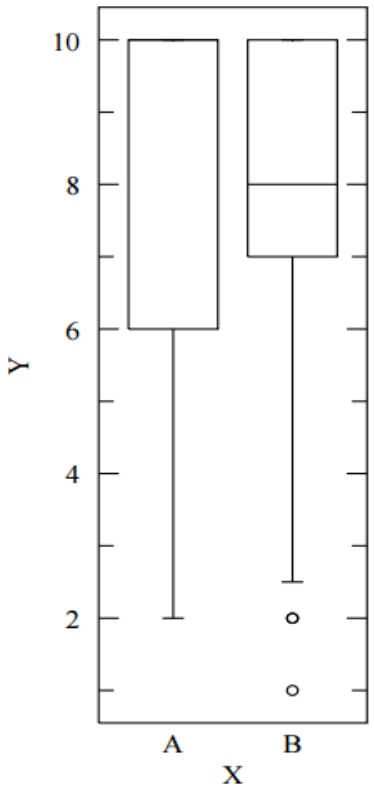

Figure 2a Achieved points in tasks Communication in English

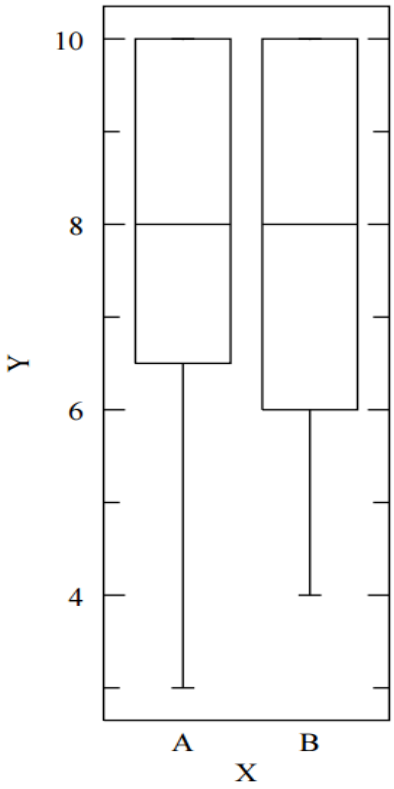

Figure 2b Achieved points in tasks English for SP

The European Commission states the benefits of CLIL as following:

- Develop intercultural communication skills;
- Prepare for internationalism;
- Provide opportunities to study content through different perspectives;
- Access subject-specific target language terminology;
- Improve overall target language competence;
- Develop oral communication skills;
- Diversify methods and forms of classroom practice;
- Increase learner motivation [3].

The positive impact of the CLIL on the mathematical performance of pupils was identified even after a short period of time [13]. Therefore teaching specialist subjects in a foreign language represents the beneficial space for the unification of a language and professional knowledge in a particular subject [10].

In other studies, the question of teacher quality in terms of language teaching and professional subject is examined; they say that the optimal solution is the cooperation of a language teacher and a teacher of the professional subject with a native speaker [2].

CONCLUSIONS

In the contemporary society the university education is the important factor for employment opportunities on the labor market. In the research we analyzed knowledge and skills level of students through their outputs from the partial tests in the following subjects: Mathematics, Communication in English and English for Specific Purposes. We concentrated our attention on the groups of students from the Faculty of European Studies and Regional Development and on the study program taught in English.

The current trend at universities is to offer to students the study programs with the subjects taught in English. The education of the future experts and scientists in the field of agriculture, biology, crop and animal production, food industry is important from many aspects. The agricultural science, research and practice require a good command of foreign languages and mathematical statistical methods.

The results of Mathematics confirm that differences between tasks of the analyzed topics (matrices and systems of linear equations, functions and derivative of a function) are significant. According to the research results we can state that there are the significant differences in students' knowledge in topics matrices and systems of linear equations. Next, we found out that the significant differences exist in students' knowledge in topics function property and function derivative.

The results of language skills of the tested students demonstrate that the differences between analyzed tasks are not significant. Based on the gained analysis and results we can conclude that students at the SUA in Nitra have the adequate abilities to achieve a command of the English language on a very good level. The cross-curricular relations could help students to understand and combine knowledge of different areas, and consequently improve the quality of university education.

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Original Paper

Mapping of migration in Slovakia

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ABSTRACT

People have been migrating for work and better life and social environment for years. Nowadays migration is taking on newer dimensions. We are especially speaking of economic migrants as well as about people from so called third countries, in which living conditions have worsened due to impact of climate changes and worse social conditions caused by higher unemployment as well as unstable political affairs and wars. In the paper we define migration, state its causes, negatives and positives. In the main part of the paper we examine how the number of migrants in Slovakia has changed lately and from which countries they come from. We found out that ratio of foreigners living in Slovakia has been changing. A couple of years ago people from European countries prevailed in Slovakia in comparison with people from so called third countries, now the ratio is becoming equal.

KEYWORDS: migration, number of foreigners, valid stay

JEL CLASSIFICATION: A10, J10, J11, J15

INTRODUCTION

"Globalization creates a situation of freedom, better condition for free activities in a globalizing world economy, enabling for entrepreneurial subjects to reach economic growth and furthermore also the growth of life quality. At the same time, it brings the whole scale of phenomena reasonably complicating a positive development of humankind on the planet." [6]. One of these is migration considered as one of the main global issues of the twenty-first century. Nowadays, vast mobility of people is bigger than at any other period of the modern history. International Organization for Migration (abbr. IOM) finds it important to manage and use development potential of migration in favor of individual migrants as well as in favor of society. If international migration is well managed it can contribute to prosperity and growth of countries of origin as well as target countries and it might be contribution to the migrants, too.

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Characteristic feature of the modern world is mobility. Integrated world markets, creation of multinational companies, quick development of communication technologies contribute to more intensive movement of students, tourists, highly qualified as well as lower qualified labour force and so forth. By changing demographic and social structure of industrialized world, the need for accepting workers and professionals from other countries emerges. Economies which want to remain competitive cannot ignore the need for changes. Therefore supporting migration for work might be the solution.

Unguided migration can result in higher social and financial costs not only on individuals but on the whole society as well. Complex and transparent attitude towards managing migration can be helpful in combat against illegal migration, minimalize negative impacts of migration and prevail integrity of migration as natural social process. Replacing illegal migration by organized so called legal migration must be priority of all the governments.

MATERIAL AND METHODS

Under the term migration we understand any movement of people regardless of reason, length or character. Liďák [2] describes migration as changes in population status of the country depending on spatial motion within which they change permanent address. In case of a movement of a person or a group of people, we speak about internal migration, if the movement is across the international boarders, we speak about international migration. It might be a movement of economic migrants, refugees, resettled person or any other movement from various reasons like family reunion, studies, etc. According to the definition of UNO an international migrant is a person who changes the country of a permanent stay. Svitačová and coll. (2014) in [5] states that migrant flow expresses number of people who in some period of time cross the boarders of a country with the aim to settle in another country.

Considering the time perspective we speak of long-term migration if the period of migration lasts for at least 12 months. If this period takes from 3 to 12 months we speak of short-term migration. An exception might be recreation, holiday, visits at friends´ and relatives, business, health care or religious pilgrimages.

The crucial expressions of the topic are defined as follows.
A migrant is a person who leaves a region or a country with the aim to settle in another region or a country.
A refugee is, according to the Geneva Agreement from 1951, a person who has legitimate concerns about persecutions from national, racial and religious reasons or because of citizenship in specific social group, or maintains specific political opinions. The person might be without citizenship who encounters out of the state of their permanent stay due to impacts of these events and who cannot stay there or does not want to go back dues to given concerns. The Slovak law order does not define the expression refugee.
A foreigner is, according to the Slovak legislation, every person who is not a citizen of the Slovak Republic.
A national of the third country (from Slovakia´s point of view) is every person who is not a citizen of the Slovak Republic nor citizen of the European Union nor Switzerland. It might be a person without citizenship as well.
An asylant is a foreigner to whom an asylum was given. According to Svitačová and coll. [5] an applicant for asylum is „a person who asks for asylum according to the Geneva Agreement

and legislation of the given country. In case of accepting the asylum the person becomes asylant."

What leads people to migrate? The main reason of migration might be so called PUSH factors and PULL factors. Among the *PUSH factors* there are especially negative economic and social environments, lack of work, environmental purposes, political instability, growing density of population, racial, ethnical, religious repressions, civil riots, armed conflict, wars etc. Among the *PULL factors* there are better living conditions, especially regarding social system, health care, education, safety, human rights, as well as higher salary and higher standard of living, work and business opportunities and moreover contacts abroad, so called migration networks.

Migration brings positives and negatives [1].
The positives of migration are:

- multiculturalism,
- equalization of demographic conditions,
- diversification of market,
- coverage of lacking professions,
- creation of new working positions,
- decreasing unemployment in the country of origin,
- transfer of know-how
- cheaper goods and services,
- GDP growth of the host country and so forth.

The negatives of migration are:

- brain-drain of qualified employees from the country of origin,
- higher unemployment,
- decreasing of salary,
- less stable labour force,
- higher social tense,
- xenophobic attitude of the public,
- potential safety threats - terrorism, racism, trafficking in human beings, peddling, crime growth etc.

„The issues" of migration are tackled by the International Organization for Migration (IOM). It is the international intergovernmental organization which was created in 1951 in Geneva. Its original mission was to help number of relocated people and refugees who, after a war, found themselves out of their homes in various European countries. During the last few decades after the Second World War, the number of migrants grew significantly and migration became one of the most significant social issues which influence society. In relation to this, spectrum of IOM activity gradually grew as well. Nowadays this organization deals with various areas of migration management throughout the world. In 1996 the agreement between the Slovak government and IOM was signed, based on which the IOM Office in Bratislava was founded.

Nowadays we assume that there are approximately 232 million of international migrants in the world which means that each 31st person in the world is presently a migrant living beyond boarders of their home country. We also assume that the number of internal migrants, that is migrants within the country of their citizenship, is 763 million. If we sum up this number with the number of international migrants it represents almost one billion of people. That is, each

seventh person in the world is a migrant. As for the attraction, migrants have especially interest in Europe. There are approximately 72 million of international migrants which is almost one third of all the migrants. Out of this sum, more than two thirds of migrants in the EU live in five countries that is Germany, Spain, Great Britain, Italy and France.

In the article we work with data gained from the database of Statistical Office of the Slovak Republic (SO SR) [7], [8] and from the Yearbooks of Ministry of Interior of the Slovak Republic [3], [4]. We use table processor Microsoft Excel 2013 and then analysis of the data thus obtained processed into spreadsheets and graphs.

RESULTS AND DISCUSSION

In Slovakia various stereotypes and myths about migration are widely spread. The most common examples are as follows.

- Foreigner is the same as refugee.
- Foreigners are economical load for the state budget.
- Foreigners take job opportunities of Slovaks.
- Foreigners spread dangerous diseases.
- Crime of foreigners is growing.
- Slovakia is as a poor country still uninteresting for foreigners and will remain transit country not a target country.
- In Slovakia there is a huge number of foreigners.

Of course there are exceptional cases to which these myths do not apply but that does not make us generalize the issue. Now we would like to deal with the myth of the last dash. In Slovakia the number of foreigners is relatively small, on December 31st, 2016 the number was 93 247 people. It is one of the lowest proportions of the foreigners compared to the overall number of inhabitants (5 435 343) which is only 1.72 % among the member countries of the European Union. Lower number is in Hungary (1.42 %), in Bulgaria (0.76 %), Croatia (0.75 %), Lithuania (0.74 %), Romania (0.37 %) and the least foreigners are in Poland (0.27 %). In the Czech Republic it is 4.12 % and in Austria up to 12.31 %.

As mentioned above in Slovakia on December 31, 2016 there were 93 247 foreigners of which 41 232 were citizens of the third countries and 52 015 were citizens of the EU. According to the size of civic communities it looks like it is shown in the Table 1. As the table shows, there are most people from Ukraine, the Czech Republic and Hungary. In the next Table 2 we can see data about number of citizens of the Slovak Republic, number of foreigners and the share of foreigners to the overall number of citizens from 2004 to 2016. Even though the increase in number of citizens in Slovakia represents almost 1 %, within the foreigners this number is a lot higher, exactly 4.22 times higher. In 2004 the share of foreigners to all the citizens of Slovakia represented only 0.41 %, in 2016 1.72 % already. Even from the Figure 1 we see that this trend has rising tendency and it is assumed that it might grow further.

Table 1 First 10 biggest civic communities in Slovakia in 2016

Ranking	Country of origin	Number of foreigners
1.	Ukraine	13 024
2.	the Czech Republic	10 317
3.	Hungary	7 813
4.	Serbia	7 232
5.	Romania	6 907
6.	Poland	5 562
7.	Germany	4 380
8.	Russia	4 035
9.	Italy	2 757
10.	Vietnam	2 580

Source: A yearbook of Ministry of Interior of the Slovak Republic [4], own elaboration

Table 2 Number of foreigners living in Slovakia up to December 31, of the given year

	2004	**2007**	**2009**	**2011**	**2014**	**2016**
Overall number of citizens of Slovakia	5 384 822	5 400 998	5 424 925	5 404 322	5 421 349	5 435 343
Overall number of foreigners	22 108	41 214	58 322	66 191	76 715	93 247
Share of all foreigners on the overall number of citizens in Slovakia	0.41 %	0.76 %	1.08 %	1.23 %	1.41 %	1.72 %

Source: Data form [3], [4], [7], [8], own elaboration

A stay of foreigner on the area of the Slovak Republic can be:

- temporary,
- permanent,
- tolerated.

Temporary stays enables a citizen of the third country stay, travel and repeatedly enter the area of the Slovak Republic in a specific time period given by the police department. The purpose might be business, employment, study, research and development**,** reunion of the family and so on.

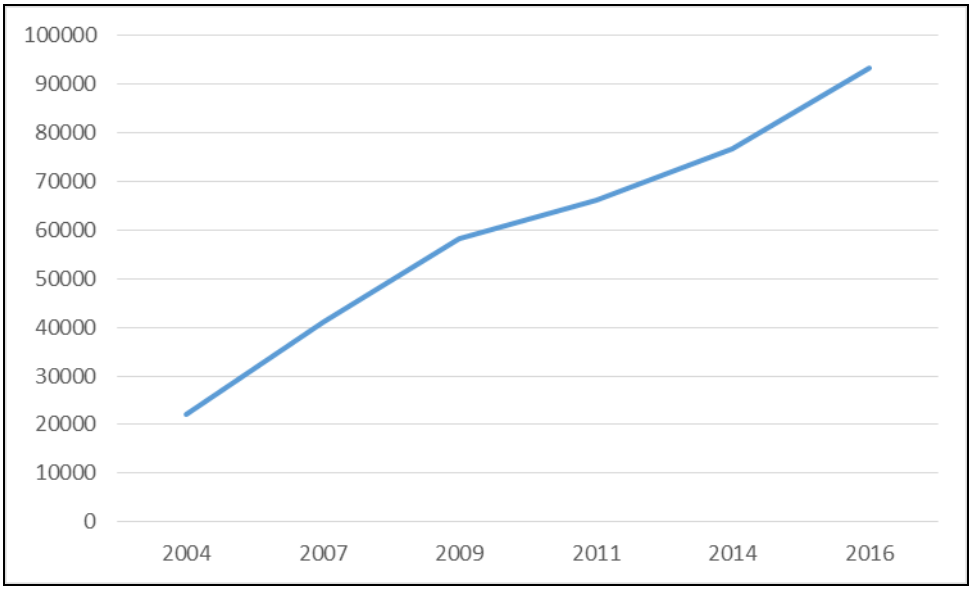

Figure 1 Overall number of foreigners from 2004 to 2016
Source: Data from [3], [4], own elaboration

Permission for a permanent stay enables the citizen of the third country stay, travel and repeatedly enters the area of the Slovak Republic in a specific time period given by the police department unless the law provides otherwise.

The police department grants a stay to a citizen of the third country if there is no reason to deny a request in such way that it does not fulfil the conditions for granting the tolerated stay, if there is an obstacle of their administrative expulsion, if they were given permanent shelter or if their travel is not possible and their restraint is not effective or any other reasons.

In the following part we will do a research on the number of foreigners living in Slovakia for the past 4 years, i. e. from 2014 to 2016. Table 3 shows number of citizens from the third countries, citizens of the EU countries and all the foreigners together living in Slovakia for the past 4 years. As the Figure 2 shows in all the categories in given years there is a significant growth in the number of foreigners.

Table 3 Number of accepted stays up to December 31, of the given year

	2013	**2014**	**2015**	**2016**
Citizens of the third countries	26 157	29 171	35 261	41 232
EU citizens	45 492	47 544	49 526	52 015
Sum	**71 649**	**76 715**	**84 787**	**93 247**

Source: Data from [3], [4], own elaboration

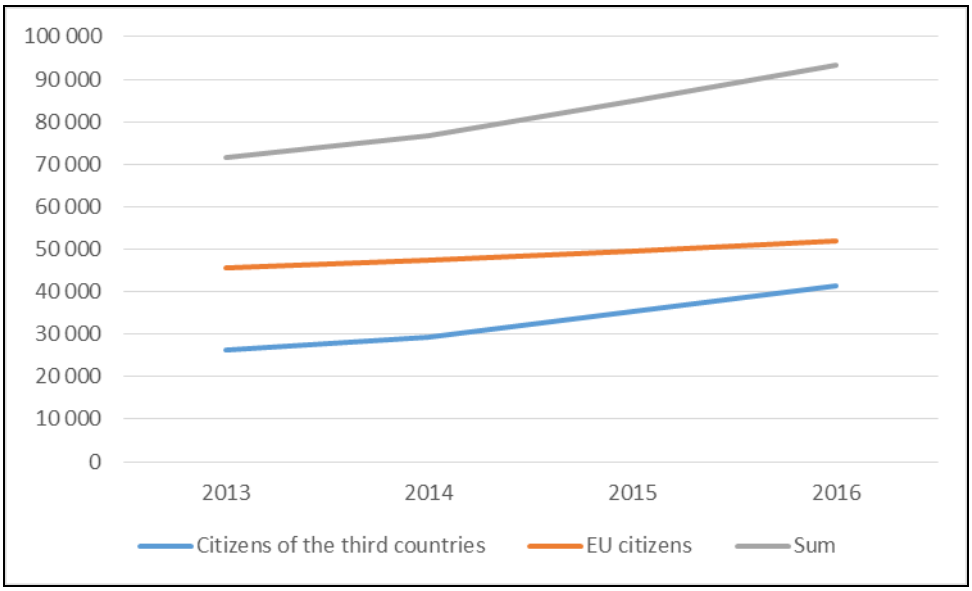

Figure 2 Number of accepted stays for foreigners up to December 31, from 2014 to 2016
Source: Data from [3], [4], own elaboration

The next Table 4 shows number of accepted stays for citizens of the third countries up to December 31, from 2014 to 2016. We selected the countries of which the number exceeds at least 1000 people. In 2016 even inhabitants of Iran were included with the number of citizens 1056.

Table 4 Numer of accepted citizens of the third countries up to December 31, of the given year

Citizenship	**Number of stays**			
	2013	**2014**	**2015**	**2016**
Ukraine	6898	8033	10706	13024
Serbia	4021	4648	5528	7232
Russia	2633	2976	3532	4035
Vietnam	2089	2180	2307	2580
China	1926	2024	2134	2346
Korea	1528	1557	1590	1638

Source: Data from [3], [4], own elaboration

As we can see from the Table 4 and from the Figure 3, the biggest growth of accepted stays of foreigners – citizens of the third countries, the most of them are from Ukraine and Serbia.

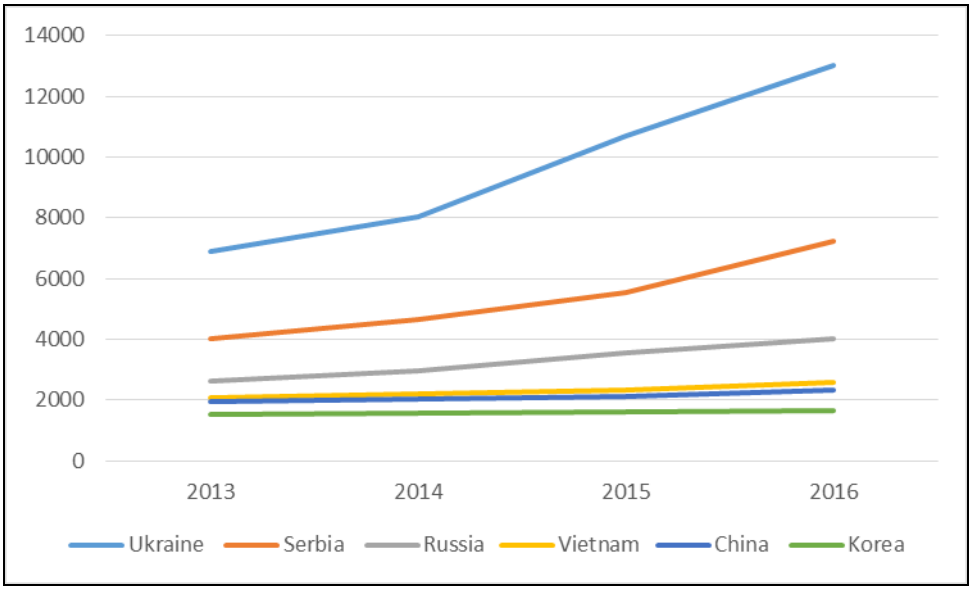

Figure 3 Number of accepted stays for citizens of the third countries up to December 31, from 2013 to 2016
Source: Data from [3], [4], own elaboration

CONCLUSIONS

As it turns out from the previously mentioned parts of the article, migration belongs to one of the main issues of modern globalized world. This does not apply to the most developed countries of the European Union like Germany, France and Great Britain but Slovakia is concerned about this issue as well. As we have shown, the number of immigrants in Slovakia for the past years is still growing whereas the numbers of citizens of the EU with the citizens of the third countries is almost equal. In 2014 it was 63.5 % compared to 36.5%, in 2016 it was 55.8 % compared to 44.2 %. Thanks to the legislation of Slovakia and attitude of the government, the numbers of immigrants are relatively low compared to other countries.

Not to forget that environment has a long-term influence on migration flows. People have long left areas with inappropriate or worse natural conditions. We are expecting that migration caused by the changes in environment will grow. Influence of climate changes and worse environment will, for migrants, mean a survival.

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Original Paper

High frequency trading regarding trade in goods of agricultural origin

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ABSTRACT

This work presents the usage model in high frequency trading mechanisms concerning goods of agricultural origin. The aim of this study is to verify this type of solution on the basis of historical data covering the period from April 1st, 2013 to April 30th, 2014. The most common ratios, i.e. Annualised Return, Investment Risk, Maximum Drawdown, as well as Information Ratio have been chosen as the measures of strategy usefulness. Options varying in both the horizon and the time interval have been considered. The best results have been achieved with respect to the historical time horizon for appointing the benchmark proportion of securities and exceeding one day, as well as one-minute time interval. It is worth emphasizing that the investment issue constitutes a very complex problem influenced by a large number of factors. Owing to this fact, no universal mode of conduct guaranteeing profits may be unequivocally indicated. It is possible only to define a scenario, which shall prove effective with a substantial degree of probability.

KEYWORDS: frequency trading, trade, origin, simulation, model

JEL CLASSIFICATION: Q02, Q13, Q17

INTRODUCTION

Any human activity makes a permanent process of decision making. The choice of the appropriate strategy constitutes an essential factor contributing to the future development and its rate. Nowadays, farmers worry not only about what shall be produced, but also about when, to whom and in which form goods shall be sold. Therefore, they are increasingly benefiting from forms of securing themselves against a future change in prices or exchange rates. Such a solution allows farmers to focus all their efforts on production, not on time-consuming searching for customers. Including future contracts constitutes one form of such a security, was presented by Dunis and others [7]. This work presents a mechanism of further trading in such a financial instrument, i.e. algorithmic trading, and more specifically pair trading - one of its forms, both of which do not require the farmer's direct time involvement. .

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One example is the method of pairs trading. This approach allows for the construction of a trading strategy where trades are entered when the process reaches an extreme value and exited when the process reverts to some equilibrium value. This technique was developed in 1980 by a team scientist lead by a Wall Street quant Nunzio Tartaglia [8]. Since then in the world published many bibliographic items describing strategies based on pairs trading. Huck [10] proposed a general and flexible framework for pairs selection. He used multiple return forecasts based on bivariate information sets and multi-criteria decision techniques. Bertram [2] presented analytic formulae and solutions for calculating optimal statistical arbitrage strategies with transaction costs. The author assumed that the traded security had been described by an Ornstein-Uhlenbeck process. Broussard and Vaihekoski [3] investigated the practical issues of implementing the self-financing pairs portfolio trading strategy. Using data from the Finnish stock market over the period 1987–2008, authors found pairs trading to be profitable even after allowing for a one day delay in the trade initiation after the signal. Tourin and Yan [13] proposed a model for analyzing dynamic pairs trading strategies using the stochastic control approach. The model was explored in an optimal portfolio setting, where the portfolio consisted of a bank account and two co-integrated stocks and the objective was to maximize for a fixed time horizon, the expected terminal utility of wealth.

The impact of time on making decisions concerning efficiency of high frequency trading on the basis of macroeconomic data was researched, among others, by Scholtus and others [12], who had found out that delay of only a few seconds in decision making, could significantly influence the financial result of the adopted strategy. At the micro scale the research, which detected volatility dynamics and the level of financial time series complexity, was conducted by Aghamohammadi and others [1]. In 2009 Izumi and others used computer simulations to evaluate the investment strategy. The authors found out that not only the strategy itself but also the way it is connected with other strategies, i.e. dependance on other players' moves, influences the achieved result. Chang and co-authors [4] used artificial neural networks for detecting purchase and sale signals on the securities market. The technical analysis combined with monitoring sale price/volume diagrams, in which artificial neural networks were applied for choosing the purchase/sale moment, was presented by Chavarnakul and Enke [5]. Gradojevic and Gencay [9] adopted fuzzy logic for the needs of evaluating the investment risk and choosing a strategy. Fuzzy sets were also used in Chang [4] and others with respect to cyclical stock exchange investments. The cluster analysis with the help of the Support Vector Machine (SVM - the method of machine learning) constituted the subject of research conducted by Choudhury and others [6]. In 2013 genetic algorithms also constituted the subject of Mabu and co-authors' work. Here the team's work was concentrated on using the GAs for the needs of decision trees. Kluger and McBride [11] showed implementation of agent-based systems for exploring investment models in intraday investments.

The pork contract - corn contract pair of instruments listed on the Chicago Mercantile Exchange (CME) over the course of one year (from April 1st 2013 to March 31st 2014) constitutes the subject of research. These securities are listed in electronic continuous sessions and are characterized by high liquidity.

MATERIAL AND METHODS

Objectives and Scope of Work

The main aim of this study is to verify the usefulness of the mechanism of statistical arbitrage [7] concerning trade in goods of agricultural origin in High Frequency Trading.
The scope of work includes:

- performance of the investment algorithm prototype based on the mechanism of statistical arbitrage with its implementation in the Matlab environment;
- presentation of investment details regarding quick purchase/sale mechanisms;
- research of the influence of the time interval and time horizon length on quality of the strategy;
- determination of the most important statistics concerning respective options, i.e. Annualised Return, Investment Risk, Maximum Drawdown and Information Ratio (IR).

The pork contract - corn contract pair of instruments listed on the Chicago Mercantile Exchange (CME) over the course of one year (from April 1st 2013 to March 31st 2014) constitutes the subject of research. These securities are listed in electronic continuous sessions and are characterized by high liquidity.

Methodology of work

Attempts to use temporary price fluctuations of the respective products have been included in the statistical arbitrage in high frequency trading in goods of agricultural origin. The assumption that relations between prices of products belonging to the same sector of economy should be characterized by stability for a long time has been the starting point. The relation stability is understood here as the quotient of prices of two products. In order to determine the investment strategy, the most important thing to do has been to calculate the ratio of goods prices ensuring their balance, as well as the sizes of deviation from the balance level, constituting signals of overvaluing one and undervaluing the other commodity. However, the first step in practical applications is to check the co-integration.

Engle-Granger Co-integration Model

Co-integration is the attribute of time series used in econometrics, which takes place when two series are not stationary but their linear combination is stationary, i.e. irrespective of time. The term was suggested and introduced into use by the econometricians Robert Engle and Clive Granger in the article published in 1987 under the title *Co-integration and Error Correction: Representation, Estimation and Testing*. They were, among other things, awarded the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 2003 for this achievement.
While creating the co-integration model, one should at first create pairs of instruments (most often securities from the same sector, e.g. cereal prices, meat prices, coffee and tea prices, etc.), and then their long-term balance is checked with the help of the co-integration vector, i.e.:

$$Y_t = \beta X_t + \varepsilon_t$$

where
Y_t - dependent variable,
X_t - independent variable,
β - co-integration (similarity) coefficient,
ε_t - random error.

Owing to the fact that estimation of the co-integration coefficient is burdened with an error, adaptive methods of smoothing are used in practical applications:
$Y_t = \beta_t X_t + \varepsilon_t$ and $\beta_t = \beta_{t-1} + \eta_t$
where η_t - is the random error in the Wiener process.

RESULTS AND DISCUSSION

For the purpose of this work the corn and pork contracts listed on the Chicago Mercantile Exchange have been adopted for analysis (as co-integrated commodities). The analysis of research results shall begin with presenting possibilities of the statistical arbitrage algorithm in high frequency trading. The following has been adopted as the input data in table 1.

Table 1 Simulation Input Parameters

Parameter	Value
startDate	07-04-2014
start time	09:30
endTime	16:00
closeTime	15:45
N	1
sym_len	2
Interval	1
Capital	10000
Times	10
oSTreshold	-1
oLTreshold	1
cTreshold	0.5

According to the procedure presented in the section concerning work methods and the operation of the algorithm prototype itself, April 7th 2014 was treated as the model day for determining the constant proportion between pork and corn. Investing started only on April 8th 2014. Due to the fact that the value of the *Sym_len* parameter is 2, the strategy covers also the following day, i.e. April 9th. Investing took place from 09.30 a.m. to 4.00 p.m. of the local time where the time interval of 1.10 was adopted as the minimum number of units of the respective commodities. The initial capital was established at $ 10,000.00. The input parameters were set in a way that the result presentation was as transparent as possible, thus, the number of units were significantly decreased in relation to the actual ones. Opening positions were set to the value of the standard deviation, whereas closing - to their half. The actual exchange quotations of pork and corn are presented in Figure 1.

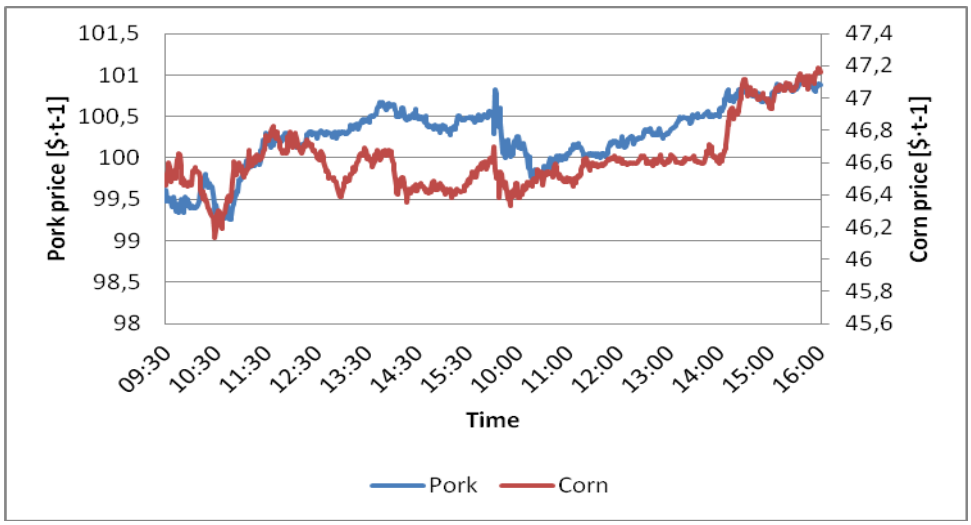

Figure 1 Pork and corn quotations in the successive time moments

The signals of position opening (closing) have been calculated on the basis of these data (Figure 2).

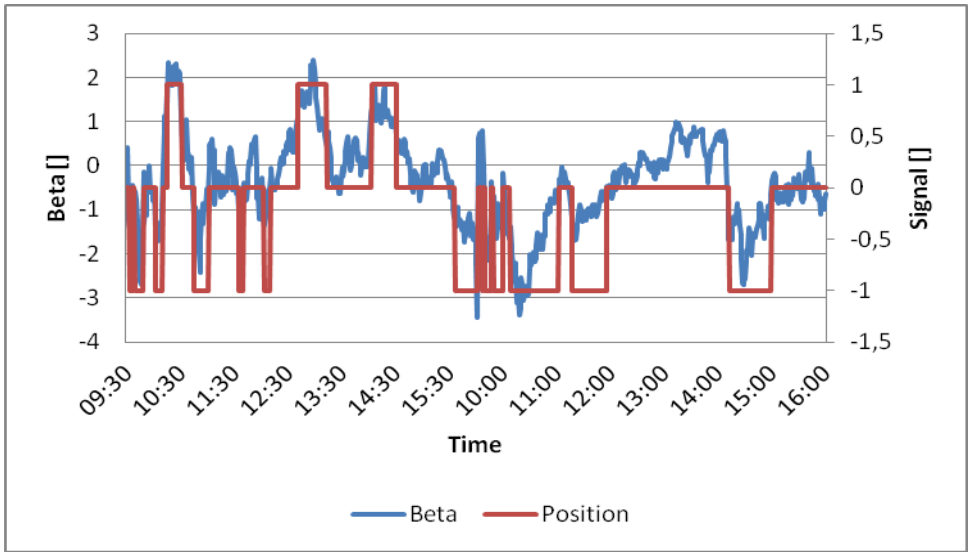

Figure 2 Standardised differences of the forecast value and the actual proportion of goods prices (Beta) and the position held by the investor

In order to illustrate the operation of the investment strategy, the original time moments have been collected in Table 2.

At 09:33 the standardised Beta parameter value, i.e. the difference between the forecast and real ratio of the pork price to corn price, decreased below the *oSTreshold* level, but since it was only the first occurrence, the investor took next actions. At 09.34 the Beta coefficient value decreased below the *oSTreshold* level again, but because it was the second consecutive occurrence, the investor decides to purchase pork counting on its momentary weakness, by speculating its disproportionate increase in relation to the corn price.

Table 2 The example of investment strategy operation

Time	Corn	Pork	Portfolio	Capital	Beta	Position	Number of units	
							Corn	Pork
9:33	46.52	99.47	10000	10000	-0.8791	0	0	0
9:34	46.4597	99.61	10000	10000	0.4173	0	0	0
9:35	46.53	99.4	10000	10000	-0.8960	0	0	0
9:36	46.555	99.4	10000	10000	-1.2501	0	0	0
9:37	46.6	99.51	364.17	9999.92	-1.4967	-1	-100	50
9:38	46.57	99.51	364.17	10002.92	-1.1868	-1	-100	50
9:39	46.515	99.50	364.17	10007.92	-0.6677	-1	-100	50
9:40	46.48	99.47	10009.89	10009.89	-0.4470	0	0	0
9:41	46.52	99.4	10009.89	10009.89	-1.1788	0	0	0
9:42	46.55	99.4	381.31	10009.81	-1.1977	-1	-100	50
9:43	46.5199	99.53	381.31	10016.02	-0.5782	-1	-100	50
9:44	46.5	99.45	381.31	10014.01	-0.7594	-1	-100	50
9:45	46.5	99.350	381.31	10008.82	-1.2606	-1	-100	50
9:46	46.56	99.350	381.31	10002.82	-1.8788	-1	-100	50
9:47	46.62	99.3	381.31	9998.31	-2.3494	-1	-100	50
9:48	46.63	99.3	381.31	9995.31	-2.6406	-1	-100	50
9:49	46.66	99.4	381.31	9997.81	-2.4136	-1	-100	50
9:50	46.65	99.36	381.31	9994.56	-2.7147	-1	-100	50
9:51	46.58	99.4	381.31	10007.31	-1.4317	-1	-100	50
9:52	46.47	99.5	10019.73	10019.73	-0.1510	0	0	0

The investor simultaneously decided to sell corn for a short while (currently the investor does not own this commodity but is bound with the brokerage agreement concerning this type of operation, according to which the only action the investor is required to conduct is to provide the broker with the appropriate amount of corn until the end of the session day). In the event of any signal of position opening, the capital available for purchase (and short sale) was divided into two equal parts and then the number of possible commodity units (the block multiple - 1 block constitutes of 10 tonnes) was calculated. In that case it was possible to purchase 5 blocks of 10 tonnes of pork (at the price of $ 995.15 per each block) and to conduct short sale of 10 blocks of 10 tonnes of corn (at the price of $ 466.00 per each block). After the operation and deduction of transaction costs, $364.17 remained available. Three minutes later the standardised Beta value increased above - 0,5. Thus, the investor was able to take the position. The investor sold pork at the price of 99.476 per tonne (so, in case of traditional investing the investor would have suffered a loss) and repurchased corn at the price of 46.48 per tonne (here the investor made a profit by conducting sale at a higher price than purchase). As a result of both transactions the investor had earned $ 9.89 in total. The next position- opening- occurred at 09:39. However, that time the investor had to wait for the closing position until 09:49. Considering transaction costs, the investor's profit rose to $ 9.92. Although both speculations were right, the latter amount was only slightly higher than the previous one. Purchase of pork at $ 99.47 and sale at $ 99.51 and short sale of corn at $46.55 and purchase at $ 46.47. Additionally, the capital value (the sum of available cash extended by the commodity value) was calculated constantly, therefore, the investor was able to withdraw from investment, having recorded a sudden loss.

Table 3 The closing mechanism - a position before the end of quotations

Time	Corn	Pork	Portfolio	Capital	Beta	Position	Number of units	
							Corn	Pork
4:45	46.563	100.4(	382.515	10052.22	-1.4396	-1	-100	50
5:45	46.565	100.4(	10052.94	10052.94	-1.4522	-1	0	0

Table 3 presents the mechanism of operation of the system “closing” - the investor's positions at the end of the listing day. Although at 15:45 the Beta parameter value was below – *oLTreshold* (-0,5), the position “closing” took place as a result of the priority of the *closeTime* parameter (15:45).

CONCLUSIONS

As shown in the work, the model of the statistical arbitrage with high frequency trading in goods of agricultural origin may constitute an interesting alternative for traditional selling methods. The agricultural producer who decides to sell goods with the help of a future contract shall receive payment for it before delivering it to the stock exchange operator. The agricultural producer's further active participation in the goods trading, for which they have already received remuneration, makes a benefit of the presented solution.

Due to the verification of usefulness of the following stock exchange ratios: Annualised Return, Annualised Volatility, Information Ratio and Maximum DrawDown has been demonstrated that the model of algorithmic trading may be a good tool for conducting investments. It is characterised by significantly lower risk of loss suffering and drawdown.

Among the analysed time intervals (the one-, five-, ten- and fifteen-minute ones), the longest interval has resulted in the highest, 20% annualised return. A slightly lower annualised return has been recorded in case of the one-minute interval (nearly 18 %), whereas the other intervals have turned out to be significantly worse. None of these investments have been evaluated as good (IR < 0.5), although the one-, ten- and fifteen-minute intervals have undoubtedly been favourable. The high investment risk has had a negative influence on their evaluation. The relations between the time interval and the strategy quality cannot be clearly defined on the basis of the conducted analyses.

The significant improvement of the quality has been observed as a result of lengthening the time horizon. Owe to this procedure, each strategy has been evaluated as very good or unusually good. However, it has not been possible to specify the cause and effect relationship between the time horizon and the annualised return with the risk level yet.

Further analyses would require conducting research with the help of a multi-criteria optimisation method of the simultaneous influence of the time horizon length and the time interval. The level of opening and closing position (*oSTreshold, oLTreshold, cTreshold*) would be an additional element, which might have an impact on the investor’s profit maximisation. The increase in the number of paired securities (the choice of pairs made among a set of available pairs) could potentially ensure the improvement of quality.

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