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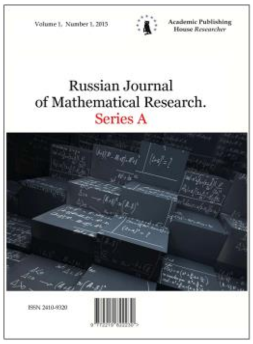

Proofs of Andrica and Legendre Conjectures

Samuel Bonaya Buya [a, *]

[a] Ngao girls' secondary school, Tana River County, Kenya

Abstract

In this research proof of Legendre conjecture is presented. The proof is based on a property possessed exclusively by all prime numbers. That is, the positive square-root of any prime number is an irrational number that always lies between two consecutive positive integers. This property excludes the number one from the set of prime numbers. Not all composite numbers possess this sure property possessed by all prime numbers. It is this special property of prime numbers special property of prime numbers that makes Legendre conjecture a sure law for all prime numbers.

In the process of seeking to prove Legendre's conjecture the prime gap problem is resolved and Riemann hypothesis reviewed in the light of these findings.

Keywords: proof of Legendre's conjecture, an exclusive property of prime numbers, number theory, Riemann hypothesis, on differences between consecutive primes.

1. Introduction

The properties of prime numbers have been studied for many centuries. Euclid gave the first proof of infinity of primes. Euler gave a proof which connected primes to the zeta function. Then there was the Gauss and Legendre's formulation of the prime number theorem and its proof by Hadamard and de la Vallee Poussin. Riemann further came with some hypothesis about the roots of the Riemann-zeta function.

Many others have contributed towards prime number theory.

Legendre's conjecture, proposed by Adrien-Marie Legendre states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers. Up to 2017 the conjecture had neither been proved nor disproved.

In this research a method will be presented of proving Legendre's conjecture. The proof is based on an exclusive properties of prime numbers not generally shared with composite numbers. A square root property of prime numbers will be discussed that also implies the truthfulness of Legendre's conjecture.

Relevance

The research aims at furthering our understanding of the prime gap problem as proposed in Legendre, Oppermann, Andrica conjectures and even Riemann hypothesis.

* Corresponding author
E-mail addresses: sbonayab@gmail.com (S. Bonaya Buya)

2. Discussion

Theorem

The square-root of every prime number is an irrational number in the gap between two consecutive integers.

Proof

First by definition a prime number is a natural number greater than 1 that cannot be formed by multiplying two smaller natural numbers each of which is greater than one. The primality property of prime numbers does not permit them to be factorized to two identical natural factors ($n \times n$), thus the square-root of a prime number is always an irrational number. All irrational numbers, including square-roots of prime numbers lie between two consecutive integers.

Thus:

$$1 < \sqrt{2} < 2$$
$$1 < \sqrt{3} < 2$$
$$2 < \sqrt{5} < 3$$
$$2 < \sqrt{7} < 3$$
$$3 < \sqrt{11} < 4$$
$$3 < \sqrt{13} < 4$$
$$4 < \sqrt{17} < 5$$
$$4 < \sqrt{19} < 5$$
$$4 < \sqrt{23} < 5......$$
$$n < \sqrt{p_i} < n+1$$

Deriving Legendre's conjecture

If the square-root of a general prime number p_i lies between two consecutive integers, n and n+1, then it follows mathematically that the prime number p_i lies between two consecutive square numbers. That is to say:

$$n < \sqrt{p_i} < n+1 \equiv n^2 < p_i < (n+1)^2$$

Proof of Legendre's conjecture

In order to prove Legendre's conjecture there is need to show that between every pair of consecutive positive integers there exists a radical number that is equal to the positive square-root of a prime number.

The above results from the theorem also suggest that the gap between square-roots of consecutive primes is less than 1. That is:

$$\sqrt{p_{j+1}} - \sqrt{p_j} < 1 \rightarrow p_{j+1} < 1 + p_j + 2\sqrt{p_j}$$
$$\rightarrow p_{j+1} - p_j < 1 + 2\sqrt{p_j}$$

(1)

Proof of Legendre conjecture thus involves resolving the prime gap problem. The inequality 1 is a statement of Andrica conjecture. The conjecture suggests a smaller gap between successive primes than Legendre conjecture. Thus proof of Andrica conjecture would also imply validity of Legendre conjecture.

The above inequality 1 suggests that the maximum possible gap tends to infinity as primes tend to infinity.

That is:

$$g_j = p_{j+1} - p_j < 1 + 2\sqrt{p_j} = 1 + p_j^{1/2+\log 2/\log p_i}$$

(2)

$$\lim_{j\to\infty} g_j \to \infty$$

$$\to \frac{g_j}{p_j} < \frac{1+2\sqrt{p_j}}{p_j}$$

(3)

$$\to \lim_{j\to\infty} \frac{g_j}{p_j} = 0$$

(4)

$$\lim_{j\to\infty} g_j < p_j^{1/2}$$

(5)

Thus the average gap in between two primes is given by the inequality below:

$$\frac{\sum g_j}{n} < \frac{1}{n}\sum_{j=1}^{n}(1+2\sqrt{p_i})$$

(6)

Thus the number of primes less than a given natural number x is given by the inequality:

$$\pi(x) > \frac{x}{\frac{1}{n}\sum_{i=1}^{n}(1+2\sqrt{p_i})} \wedge p_i \le x$$

(7)

Hoheisel (1930) was the first to show that there is a constant θ<1 such that:

$$\pi(x+x^{\theta}) - \pi(x) \sim \frac{x^{\theta}}{\log(x)} as: x \to \infty$$

Hence showing that

$$g_n < p_n^{\theta}$$

(8)

Hoheisel obtained the possible value 32999/33000 for θ. These results were later improved by Heilbronn to θ = ¾ +ε, for any ε>0 and by Chudakov. Other major improvements were introduced by Ingham, Huxley (1972), Baker, Harman and Pintz (2001).

Ingham showed that for some positive constant c,

$$\zeta(1/2+it) = O(t^{c})$$

then:

$$\pi(x+x^{\theta}) - \pi(x) \sim \frac{x\theta}{\log(x)}$$

For any $\theta > (1+4c)\big/(2+4c) = 1\big/2 + \frac{2c}{2+4c}$

The Lindelöf hypothesis implies that Ingham's formula holds for c any positive number.

If we take if $c = \log 2/\log p_i$ then

$$(\theta > (1+4\log 2/\log p_i)\big/(2+4\log 2/\log p_i))$$

This would mean that

$$g_i < 1 + p_j^{(1+4\log 2/\log p_j)/(2+4\log 2/\log p_j)} = 1 + p_j^{1/2 + \frac{2\log 2/\log p_j}{2+4\log 2/\log p_j}} < 1 + p_j^{1/2+\log 2/\log p_j} = 1 + 2\sqrt{p_j}$$

(9)

The inequality 8 is in agreement with inequality 2. The inequality 1 was assumed from observations made from the proposed theorem in section 2. No proof was presented for deriving the inequality; instead the Lindelöf hypothesis was used in conjunction with Ingham's formula to arrive at inequality 9. As such the above proposed proof has a gap because it is based on an unproved hypothesis. There is need for a more rigorous proof of the equality 1 above.

Analytical solution of the gap between consecutive primes and proof of Andrica and Legendre conjectures

Consider two consecutive primes, p_{j+1}, p_j

If the primes are expressible in terms of a variable y such that:

$p_{i+1} = 2y + 2n + 1 \wedge p_i = 2y + 1$

Then the gap between the two primes is 2n and:

$$y = \frac{p_j - 1}{2}$$

$$g_j = p_{j+1} - p_j = 2n \to n = \frac{p_{j+1} - p_j}{2}$$

$$\sqrt{p_{j+1}} - \sqrt{p_j} = \sqrt{(2y+2n+1)} - \sqrt{(2y+1)}$$

$$= (2y+2n+1)^{1/2} - (2y+1)^{1/2}$$

$$= \{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2}(2n+1) - \tfrac{1}{8}(2y)^{-3/2}(2n+1)^2 + \ldots\} - \{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2} - \tfrac{1}{8}(2y)^{-3/2} + \ldots\}$$

$$< (2y)^{-1/2} n = \frac{g_j}{2\sqrt{p_j - 1}} = \frac{p_{j+1} - p_j}{2\sqrt{p_j - 1}} = \frac{(\sqrt{p_{j+1}} + \sqrt{p_j})(\sqrt{p_{j+1}} - \sqrt{p_j})}{2\sqrt{p_i - 1}}$$

$$\frac{\sqrt{p_{j+1}} + \sqrt{p_j}}{2\sqrt{p_j - 1}} < 2$$

$$\frac{\sqrt{p_{j+1}} - \sqrt{p_j}}{2\sqrt{p_j - 1}} < \frac{1}{2}$$

$$\lim_{p_j \to \infty} \frac{\sqrt{p_{j+1}} + \sqrt{p_j}}{2\sqrt{p_j - 1}} = 1$$

$$\lim_{p_j \to \infty} \frac{\sqrt{p_{j+1}} - \sqrt{p_j}}{2\sqrt{p_j - 1}} = 0$$

$$\sqrt{p_{j+1}} - \sqrt{p_j} < \frac{g_j}{2\sqrt{p_j - 1}} < 1$$

(10)

To clarify the derivation of inequality 10, it should be noted that:

$$\{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2}(2n+1) - \tfrac{1}{8}(2y)^{-3/2}(2n+1)^2 + \ldots\} - \{(2y)^{1/2} + \tfrac{1}{2}(2y)^{-1/2} - \tfrac{1}{8}(2y)^{-3/2} + \ldots\} <$$

$$\tfrac{1}{2}(2y)^{-1/2}(2n+1)\} - \tfrac{1}{2}(2y)^{-1/2} = \tfrac{1}{2}(2y)^{-1/2}$$

It should be noted again that:

$$(a+b)^n - (a+1)^n = \{a^n + na^{n-1}b + \frac{n(n-1)a^{n-2}b^2}{2!} + \ldots..\} - \{a^n + na^{n-1} + \frac{n(n-1)a^{n-2}}{2!} + \ldots..\}$$

There is need to improve on the above inequality 10

Theorem (gap between successive prime numbers)

The size of the gap between successive prime numbers is governed by the inequality

$$p_{j+1} > \frac{g_j^{\ 2}}{4} + g_i + 1$$

(11)

Proof

The above theorem proposes that there exists a Diophantine equation governing gaps between successive primes given by:

$$g_j^{\ 2} + 4g_j + 4(p_{j+1} + 1 - s) = 0$$

$$s \in N$$

(12)

The solution of the above Diophantine equation is given by:

$$g_j = -2 + \sqrt{4 - (p_{j+1} + 1 - s)}$$

$$s > p_{j+1}$$

(13)

$$p_2 = 3; s = 9 \wedge g_1 = 1$$

For $p_3 = 5; s = 16 \wedge g_2 = 2$

$$p_4 = 7; s = 20 \wedge g_3 = 2....$$

The above theorem is valid since it enables generation of a Diophantine quadratic equation that accounts for gaps between successive primes.

The inequality 10 can be rewritten for an easy proof of Andrica and Goldbach conjecture using the above theorem.

$$\sqrt{p_{j+1}} - \sqrt{p_j} < \frac{g_j}{2\sqrt{p_j - 1}} = \frac{g_j}{2\sqrt{p_{j+1} - g_j - 1}} = \frac{1}{2\sqrt{\frac{p_{j+1} - g_j - 1}{g_j}}}$$

$$p_{j+1} > \frac{g_j^{\ 2}}{4} + g_j + 1 \rightarrow \sqrt{p_{j+1}} - \sqrt{p_j} < 1$$

Thus Andrica conjecture is proved. From the foregoing analysis the Andrica conjecture follows from the Legendre conjecture since it narrows the gap between the primes proposed in the Legendre conjecture.

By the theorem 11 above if

1) $g_i = 1 \rightarrow p_2 > 2.25$

2) $g_i = 2 \Rightarrow p_{i+1} > 3$

3) $g_i = 4 \Rightarrow p_{i+1} > 9$

$g_i = 6 \Rightarrow p_{i+1} > 16$

Further analysis

From inequality 10:

$$g_j < 2\sqrt{p_j - 1}$$

(14)

Thus the gap between primes can be further reduced than that stipulated by Andrica Conjecture

Thus the average gap between primes is given by the inequality:

$$g < \frac{1}{n}\sum_{j=1}^{j=n}\sqrt{p_j - 1}$$

(14)

Thus:

$$\frac{1}{10}\sum_{j=1}^{j=10}\sqrt{p_j - 1} = \frac{1}{10}(\sqrt{1} + \sqrt{2} + \sqrt{4} + \sqrt{6} + ... + \sqrt{28}) = 3.180830383$$

Thus the inequality 14 predicts that the average gap of prime numbers between 1 and 30 is less than 3.181. The prime number theorem stipulates that the average gap between 1 and 30 is about equal to $\ln 30 = 3.401197382$. Thus the results above generally agree with the prime number theorem.

From inequality 13 the gap between primes can be represented by the equation:

$$(n(g_j + \lambda))^{ix} = (p_j - 1)^{\frac{1}{2}+ix}$$

(15)

Thus from equation 15:

$$p_j = (n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1$$

(16)

$$\frac{1}{1 - p_j^{-(\frac{1}{2}+ix)}} = \frac{1}{1 - ((n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1)^{-(\frac{1}{2}+ix)}}$$

(17)

$$\zeta(s) = \prod_{j=1}^{j=x}\frac{1}{1 - ((n(g_j + \lambda))^{\frac{ix}{\frac{1}{2}+ix}} + 1)^{-(\frac{1}{2}+ix)}}$$

(18)

Thus the gap stipulated in inequality 13 implies Riemann hypothesis.

From the results of equation 13:

$$\log_e x = \frac{\vartheta}{n}\sum_{j=1}^{n}\sqrt{p_j - 1} \rightarrow$$

$$x^{\frac{n}{\theta}} = e^{\sum_{j=1}^{n}\sqrt{p_j - 1}} \rightarrow$$

$$x = \sqrt[\frac{n}{\theta}]{e^{\sum_{j=1}^{n}\sqrt{p_i - 1}}} = (e^{\sum_{j=1}^{n}\sqrt{p_j - 1}})^{\frac{\vartheta}{n}} = e^{\frac{\vartheta}{n}\sum_{j=1}^{n}\sqrt{p_j - 1}}$$

(19)

Using equation 19:

$$\varsigma(s)=\sum_{x=1}^{\infty}\frac{1}{(\sqrt[\frac{\vartheta}{n}]{e^{\sum_{j=1}^{n}\sqrt{p_j-1}}})^s}=\sum_{x=1}^{\infty}(\frac{1}{(e^{\sum_{j=1}^{n}\sqrt{p_j-1}})^{\frac{\vartheta}{n}}})^s=\sum_{x=1}^{\infty}e^{-s\frac{\theta}{n}\sum_{j=1}^{n}\sqrt{p_j-1}}=\sum_{k=1}^{\infty}e^{-s\ln x}$$

(20)

If we take $s=\frac{1}{2}+it$ then the equation 20 takes the form:

$$\varsigma(s)=\sum_{x=1}^{\infty}e^{-(\frac{1}{2}+it)\frac{\theta}{n}\sum_{j=1}^{n}\sqrt{p_j-1}}=\sum_{x=1}^{\infty}e^{-(\frac{1}{2}+it)g_x}=\sum_{x=1}^{\infty}e^{-g_k/2}(\cos g_x t+i\sin g_x t)$$

$$=\sum_{x=1}^{\infty}e^{-\ln x/2}(\cos t\ln x+i\sin t\ln x)=\sum_{x=1}^{\infty}\frac{(\cos(t\ln x)+i\sin(t\ln x))}{x^{1/2}}$$

(21)

In the functional trigonometric series of the form 21 above the non-trivial zeroes of Riemann hypothesis can be derived and Riemann hypothesis verified.

3. Conclusion

Prime numbers have an exclusive property of which Legendre conjecture is a derivative. The square-root of every prime number is an irrational number between two consecutive integers. This property can be used to prove both Legendre's and Andrica conjecture through provable derivative inequality. It is possible to reduce the gap between further to less than that stipulated in Andrica conjecture. A function has been achieved that accounts the non-trivial zeroes of the Riemann hypothesis.

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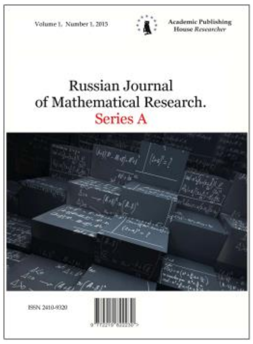

Does The Gravitational Weight Vary With the Area of a Flat-Sheet? A Comparative Study of Area Measurements: With Special References to Simpson's Rule and Bhattacharya's Theorem

Asis Ghosh [a, *], Syama P. Bhattacharya [b]

[a] West Bengal University of Technology, Kolkata, West Bengal, India
[b] Howard university, Washington D.C. and University of Maryland, Baltimore City, USA

Abstract

A few methods are known for the measurements of bended flat areas. But the problems arise when the areas are bended with many bends and deflections. In this situation we have to depend on Simpson's rule. If we follow this rule however, we can measure areas with certain limitations. Therefore accurate results are not always achieved. Recently a theorem has been reported which is based on conversion of gravitational weight of a map occupied by the area of the flat surface. This procedure appears to be simple, prompt and accurate. However a comparative study of area measurements is therefore needful to determine the validity of this theorem.

Keywords: Simpson's rule, Uniform density, $I=W10^2$, Plotted plan, Parabola.

1. Introduction

The curiosity for the measurements of multiple bended flat surfaces was initially originated in the mind of men since the time of third century B.C. (1). We recall the work of Archimedes (287B.C.-212 B.C.) The work of Archimedes on geometrical construction and the measurement of flat and spherical surfaces(2). Many well-known scientists, such as Isaac Newton(1643), (3). Pythagoras (6 century B.C.), Euclid(3rd century B.C.), Ptolemy (2nd century A.D.) , Heron of Alexandria (dates not known) etc gifted their contributions in the technology of surface measurements. In fact, many scientists were then involved in this investigation, but one accurate method was never been reported. However various improved methods are now available, but these methods can be used only in the measurement of simple bended flat areas. (4,5). But actual objective of this investigation was to develop an easy procedure which will correctly measure all kinds of flat areas irrespective of sizes and geometric forms. (6).

Although a perfect method of such measurements is not yet known to us, but a few ongoing procedures which are somehow in practical use have attracted our attention. These are computations of areas from the mid-ordinate rule (3) and the application of Simpson's rule. (7) Surprisingly, a recent report of a theorem $I=W10^{2}$ claims that accurate measurements of areas surrounded by many bends and deflections are possible. (6) This theorem is based on the findings that gravitational weight (GW) of a flat sheet (FS) is directly proportional with the area of the FS which of course should be of uniform density. Now we feel it is necessary to compare the claims of this theorem with the other existing methods. Also, validity of this theorem and it's importance in the area measurements are required for careful verification.

[*] Corresponding author
E-mail addresses: syamaprasadbhattacharya32@gmail.com (A. Ghosh)

2. Materials and methods

The measurements of unknown areas bended by many bends and deflections may be done somehow with the help of some known methods.

A reduced size map of a water-land surrounded by a border line with numerous bends and deflections was marked for the determination of its area (Figure 1). A small size bended land was also present in the center of the large bended land. So the determination of the area of the water-land was done by subtracting the area of the central land from the area of the total land. Initially we selected eight different methods (Table 1) for these measurements. But actually we were able to work with four different methods only. These were: graphical method, mid-ordinate rule, Simpson's rule and Bhattacharya's theorem. We were unable to use the rest of the methods, such as: Average ordinate rule, Trapezoidal rule, Field notes and Plotted plan because the entire area can not be divided into regions of convenient geometrical shapes. Another reason was the ends of the ordinates drawn inside the map can not be assumed as straight lines.

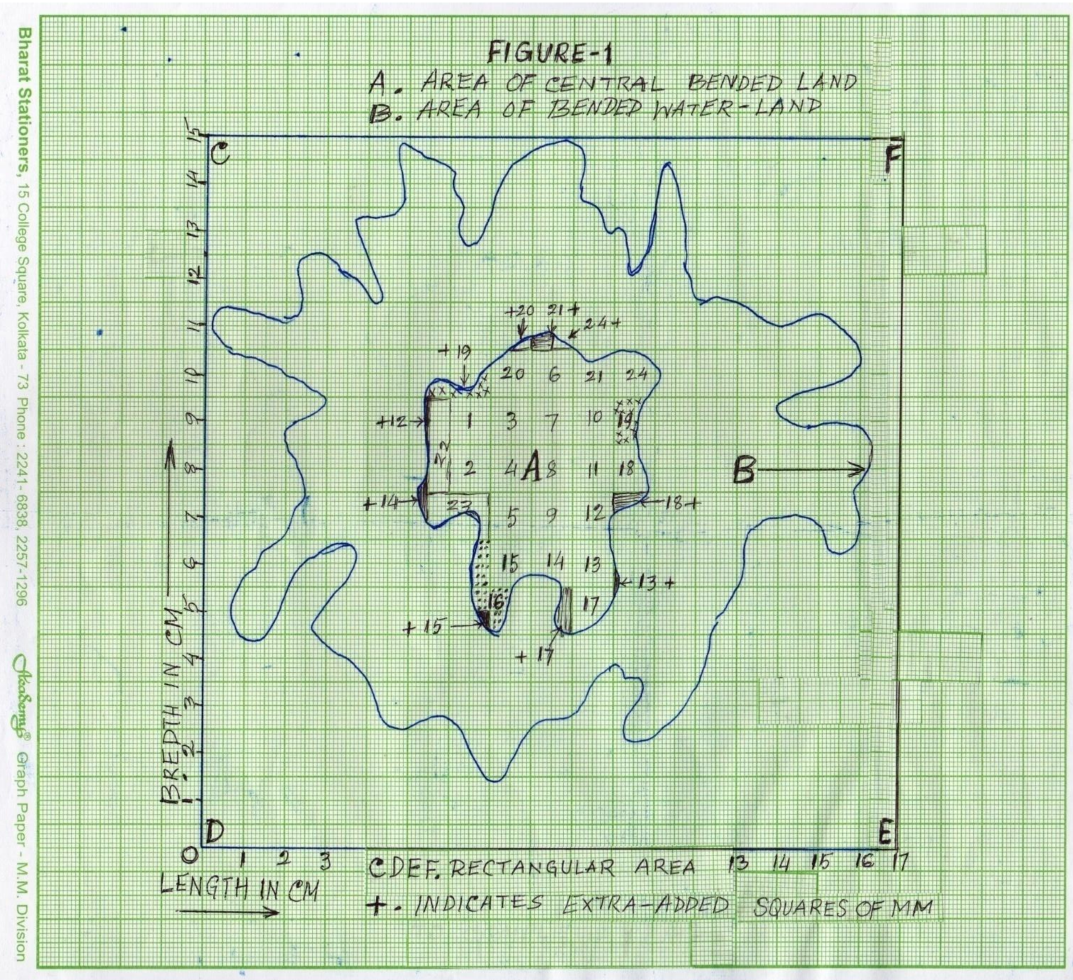

Fig. 1. Maps of the central high-land surrounded by the water-land showing the border lines with many bends and deflections

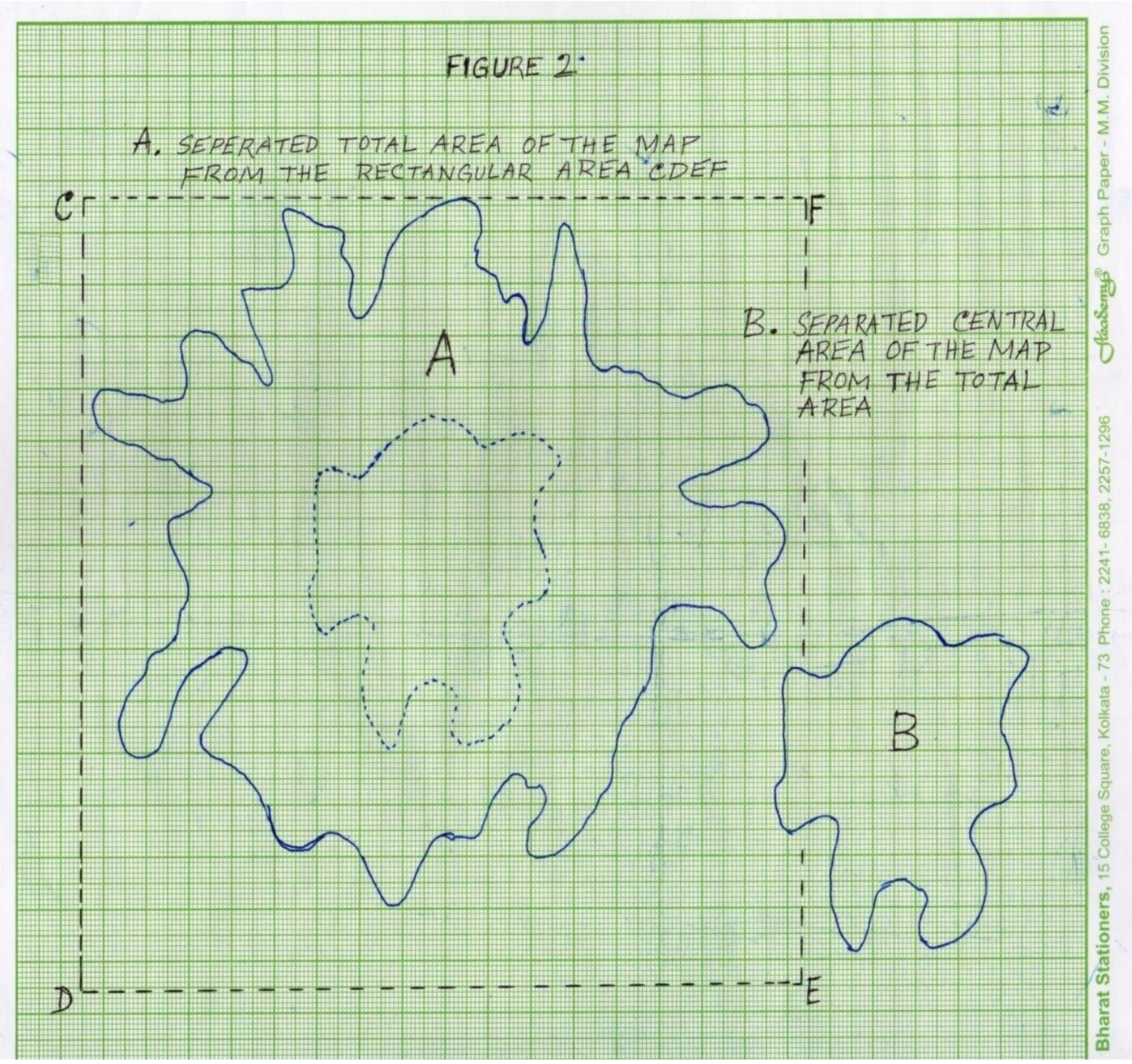

Fig. 2. The different areas of the total map were separated for the determination of gravitational weights

i. Graphical method:

The total map was drawn onto a graph paper by tracing the map after inserting a carbon paper between the map and the graph paper. The total number of squares in centimeters were counted easily. But we had the problems to count near the border line where the bends and deflections were present. However we followed the give and take principle (Figure 1) Moreover, some assumptions were required to complete the countings. In this method eventually correct measurements were not possible due to many bends and deflections in the boundary line. However approximately correct results were obtained in the measurements of central land. (Table 1)

ii. The Mid-Ordinate Rule:

We determined the area of the central land according to this rule Area=Common distance * sum of mid-ordinates.

The total bended area was first measured by combining this method with graphical method, particularly at the regions of boundary line. Here we also used the give and take principle. However long time was required to complete one task. Even some unavoidable minor errors were present in counting. The data were recorded in Table-1. However approximate results were obtained in the measurement of central area. But we were unable to count correctly due to irregular shape of the border line.

iii. The Average Ordinate Rule:

We attempted to measure all the areas according to the formula:

Area = $\frac{\text{Sum of ordinates}}{\text{no. of ordinates}}$ * length of the base line. It appeared that this formula by itself only can not help in the measurements of our proposed areas; because the bended border line can not be assumed as the straight lines. Therefore the total area was measured with the help of average ordinate rule combining with the graphical method. But still the results were not found to be correct. More over extended long time was required to complete the calculations. Therefore this method was considered as an unsuitable one for our tasks.

iv. From Field Notes:

According to this procedure it was not possible to draw geometrical figures within the bended areas. So our attempts for the measurements were discontinued.

v. The Trapezoidal Rule:

We were unable to apply this rule for the measurements; because we could not assume boundaries between the ends of ordinates to be straight. So, the areas between the base line and the irregular boundary line can not be considered as trapezoid. Therefore we did not proceed further for the measurements.

vi. Simpson's Rule

To apply this rule we drew a base line in the middle of the total map. The ordinates were drawn from both sides of the base line. The boundaries between the ends of ordinates were assumed to form the arc of parabolas. The number of ordinates were odd numbers and divisions were even numbers. The areas were calculated according to the formula:

Area = $\frac{\text{Common distance}}{3}$ { first ordinate + last ordinate + 2 × (Sum of remaining odd ordinates) + 4 × (Sum of even ordinates) }. In this measurement we had to ignore some strips of areas near the arcs of parabolas. The data of Simpson's rule are written in Table 1. The countings were performed separately by five persons. But correct results were not obtained. In the measurements of the central land, it were easier tasks and the results were also approximately correct. This was due to simple bended structure of the central land.

vii. Bhattacharya's Theorem(6):

While using this theorem for the measurement of irregular areas four straight lines were drawn on the four sides of the map to form a rectangle (Figure 1). The lengths of these lines were measured and written in centimeters. The scale of the map shows the exact ratio between the size of the map and that of the original bended lands.

A piece of polyethylene sheet (PES) was used for taking a photographic image of the map onto it. The outline of the photographic picture of the map was drawn on the PES and then it was separated from the PES sheet by cutting it with an electric needle. The GW of this portion of PES was measured in an electronic balance(6).

3. Results

i. Typical procedure of calculations

Proportional weight of the unknown bended area of the PES:

The weight of PES (total map) = 950 mg which is marked as W_1 and the weight of the known rectangular area of PES = 1.875 gm. Which is marked as W_2. The ratio of the weights of total unknown and known rectangular areas will be $\frac{W1}{W2}$, say, this ratio = W

So W = $\frac{W1}{W2} = \frac{0.950}{1.875}$ = 0.506

The index (I) of the unknown bended total area of the map is,

I = $\frac{W1}{W2}$ * 100 = 0.506 * 100 = 50.6 %

ii. Area of the total map:

Area of the rectangle = 15 cm * 17 cm = 255^2 cm

GW of the PES (rectangle) = 1.875 gm (measured in a balance)

Therefore 1.875 gm of PES = 255^2 cm of area

Then 1 gm of PES = $\frac{255}{1.875}$ = 136^2 cm of area

GW of the PES (total map) = 0.950 gm (measured in a balance)

Hence 0.950 gm PES = 0.950 * 136 = 129.2^2 cm (area of the total map)

iii. Area of the Central land:

GW of the PES of the central land = 0.2 gm (measures in a balance)

1 gm of PES = 136² cm of area

0.2 gm of PES = $\frac{255}{1.875}$ * 0.2 = 27.2² cm (area of the central land)

iv. Area of The Water Land

Total land minus the central land

129.2 – 27.2 = 102² cm (area of the water land)

All experiments including calculations were separately performed by five different persons (one author and four surveyors). Each experiment was repeated five times and the average number was recorded in Table 1.

1. The Statistical Procedure

The standard deviation of the means and the t-tests for the

significance at the level P<0.5 were determined adopting the procedure described in SPSS computer program. (6)

Table 1. Measurements of Bended Areas (Results are expressed in square centimeters)

	Graphical Method 1.	Mid Ordinate rule 2. 5	Average Ordinate rule 2.	Trapezoidal rule 2.	Field notes 2.	Plotted plan 3.	Simpson's rule 1, 4, 5	Bhattacharya's theorem 6.
Total area	118.4 ± 2.35	131.2 ± 4.25	N.M.	N.M.	N.M.	N.M.	139.4 ± 5.75	129.2 ± 0.95
Central area	25.43 ± 1.8	24.6 ± 3.8	N.M.	N.M.	N.M.	N.M.	28.2 ± 4.55	27.2 ± 0.05
Water area	93.37 ± 2.85	106.6 ± 4.85	N.M.	N.M.	N.M.	N.M.	110.7 ± 5.25	102.0 ± 0.88

N.M. Not measurable

1. Takes long time to complete.

2. Many bends and deflections are present in the boundaries between the ends of ordinates; can not be assumed to be straight lines.

3. The entire area can not be divided into regions of convenient shapes.

4. The boundaries between the ends of ordinates can not be assumed to form an arc of parabolas.

5. Some difficult portions of bended areas were measured by graphical method.

6. Takes short time to complete. No problem for many bends and deflections.

4. Discussion

All the methods written in this paper for the measurements of irregularly bended flat areas were examined separately by five different experts. Our objective was to select just one dependable method. So far, we examined eight methods. Except one, all the remaining methods were noticed to be attached with one or more unsolved questions. Sometimes one or more requirement to design a specific structure created problems in the measurements. As an example: Computation of area from plotted plan. Here the entire area was required to divide into regions of convenient geometrical figures, such as: triangles, squares, trapeziums, parallel lines etc. In this case, the entire area was changed to a particular shape in order to make easy measurements. However the graphical procedure was noticed to be an easy method for the measurement of simple bended areas. But long time was required to complete just-one measurement. Some assumptions were frequently needed when countings were done near the bends and deflections of the areas. When careful calculations were done with graphical procedure nearly correct results were obtained (Table 1).

When the graphical method was combined with mid-ordinate rule the time required for calculations was greatly reduced and the results were approximately correct. But the mid-ordinate rule without combination of graphical method appeared to be difficult to complete the measurements.

Simpson's rule is a well-known method for the measurements of bended flat areas. In this rule the boundaries between the ends of ordinates were assumed to form arcs of parabola. This rule was applicable only when the number of divisions of an area was even, i.e. the number of ordinates were odd, it gave nearly accurate results as long as the bends were considered as parabolas. However up to certain limits Simpson's rule was applicable to obtain nearly accurate results. When the areas appeared to have many bends and deflections (Figure 1) Simpson's rule became difficult to apply. Additionally prolonged time was required to complete a measurement.

An alternative method has been reported recently which is based on a theorem $I = W10^2$. (6)

We examined this theorem and followed the method repeatedly on various types and sizes of multiple bended flat areas. We noticed that the procedure was prompt, simple, short and accurate. In this procedure we never had to do any complicated calculations or assumptions.

Now if we compare the results of eight experiments (Table 1) we clearly see the standard deviation (SD) of the means of all results except one are highly significant. Whereas insignificant SD of the means are noticed in all the results obtained by applying Bhattacharya's theorem indicate the accuracy of the findings obtained in the measurements.

So, we can write with confidence that the unique method based on Bhattacharya's theorem can be used for the accurate measurements of any flat areas with many bends and deflections.

5. Conclusion

Does the gravitational weight vary with the area of a flat sheet ? The answer is yes. Based on this report, we hope the old problem of two thousand years in area measurements should no longer exist.

6. Acknowledgements

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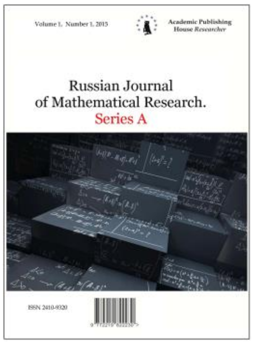

On Solutions of Differential Inclusions with Almost Convex Right-Hand Side

Rafik A. Khachatryan [a, *]

[a] Yerevan State University, Armenia

Abstract

In the paper the question of the existence of a differential inclusion $\dot{x}(t) \in a(x)$ under the initial condition $x(t_0) = x_0$ is considered. It is assumed that a multivalued mapping a continuous and the sets $a(x)$ are almost convex.

Keywords: Set-valued mapping, almost convex, differential inclusion.

1. Введение

Дифференциальным включением называется соотношение вида

$$\frac{dx}{dt} \in a(x). \quad (1)$$

Рассмотрим вопрос существования решения дифференциального включения с начальным условием $x(t_0) = x_0$. В зависимости от свойств многозначного отображения (непрерывность в том или ином смысле) решения диффереціального включения (1) обладает различными дифференциальными свойствами.

Пусть вектор функция $x(t)$ определена на интервале или отрезке $J(t_0 \in J)$ и $x(t_0) = x_0$. Она называется классическим решением, если всюду на J имеет непрерывную производную и удовлетворяет включению (1).

Вектор функция $x(t)$ называется решением Каратеодори дифференциального включения (1) на интервале J, если она на интервале J абсолютно непрерывна и почти всюду удовлетворяет включению (1).

Решение включения (1) с выпуклозначной правой частью при предположении полунепрерывности сверху многозначного отображения впервые было рассмотрено Зарембой в своей статье (Zaremba, 1934). Паратингентной производной $Dx(t_0)$ функции $x(t)$ в точке t_0 называется совокупность всех пределов:

$$\lim_{\substack{t_k \to t_0 \\ x_k \to x_0}} \frac{x(t_k) - x(s_n)}{t_k - s_n}, \quad s_n \neq t_k.$$

* Corresponding author
E-mail addresses: khachatryan.rafik@gmail.com (R.A. Khachatryan)

Заремба определял решение как непрерывную функцию, паратингентная производная $Dx(t)$ которой всюду удовлетворяет включению:

$$Dx(t) \subseteq a(x(t)) \qquad (2)$$

Верна следующая теорема.

Теорема 1 (Zaremba, 1934)**,** Пусть многозначное отображение $a : R^n \to 2^{R^n}$ с выпуклыми замкнутыми значениями полунепрерывно сверху и существует число $C > 0$ такое, что $\|y\| < C \ \forall y \in a(x)$, $x \in R^n$.

Тогда существует липшицева функция $x(t)$ с начальным условием $x(0) = x_0$, паратингентная производная $Dx(t)$, которой всюду удовлетворяет включению:

$$Dx(t) \subseteq a(x(t)), \quad t \in [0,1].$$

Дифференциальное включение с обобщенными производными (контингентными производными) было рассмотрено Вазевским (Wazewski, 1961).

Доказано, что если для любого x множество $a(x)$ – выпуклый компакт и отображение а непрерывно, то включение (2) равносильно включению в контингентциях и дифференциальному включению (1).

Для дифференциальных включений с невыпуклой правой частью первая теорема существования классического локального решения была доказана Филипповым, при условии, что правая часть удовлетворяет условию Липшица (Филипов, 1967). Затем им же была доказана теорема существования решения Каратеодори с непрерывной правой частью (Филипов, 1971). А в статье (Филипов, 1977) построен пример дифференциального включения вида (1), где многозначное отображение а с невыпуклыми значениями непрерывно, не удовлетворяет условию Липшица и включение (1) не имеет классическое решение. В настоящее время имеются достаточно содержательных и подробных монографии и статей целиком или в значительной степени излагающих проблемы существования решений (классические или в смысле Каратеодори) дифференциальных включений. К числу таких работ можно отнести: Ж.П. Обен (Aubin, Celina, 1984), А.А.Толстоногое (Толстоногов, 1986), Е.С. Половинкин (Полованский, 2015), А.Д. Иоффе (Ioffe, 2017), В.И. Благодатстких (Благодаских, Филипов, 1985), В.Д. Гельман (Барисович и др., 2005).

Однако, в литературе достаточно мало работы посвящены дифференциальными включениями с обобщенными производными.

В настоящей статье рассматривается вопрос существования липшицевой функций $x(t)$, $t \in [0,1]$ с начальным условием $x(0) = x_0$, паратингентная производная которой удовлетворяет всюду включению (2). Предполагается, что множества $a(x)$ являются почти выпуклыми, а отображение а непрерывно. Оказывается, что в этом случае дифференциальное включение (1) не равносильно включению (2) и оно вообще говоря не имеет классическое решение. Доказывается существование липшицевой функции, удовлетворяющей включению (2) всюду.

Понятие почти выпуклости было введено в работах (Остапенко, 1982; Остапенко, 1983). Потребность изучения таких множеств возникла в теории диффернциальных играх (Остапенко, 1982).

2. Методы исследования

В статье применены методы выпуклого и негладкого анализа. Ключевую роль здесь играет теорема Арцела о компактности.

В дальнейшем $B_\gamma(a)$ - замкнутый шар с центром а радиуса; $M \subseteq R^n$ замкнутое множество: $diam(M)$ диаметр множества М; $conv\{M\}$ - выпуклая оболочка множества М.

Определение 1 (Остапенко, 1982). Множество $M \subseteq R^n$ удовлетворяет условию почти выпуклости с константой $B \geq 0$, если для любых

$x_j \in M$, $\lambda_j \ge 0$, $j \in J$,

где J - конечное множество индексов, таких, что $\sum_{j \in J} \lambda_j = 1$, выполняется

$\sum_{j \in J} \lambda_j x_j \in M + \theta r^2 B_1(0)$.

где $r \equiv \max_{i,j \in J} \|x_i - x_j\|$.

Заметим, что если $\theta = 0$, то M -выпуклое множество. Класс почти выпуклых множеств достаточно широк.

Пример 1. Множество $M = \{a, b\}$ состоящих из двух точек почти выпукло. Действительно, имеем

$$conv\{a,b\} \subseteq M + \frac{1}{2\|a-b\|}\|a-b\|^2 B_1(0),$$

т.е. в этом случае константа почти выпуклости θ можно выбрать $1/(2\|a-b\|)$ (см. Рисунок 1).

Пример 2. Дуга на окружности является почти выпуклым множеством. Это непосредственно следует из достаточного условия почти выпуклости, доказанный в (Остапенко, 1983) (см. теорема 2). Найдем константа почти выпуклости. Предположим, что дуга M меньше полуокружности и множество $Q = \{x_1, x_2, ..., x_k\}$ находится на этой дуге.

Пусть $A = x_1$, $B = x_k$, $d = diam(Q) = AB$. Тогда множество $conv\{Q\}$ находится на a-окрестности множества M, где a = CD (см. Рисунок 1).

Имеем

$$DC = R - \sqrt{R^2 - \frac{d^2}{4}}.$$

Теперь число θ выберем из неравенства:

$$DC \le \theta d^2,$$

т.е.

$$\frac{1}{R + \sqrt{R^2 - \frac{d^2}{4}}} \le \theta.$$

Очевидно, что этому неравенств удовлетворяет числа $\theta \ge 1/2R$. Если дуга больше полуокружности, то она почти выпукла по теореме 3 (Остапенко, 1983) с некоторой константой θ. Тогда, как видно из рисунка 1, если $Q = \{a, b\}$, то множество $conv\{Q\}$ находится в β – окрестности дуги, где $\beta = \|a-b\|/2$.

Таким образом

$$\theta \ge \frac{1}{2\|a-b\|}.$$

Значит, если $\|a-b\| \to 0$, то $\theta \to \infty$.

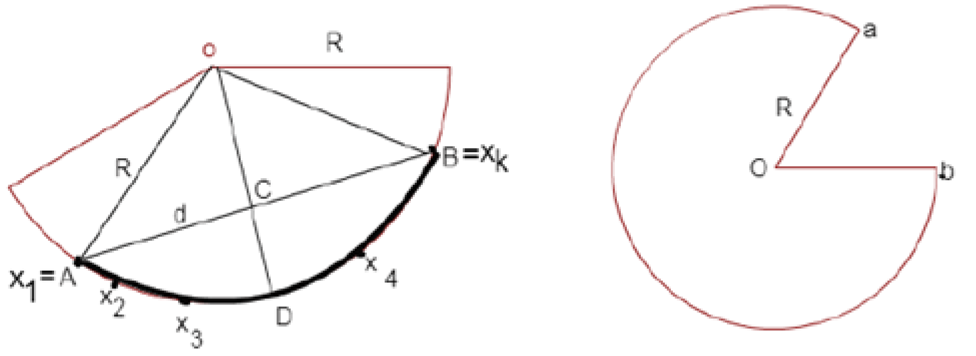

Рис. 1. Множество $conv\{Q\}$ находится на a-окрестности множества M, где a = CD

Пример 3. Окружность M с радиусом R является почти выпуклым множествим с константой $\theta \ge 1/(\sqrt{3}\,R)$. Действительно, пусть множество $Q \equiv \{x_1, x_2,..., x_k\} \subset M$ (см. Рисунок 1). Рассмотрим две случаи. Если $0 \notin conv\{Q\}$. Это означает, что множество находится в некоторой полуокружности. Тогда из примера 2 следует включение:

$$conv\{Q\} \subseteq M + \frac{1}{2R}(diam(Q))^2 B_1(0). \qquad (3)$$

Если $0 \in \operatorname{int} Q$. Тогда в $convQ$ существует некоторый остроугольный треугольник, содержащий внутри себя центр окружности о. Значит, окружность описана этому треугольнику. Следовательно, длина некоторой стороны треугольника больше или равно $R\sqrt{3}$. Отсюда

$$diam(Q) \ge \sqrt{3}R.$$

Очевидно, что множество Q находится в R-окрестности множества M .

Теперь выберем число θ из условия

$$R \le \theta(diam(Q))^2 \qquad (4)$$

Это неравество имеет место, если $\theta \ge 1/(\sqrt{3}R)$. Если точка O находится на границе множества $conv\{Q\}$, то $diam(Q) = 2R$. Тогда неравенство (4) выполняется, если $\theta \ge 1/2R$. В общем случае, имея ввиду и включение (3), имеем

$$conv\{Q\} \subseteq M + \frac{1}{\sqrt{3}R}(diam(Q))^2 B_1(0).$$

Отсюда M – почти выпуклое множество с константой $1/(\sqrt{3}R)$.

В дальнейшем мы используем следующее свойство почти выпуклых множеств.

Предложение 1 (Остапенко, 1983). Теорема 3, Следствие 3). Если M почти выпуклое множество с константой θ, то для любого $\varepsilon < 1/(16\theta)$ множество $M + B_\varepsilon(0)$ почти выпукло с константой 4θ.

Напомным теперь определения многозначного отображения. Пусть 2^{R^n} совокупность всех непустых подмножеств из R^n.

Отображение $a: R^n \to 2^{R^n}$ называется полунепрерывным снизу в $x_0 \in R^n$, если для любого $\varepsilon > 0$ существует такое $\delta > 0$, что

$$a(x_0) \subseteq a(x) + B_\varepsilon(0) \;\; \forall x \in B_\delta(x_0).$$

Отображение $a:R^n \to 2^{R^n}$ называется полунепрерывным сверху в $x_0 \in R^n$, если для любого $\varepsilon > 0$ существует такое $\delta > 0$, что

$$a(x) \subseteq a(x_0) + B_\varepsilon(0)\ \forall x \in B_\delta(x_0).$$

Если отображение a полунепрерывно снизу и сверху в x_0, то оно называется непрерывным в этой точке (см. Барисович и др., 2005), Определение 1.2.43, с. 38: определение непрерывности в смысле Хаусдорфа).

Основным результатом статьи является следующая теорема.

Теорема 2. Пусть

1) многозначное отображение $a:R^n \to 2^{R^n}$ с компактными значениями непрерывно;

2) существует число $\theta \ge 0$ такая, что для любого $x \in R^n$ множество $a(x)$ почти выпукло с констатой θ;

3) существует константа C > 0 такая, что для любого $y \in a(x)$

$$\|y\| \le C(1 + \|x\|)\ \ \forall x \in R^n.$$

Тогда существует липшицевая функция $x(t)$, паратингентная производная $Dx(t)$ которой **всюду** удовлетворяет включению

$$Dx(t) \subseteq a(x(t)),\ \ t \in [0,1].$$

с начальным условием $x(0) = x_0$.

Доказательство. Доказательство аналогично теореме Зарембы. Отметим ключевые моменты доказательства. Разобьем отрезок $[0,1]$ на 2^m (m-натуральное число) частей и положим $\delta = 2^{-m}$. Запишем вместо соотношения (1) разностное включение

$$x_\delta(t+\delta) \in x_\delta(t) + \delta a(x_\delta(t)), \qquad (5)$$

$$t = 0, \delta, 2\delta, \ldots, (2^m - 1)\delta.$$

Решение $x_\delta(t)$ разностного включения (3) построим шаг за шагом. Положим $x_\delta(0) = x_0$. В первом шаге в качестве $x_\delta(\delta)$ выберем произвольный элемент множества $x_\delta(0) + \delta a(x_\delta(0))$. Во втором шаге в качестве $x_\delta(2\delta)$ выберем такой элемент множества $x_\delta(\delta) + \delta a(x_\delta(\delta))$, что

$$\left\| \frac{x_\delta(\delta) - x_\delta(0)}{\delta} - \frac{x_\delta(2\delta) - x_\delta(\delta)}{\delta} \right\| = \min_{u \in x_\delta(\delta) + \delta a(x_\delta(\delta))} \left\| \frac{x_\delta(\delta) - x_\delta(0)}{\delta} - \frac{u - x_\delta(\delta)}{\delta} \right\| \qquad (6)$$

Аналогично, построим точки $x_\delta(3\delta), \ldots, x_\delta(1)$.

Доопределим $x_\delta(t)$ для всех [0,1], построив линейную интерполяцию:

$$x_\delta(t) = x_\delta(k\delta) + (t - k\delta) \cdot \frac{x_\delta((k+1)\delta) - x_\delta(k\delta)}{\delta},\ \ t \in [k\delta, (k+1)\delta],$$

$$k = 0, 1, \ldots, (2^m - 1).$$

Из разностного включения (5) следует, что

$$\|x_\delta(t+\delta)\| \le (1 + \delta C)\|x_\delta(t)\| + \delta C$$

Отсюда получим

$$\|x_\delta(t)\| \le e^{Ct}(1 + \|x_0\|) - 1 \le e^C(1 + \|x_0\|) - 1 \equiv \nu.$$

Значит, для любого $y \in a(x_\delta(t))$

$$\|y\| \le C(1 + \|x_\delta(t)\|) \le C(1 + v) \equiv L$$

Поэтому

$\|x_\delta(t+\delta)-x_\delta(t)\| \le \delta L$, $t=0,\delta...$ (7)

Отсюда

$\|x_\delta(t_1)-x_\delta(t_2)\| \le L|t_1-t_2|$, $\forall t_1,t_2 \in [0,1]$.

Поэтому в силу теоремы Арцела из последовательности $\{x_\delta(t)\}$ функций можно выбрать равномерно сходящуюся подпоследовательность. Без ограничения общности можно предполагать, что сама последовательность $x_\delta(.)$ сходится к некоторой функции $x_0(.)$, причем

$$\gamma(\delta)=\max_{0\le t\le 1}\|x_\delta(t)-x_0(t)\| \to 0.$$

Покажем, что предельная функция $x_0(t)$ удовлетворяет условию Липшица с константой L. В самом деле,

$$\|x_0(t_1)-x_0(t_2)\| \le \|x_0(t_1)-x_\delta(t_1)\| +$$
$$+\|x_\delta(t_1)-x_\delta(t_2)\|+\|x_\delta(t_2)-x_0(t_2)\| \le$$
$$\le \gamma(\delta)+L|t_1-t_2|+\gamma(\delta).$$

Устремляя δ к нулю, получим требуемый результат.

Так как

$$\|x_\delta(t)-x_0(t_0)\| \le \|x_\delta(t)-x_\delta(t_0)\| +$$
$$+\|x_\delta(t_0)-x_0(t_0)\| \le L|t-t_0|+\gamma(\delta),$$

то в силу полунепрерывности сверху отображения a спаведливо включение

$$a(x_\delta(t)) \subseteq a(x_0(t_0))+\varepsilon B_1(0)$$

как только разность $|t-t_0|$ и δ достаточно малы. Предположим, что это включение выполняется, если выполнены следующие условия $|t-t_0|<\gamma$, $\delta<\delta_0$.

Покажем, что паратингентная производная $Dx(t)$ удовлетворяет включению (2) всюду на отрезке [0,1]. Пусть $t_0 \in (0,1)$ и t_1,t_2-фиксированные точки, имеющие вид $t_1=k_1\delta_1$, $t_2=k_2\delta_1$, $\delta_1<\delta$, причем

$$|t_1-t_0|<\gamma,\ |t_2-t_0|<\gamma.$$

Так как $\delta=2^{-m}$, то при $\delta<\delta_1$ точки t_1,t_2 будут входить в разбиение отрезка [0,1].

Имеем

$$\frac{x_\delta(t_2)-x_\delta(t_1)}{t_2-t_1}=\sum_{t_1,t_1+\delta,\dots,t_2-\delta}\frac{\delta}{t_2-t_1}\cdot\frac{x_\delta(t+\delta)-x_\delta(t)}{\delta}.$$

Для достаточно малых δ имеем

$$y_t \equiv \frac{x_\delta(t+\delta)-x_\delta(t)}{\delta} \in a(x_0(t_0))+\varepsilon B_1(0),\ \forall t=t_1,t_1+\delta,\dots,t_2-\delta.$$

Так как множество $a(x_0(t_0))$ почти выпукло с константой θ, то по предложению 1 при $\varepsilon<1/(16\theta)$ множество $a(x_0(t_0))+\varepsilon B_1(0)$ почти выпукло с константой 4θ. Поэтому согласно определению 1 почти выпуклости имеем

$$\frac{x_\delta(t_2)-x_\delta(t_1)}{t_2-t_1} \in a(x_0(t_0))+\varepsilon B_1(0)+4\theta\max_{t,p}\|y_t-y_p\|^2 B_1(0) \quad (8)$$

Так как отображение a непрерывно по Хаусдорфу, то оно равномерно непрерывно на компактном множестве $B_\nu(0)$. Это означает, что для любого $\varepsilon > 0$ существует $\Delta > 0$ такое, что если $x', x'' \in B_\nu(0)$, то

$$\|x' - x''\| < \Delta \Rightarrow a(x') \subseteq a(x'') + B_\varepsilon(0).$$

Отсюда, если положим $x' = x_\delta(t+\delta)$, $x'' = x_\delta(t)$, то из неравенства (7) следует, что при $\delta < \Delta/L$

$$\|x_\delta(t+\delta) - x_\delta(t)\| < \Delta$$

и поэтому из соотношения (6) получим

$$\|y_{t+\delta} - y_t\| \le \varepsilon,\ t = t_1, t_1+\delta, \ldots, t_2 - \delta.$$

Отсюда для произвольных $t, p = t_1, t_1+\delta, \ldots, t_2 - \delta$ справедливо неравенство

$$\|y_t - y_p\| \le \varepsilon |t_1 - t_2|.$$

Поэтому, если $2\nu < \varepsilon$, то

$$\|y_t - y_p\|^2 \le (t_2 - t_1)^2 \varepsilon^2 < \varepsilon^4 \qquad (9)$$

Имеем

$$\frac{x_0(t_2) - x_0(t_1)}{t_2 - t_1} = \frac{x_0(t_2) - x_0(t_1)}{t_2 - t_1} + \frac{x_\delta(t_2) - x_\delta(t_1)}{t_2 - t_1} +$$

$$+ \frac{x_\delta(t_2) - x_0(t_1)}{t_2 - t_1}. \qquad (10)$$

Выберем теперь число δ настолько малым, что первое и третье слагаемое в (10) были по модулю меньше ε. Теперь имея ввиду соотношения (8)-(10), получим

$$\frac{x_0(t_2) - x_0(t_1)}{t_2 - t_1} \in a(x_0(t_0)) + 3\varepsilon B_1(0) + 4\theta\varepsilon^4 B_1(0).$$

Отсюда, поскольку ε произвольно, получим

$$Dx_0(t_0) \subseteq a(x_0(t_0)).$$

Аналогично рассматриваются случаи, когда $t_0 = 0$ или $t_0 = 1$.

3. Обсуждение результатов

Итак основным результатом статьи является следующая теорема, в которой доказывается существования решения дифференциального включения с паратингентной производной в случае, когда правая часть $a(x)$ дифференциального включения (1) принимает почти выпуклые значения, а многозначное отображение $a : R^n \to 2^{R^n}$ непрерывно по Хаусдорфу.

Замечание 1. Если ослабить условие непрерывности отображения a, то утверждение теоремы 2 становится неверным. Действительно, пусть

$$x \in R^1,\ a(0) = \{-1,1\},\ a(x) = -signx.$$

Тогда отображение а полунепрерывно сверху, множества $a(x)$ **почти выпуклы** (пример 1), но решение с начальным условием $x(0) = 0$ не существует при $t > 0$ (см. пример Благодаских, Филипов, 1985: 243).

Замечание 2. Отметим, что в условиях теоремы 2 включение 2 в паратингенциях **не равносильно** дифференциальному включению (1).

Действительно, пусть $a(x) = \{-1,1\} \cup \{2\}$, $\forall x \in R^1$. Отображение a непрерывно, а для каждого x множество $a(x)$ почти выпукло с константой $\theta = 1/2$. Рассмотрим дифференциальное включение (1) с начальным условием $x(0) = 0$ на отрезке [0,3]. Липшицева функция

$$x(t) = \begin{cases} t & \text{если} \quad t \in [0,1], \\ 2 - t & \text{если} \quad t \in [1,2], \\ 2t - 4 & \text{если} \quad t \in [2,3] \end{cases}$$

на отрезке [0,3] удовлетворяет дифференциальному включению $\frac{d}{dx} x(t) \in a(x)$ почти всюду.

Однако включение (2) не выполняется всюду, поскольку например $1.5 \in Dx(2)$, и поэтому $Dx(2) \subseteq a(x(2))$.

Решение включения (2) в паратингентциях будет например функция

$$y(t) = \begin{cases} t & \text{если } t \in [0,1] \\ 2 - t & \text{если } t \in [1,3] \end{cases}$$

Действительно имеем

$$Dy(t) = \begin{cases} 1 & \text{если} \quad t \in [0,1), \\ -1 & \text{если} \quad t \in (1,3], \\ [-1,1] & \text{если} \quad t = 1 \end{cases}$$

поэтому всюду на отрезке [0,3] им:еет место включение

$Dy(t) \subseteq a(y(t)),\; y(0) = 0$,

Замечание 3. В примере 3 (см. Филипов, 1977: 1076) многозначное отображение $a: R^2 \to 2^{R^2}$ непрерывно, для каждого x множество $a(x)$ либо дуга на окружности либо окружность, (т.е. множества $a(x)$ почти выпуклы согласно примеру 2), и показано, что дифференциальное включение вида (1) с начальным условием $x(0) = 0$ не имеет классическое решение. Таким образом существует пример дифференциального включения вида (1), где отображение a непрерывно, значения $a(x)$ почти выпуклы, но включение (1) не имеет классическое решение.

4. Заключение

Результаты статьи могут быть применены в теории дифференциалных играх и в задачах оптимального управления.

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О решениях дифференциальных включений с почти выпуклой правой частью

Рафик Агасиевич Хачатрян [a], *

[a] Ереванский государственный университет, Армения

Аннотация. В статье рассматривается вопрос существования решения включения вида $Dx(t) \subseteq a(x(t))$ при начальном условии $x(t_0) = x_0$, где $Dx(t)$ – паратингентная производная функции $x(t)$. Предполагается, что многозначное отображение a непрерывно, а множества $a(x)$ почти выпуклы.

Ключевые слова: Многозначное отображение, почти выпуклость, дифференциальное включение.

* Корреспондирующий автор
Адреса электронной почты: khachatryan.rafik@gmail.com (Р.А. Хачатрян)