Analytic computation of digamma function using some new identities

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Abstract

Motivated by the substantial development in the theory of digamma function, we derive some new identities for the digamma function. These new identities enable us to compute the values of the digamma function for fractional orders in an analogous manner. Also, we mention two errata, found in Jensen's article (An elementary exposition of the theory of the Gamma function, 1916), and present their correct forms.

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Keywords: Digamma(Psi) function; Gamma function; Euler's constant.

1. INTRODUCTION AND PRELIMINARIES

A natural property of digamma (Psi)function is as application in the theory of beta distributions-probability models for the domain [0,1]. It is used mainly in the theory of special functions in wide range of applications. Digamma functions are directly connected with many special functions such as Riemann's zeta function and Clausen's function etc.

Many authors have contributed to develop the theory of polygamma function with respect to properties [25; 9; 13; 14; 16], inequalities [2; 3; 6], monotonicity [21; 22; 23; 24], series [5; 7; 15; 27; 10; 12], and fractional calculus [1; 19; 20].

The Gamma function, $\Gamma(x)$, was introduced by Leonard Euler as a generalization of the factorial function on the sets, $\mathbb{R}$ of all real numbers, and $\mathbb{C}$ of all complex numbers. It (or, Euler's integral of second kind) is defined by

$$\begin{aligned}\Gamma(z) &= \int_0^\infty \exp(-t)t^{z-1}dt, \qquad \Re(z) > 0\\ &= \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}dt. \qquad (1.1)\end{aligned}$$

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In 1856, Karl Weierstrass gave a novel definition of gamma function

$$\frac{1}{\Gamma(z)} = z\exp(\gamma z)\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\exp\left(-\frac{z}{n}\right)\right], \tag{1.2}$$

where $\gamma = 0.5772156649015328606065120900824024310 42\ldots$, is called Euler-Mascheroni constant, and $\frac{1}{\Gamma(z)}$ is an entire function of z, and

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots..+\frac{1}{n}-\ell n(n)\right).$$

The function

$$\psi(z) = \frac{d}{dz}\{\ell n\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{1.3}$$

or, equivalently

$$\ell n\Gamma(z) = \int_1^z \psi(\zeta)d\zeta, \tag{1.4}$$

is the logarithmic derivative of the gamma function or digamma function.
$\psi^{(i)}(z)$ for $i \in \mathbb{N}$ are called the polygamma functions, and ψ has the presentation as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^{\infty}\frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt \quad (\gamma = \text{ Euler's constant}). \tag{1.5}$$

The Psi function has following series representation

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(z+n)}, \quad z \neq -1,-2,-3,\ldots \tag{1.6}$$

In 1813, Gauss [9] (see also, Jensen [13, p.146, eq.(32)]; [8, p.19, (1.7.3) eq.(29)], Böhmer [4, p.77]) discovered an interesting formula for digamma (Psi) function as follows

$$\psi(p/q) = -\gamma - \ell n(q) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + {\sum_{j=1}^{[\frac{q}{2}]}}' \left\{\cos\left(\frac{2\pi jp}{q}\right)\ell n\left(2-2\cos\frac{2\pi j}{q}\right)\right\}, \tag{1.7}$$

where $1 \leq p < q$ and p,q are positive integers, and accent(prime) to right of the summation sign indicates the term corresponding to (last term) $j = \frac{q}{2}$ (when q is positive even integer) should be divided by 2.

A different form of Gauss formula is also given in N. Nielsen [18, p. 22, an equation between equations (7) and (8)] as follows

$$\psi(p/q) = -\gamma - \ell n(q) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + \sum_{j=1}^{q-1}\left\{\cos\left(\frac{2\pi pj}{q}\right)\ell n\left(2\sin\left(\frac{\pi j}{q}\right)\right)\right\}, \tag{1.8}$$

where $1 \leq p < q$ and p, q are positive integers.

Afterwards, in 2007, an attempt was made by Murty and Saradha [17, p. 300, after eq.(4)] (see also, Lehmer [14, p. 135, after eq.(20)]) for the simplification of the above Gauss formula (1.7) as follows

$$\psi(p/q) = -\gamma - \ell n\,(2q) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + 2\sum_{j=1}^{[\frac{q}{2}]}\left\{\cos\left(\frac{2\pi p j}{q}\right)\ell n \sin\left(\frac{\pi j}{q}\right)\right\}, \tag{1.9}$$

where $p = 1, 2, 3, \ldots, (q-1), q = 2, 3, 4, \ldots; (p, q) = 1$.

Also, we have verified the results (1.7), (1.8) and (1.9) by taking different values of p and q.

Gradshteyn and Ryzhik [11, p. 904, eq 8.363(6)] recorded an erroneous formula for digamma function such that

$$\psi(p/q) \mathrel{\mathring{=}} -\gamma - \ell n\,(2q) - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + 2\sum_{j=1}^{[\frac{q+1}{2}]-1}\left\{\cos\left(\frac{2\pi p j}{q}\right)\ell n \sin\left(\frac{\pi j}{q}\right)\right\}, \tag{1.10}$$

where $p = 1, 2, 3, \ldots, (q-1), q = 2, 3, 4, \ldots; (p, q) = 1$ and the symbol $\mathring{=}$ exhibits the fact that equation (1.10) does not hold true as stated.

Some important facts, which appreciate us to work in this direction, are as follows

—We cannot compute the value of digamma function when $p > q$ or (and) $\frac{p}{q}$ is negative fraction using Gauss formula [9].

—We cannot compute the value of digamma function when $p > q$ using Jensen formula [13].

—We cannot compute the value of digamma function when $\frac{p}{q}$ is negative using Jensen [13].

—Murty and Saradha [17, p. 300] corrected a formula of Lehmer [14, p. 135] for $\psi(\frac{p}{q})$

—Some specific values of digamma function were proved transcendental by Murty and Saradha [17].

2. SOME NEW IDENTITIES FOR DIGAMMA FUNCTION

Some functional relations for digamma function, that are easily derivable from the properties of the gamma function, are recalled here. Indeed, from the formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \neq 0, \pm 1, \pm 2, \pm 3, \dots \tag{2.1}$$

taking ℓn both sides and differentiating the above equation with respect to z, we derive the some basic identities for digamma function as follows

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \quad \psi(1-z) = \psi(z) + \pi \cot(\pi z), \quad z \neq 0, \pm 1, \pm 2, \pm 3, \dots \tag{2.2}$$

$$\psi(z+n) = \frac{1}{z} + \frac{1}{z+1} + \cdots + \frac{1}{z+n-1} + \psi(z). \tag{2.3}$$

On setting $z = (1-z)$ in equation (2.2), we get

$$\psi(-z) = \frac{1}{z} + \psi(1-z). \tag{2.4}$$

On comparing the values of $\psi(1-z)$ from the equations (2.2) and (2.4), we get a new identity

$$\psi(z) + \pi \cot(\pi z) = \psi(-z) - \frac{1}{z}. \tag{2.5}$$

By setting $z = \frac{p}{q}$, $1 \leq p < q$ in equations (2.2) and (2.4), we get more identities. These identities, enable us to derive our main identities, are as follows

$$\psi\left(\frac{p+q}{q}\right) = \frac{q}{p} + \psi\left(\frac{p}{q}\right), \text{ and } \psi\left(\frac{-p}{q}\right) = \frac{q}{p} + \psi\left(\frac{q-p}{q}\right), \quad 1 \leq p < q. \tag{2.6}$$

For the sake of convenient computation of digamma function, we derive some more identities, which are simple but more applicable in the computation of digamma function for $\frac{p}{q} > 1$. For this concern, we connect the Murty and Saradha's formula for digamma function (1.11) with our above identity (2.6) and get the result as follows

$$\psi\left(\frac{q-p}{q}\right) = -\gamma - \ell n(2q) + \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + 2\sum_{j=1}^{[\frac{q}{2}]}\left\{\cos\left(\frac{2\pi p j}{q}\right)\ell n \sin\left(\frac{\pi j}{q}\right)\right\}, \tag{2.7}$$

$(p,q) = 1;\ 1 \leq p < q$. Now, we derive the identity for computation of the digamma function for negative fractions $\left(-\frac{p}{q}\right)$. For this motive, we derive the identity in the similar manner as used in the above identity and get the result as follows

$$\psi\left(\frac{-p}{q}\right) = \frac{q}{p} - \gamma - \ell n(2q) - \frac{\pi}{2}\cot\left(\frac{\pi(q-p)}{q}\right) + 2\sum_{j=1}^{[\frac{q}{2}]}\left\{\cos\left(\frac{2\pi(q-p)j}{q}\right)\ell n \sin\left(\frac{\pi j}{q}\right)\right\}, \tag{2.8}$$

$1 \leq p < q$.

3. NUMERIC COMPUTATIONS OF DIGAMMA FUNCTION

Table I. ψ- Function(Fractional Valued, $p > q$)

Ser. No.	$z = \frac{p}{q}$	$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$
1	$\frac{7}{3}$	$-\gamma + \frac{15}{4} - \frac{\pi\sqrt{3}}{6} - \frac{3}{2}\ell n 3$
2	$\frac{3}{2}$	$-\gamma + 2 - 2\ell n 2$
3	$\frac{5}{2}$	$-\gamma + \frac{8}{3} - 2\ell n 2$

Table II. ψ- Function(Positive fractional Order)

Ser. No.	$z = \frac{p}{q}$	$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$
1	$\frac{1}{2}$	$-\gamma - 2\ell n 2$
2	$\frac{1}{3}$	$-\gamma - \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ell n 3$
3	$\frac{1}{4}$	$-\gamma - \frac{\pi}{2} - 3\ell n 2$
4	$\frac{1}{5}$	$-\gamma - \ell n 10 - \left(\frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n\left(\frac{\sqrt{5}-1}{2}\right) - \ell n\frac{\sqrt{5}}{4}\}$
5	$\frac{1}{6}$	$-\gamma - \ell n 12 - \frac{\pi\sqrt{3}}{2} - \ell n\sqrt{3}$
6	$\frac{1}{8}$	$-\gamma - \frac{(1+\sqrt{2})\pi}{2} - 4\ell n 2 - \sqrt{2}\ell n(1+\sqrt{2})$
7	$\frac{1}{10}$	$-\gamma - \ell n 20 - \left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n(\sqrt{5}-2) - \ell n\sqrt{5}\}$
8	$\frac{1}{12}$	$-\gamma - \ell n 24 - (2+\sqrt{3})\frac{\pi}{2} + \{\sqrt{3}\ell n(2-\sqrt{3}) - \ell n\sqrt{3}\}$
9	$\frac{2}{3}$	$-\gamma + \frac{\sqrt{3}\pi}{6} - \frac{3}{2}\ell n 3$
10	$\frac{2}{5}$	$-\gamma - \ell n 10 - \left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n\left(\frac{\sqrt{5}+1}{2}\right) - \ell n\frac{\sqrt{5}}{4}\}$
11	$\frac{3}{4}$	$-\gamma + \frac{\pi}{2} - 3\ell n 2$
12	$\frac{3}{5}$	$-\gamma - \ell n 10 + \left(\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n\left(\frac{\sqrt{5}+1}{2}\right) - \ell n\frac{\sqrt{5}}{4}\}$
13	$\frac{3}{8}$	$-\gamma - \frac{(\sqrt{2}-1)\pi}{2} - 4\ell n 2 + \sqrt{2}\ell n(1+\sqrt{2})$
14	$\frac{3}{10}$	$-\gamma - \ell n 20 - \left(\frac{\sqrt{(10-2\sqrt{5})}}{1+\sqrt{5}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n(2+\sqrt{5}) - \ell n\sqrt{5}\}$
15	$\frac{4}{5}$	$-\gamma - \ell n 10 + \left(\frac{1+\sqrt{5}}{\sqrt{(10-2\sqrt{5})}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n\left(\frac{\sqrt{5}-1}{2}\right) - \ell n\frac{\sqrt{5}}{4}\}$
16	$\frac{5}{6}$	$-\gamma - \ell n 12 + \frac{\pi\sqrt{3}}{2} - \ell n\sqrt{3}$
17	$\frac{5}{8}$	$-\gamma + \frac{(\sqrt{2}-1)\pi}{2} - 4\ell n 2 + \sqrt{2}\ell n(1+\sqrt{2})$
18	$\frac{5}{12}$	$-\gamma - \ell n 24 - (2-\sqrt{3})\frac{\pi}{2} + \{\sqrt{3}\ell n(2+\sqrt{3}) - \ell n\sqrt{3}\}$
19	$\frac{7}{8}$	$-\gamma + \frac{(1+\sqrt{2})\pi}{2} - 4\ell n 2 - \sqrt{2}\ell n(1+\sqrt{2})$
20	$\frac{7}{10}$	$-\gamma - \ell n 20 + \left(\frac{\sqrt{(10-2\sqrt{5})}}{1+\sqrt{5}}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n(2+\sqrt{5}) - \ell n\sqrt{5}\}$
21	$\frac{7}{12}$	$-\gamma - \ell n 24 + (2-\sqrt{3})\frac{\pi}{2} + \{\sqrt{3}\ell n(2+\sqrt{3}) - \ell n\sqrt{3}\}$
22	$\frac{9}{10}$	$-\gamma - \ell n 20 + \left(\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}\right)\frac{\pi}{2} + \frac{1}{2}\{\sqrt{5}\ell n(\sqrt{5}-2) - \ell n\sqrt{5}\}$
23	$\frac{11}{12}$	$-\gamma - \ell n 24 + (2+\sqrt{3})\frac{\pi}{2} + \{\sqrt{3}\ell n(2-\sqrt{3}) - \ell n\sqrt{3}\}$

Table III. ψ- Function(Negative fractional Valued)

Ser. No.	$z=\frac{p}{q}$	$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$
1	$-\frac{2}{3}$	$-\gamma+\frac{3}{2}-\frac{\pi\sqrt{3}}{6}-\frac{3\ell n3}{2}$
2	$-\frac{3}{4}$	$-\gamma+\frac{4}{3}-\frac{\pi}{2}-3\ell n2$
3	$-\frac{1}{2}$	$-\gamma+2-2\ell n2$
4	$-\frac{1}{3}$	$-\gamma+3+\frac{\sqrt{3}\pi}{6}-\frac{3}{2}\ell n3$
5	$-\frac{1}{4}$	$-\gamma+4+\frac{\pi}{2}-3\ell n2$
6	$-\frac{5}{8}$	$-\gamma+\frac{8}{5}-\frac{(\sqrt{2}-1)\pi}{2}-4\ell n2+\sqrt{2}\ell n(1+\sqrt{2})$
7	$-\frac{3}{8}$	$-\gamma+\frac{8}{3}+\frac{(\sqrt{2}-1)\pi}{2}-4\ell n2+\sqrt{2}\ell n(1+\sqrt{2})$
8	$-\frac{1}{8}$	$-\gamma+8+\frac{(1+\sqrt{2})\pi}{2}-4\ell n2-\sqrt{2}\ell n(1+\sqrt{2})$
9	$-\frac{5}{6}$	$-\gamma+\frac{6}{5}-\frac{\pi\sqrt{3}}{2}-\frac{3}{2}\ell n3-2\ell n2$
10	$-\frac{3}{2}$	$-\gamma+\frac{8}{3}-2\ell n2$
11	$-\frac{7}{3}$	$-\gamma+\frac{117}{28}+\frac{\pi\sqrt{3}}{6}-\frac{3}{2}\ell n3$

The following errata are found in a paper of Jensen [13, p. 147] such that

$$\psi(3/5) \circeq -\gamma+\frac{\pi}{2}\sqrt{\left(1-\frac{2}{\sqrt{5}}\right)}-\frac{5}{4}\ell n5+\frac{\sqrt{5}}{4}\ell n\left(\frac{3+2\sqrt{5}}{2}\right), \tag{3.1}$$

$$\psi(4/5) \circeq -\gamma+\frac{\pi}{2}\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}-\frac{5}{4}\ell n5-\frac{\sqrt{5}}{4}\ell n\left(\frac{3+2\sqrt{5}}{2}\right), \tag{3.2}$$

where the symbol $\circeq$ exhibits the fact that each of the equations (3.1) and (3.2) does not hold true as stated.

The following are the corrected forms of above equations

$$\psi(3/5) = -\gamma+\frac{\pi}{2}\sqrt{\left(1-\frac{2}{\sqrt{5}}\right)}-\frac{5}{4}\ell n5+\frac{\sqrt{5}}{4}\ell n\left(\frac{3+\sqrt{5}}{2}\right), \tag{3.3}$$

$$\psi(4/5) = -\gamma+\frac{\pi}{2}\sqrt{\left(1+\frac{2}{\sqrt{5}}\right)}-\frac{5}{4}\ell n5-\frac{\sqrt{5}}{4}\ell n\left(\frac{3+\sqrt{5}}{2}\right). \tag{3.4}$$

4. CONCLUDING REMARK

We conclude our present investigation by observing that some new identities for digamma function have been deduced as the equations (2.6), (2.7) and (2.8) in an analogous manner. Using these new identities, we have calculated the values of digamma function for positive and negative fractional orders. Also, we have presented correct forms of two errata found in Jensen's article [13, p. 147].

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The ruin problem for a Wiener process with state-dependent jumps

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Abstract

Let $X(t)$ be a jump-diffusion process whose continuous part is a Wiener process, and let $T(x)$ be the first time it leaves the interval $(0,b)$, where $x = X(0)$. The jumps are negative and their sizes depend on the value of $X(t)$. Moreover there can be a jump from $X(t)$ to 0. We transform the integro-differential equation satisfied by the probability $p(x) := P[X(T(x)) = 0]$ into an ordinary differential equation and we solve this equation explicitly in particular cases. We are also interested in the moment-generating function of $T(x)$.

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Keywords: First exit time, Brownian motion, Poisson process, jump size, integro-differential equation.

1. INTRODUCTION

We consider the one-dimensional jump-diffusion process $\{X(t), t \geq 0\}$ defined by

$$X(t) = X(0) + \mu t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, $\{B(t), t \geq 0\}$ is a standard Brownian motion and $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ that is independent of $\{B(t), t \geq 0\}$. Thus the continuous part of $X(t)$ is a Wiener process with infinitesimal mean μ and infinitesimal variance σ^2. Moreover, there are Poissonian jumps of random size.

The random variables Y_i, $i = 1, 2, \ldots$, are assumed to be independent and identically distributed as the mixed type random variable Y whose probability density function, when $X(t) = z$, is given by

$$f_Y(y; z) = p_0 (-1)^n (n+1) \frac{y^n}{z^{n+1}} I_{(-z,0)}(y) + q_0 \, \delta(y + z) \quad \forall y \in \mathbb{R}, \tag{2}$$

where $p_0 = 1 - q_0 \in [0, 1]$, $n \in \{0, 1, \ldots\}$, $z > 0$, $I_{(-z,0)}(\cdot)$ is the indicator function of the interval $(-z, 0)$, and $\delta(\cdot)$ is the Dirac delta function. Notice that if $n = 0$, then the continuous part of Y is uniformly distributed over the interval $(-z, 0)$.

Next, let $T(x)$ be the first exit time of $X(t)$ from the interval $(0, b)$:

$$T(x) = \inf\{t > 0 : X(t) \notin (0, b) \mid X(0) = x \in (0, b)\}. \tag{3}$$

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We are interested in computing the probability of ruin

$$p(x) := P[X(T(x)) = 0]. \tag{4}$$

Remark that because the jumps are always negative and are, in absolute value, smaller than or equal to the current value of $X(t)$, there is no overshoot.

Ruin problems are important in actuarial science and in mathematical finance, in particular. Such problems for jump-diffusion processes have been considered, among others, by Cai and Xu (2006), Jiang and Yan (2006), Gerber and Yang (2007), and Yin *et al.* (2013).

In Section 2, we will give the integro-differential equation satisfied by the function $p(x)$, and we will show that it is possible, under some assumptions, to transform this equation into an ordinary differential equation (o.d.e.). The resulting o.d.e. will be solved explicitly in particular cases.

In Section 3, we will turn to the problem of computing the moment-generating function of $T(x)$. We will conclude this paper with a few remarks in Section 4.

2. PROBABILITY OF RUIN

Assume that $g(x)$ is a twice continuously differentiable function. The infinitesimal generator of the process $\{X(t), t \geq 0\}$ defined in (1) is (see Kou and Wang 2003)

$$\mathscr{L}g(x) = \frac{1}{2}\sigma^2 g''(x) + \mu\, g'(x) - \lambda\, g(x) + \lambda \int_{-\infty}^{\infty} g(x+y)\, f_Y(y;x)\, \mathrm{d}y \tag{5}$$

for $x \in (0,b)$. With the density function of Y defined in Eq. (2), then Eq. (5) becomes (since $p(0) = 1$)

$$\begin{aligned} \mathscr{L}g(x) \;=\; & \frac{1}{2}\sigma^2 g''(x) + \mu\, g'(x) - \lambda\, g(x) \\ & + \lambda\, p_0\, (-1)^n\, (n+1)\, \frac{1}{x^{n+1}} \int_{-x}^{0} g(x+y)\, y^n\, \mathrm{d}y + \lambda\, q_0. \end{aligned} \tag{6}$$

If we assume that the conditional transition density function

$$p(x_1,t;x_0,t_0) := \lim_{d x_1 \downarrow 0} \frac{P[X(t) \in (x_1, x_1 + d x_1) \mid X(t_0) = x_0]}{d x_1} \tag{7}$$

exists for $t > t_0$ (see Gihman and Skorohod 1972), then we can write that the function $p(x)$ defined in (4) satisfies the integro-differential equation

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\,\sigma^2\,p''(x)+\mu\,p'(x)-\lambda\,p(x) \\
& +\lambda\,p_0\,(-1)^n\,(n+1)\,\frac{1}{x^{n+1}}\int_{-x}^{0}p(x+y)\,y^n\,\mathrm{d}y+\lambda\,q_0
\end{aligned} \tag{8}
$$

for $x\in(0,b)$. The boundary conditions are

$$
p(0)=1 \quad \text{and} \quad p(b)=0. \tag{9}
$$

Next, we rewrite the generalized backward Kolmogorov equation (8) as follows:

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\sigma^2\,p''(x)+\mu\,p'(x)-\lambda\,p(x) \\
& +\lambda\,p_0\,(-1)^n\,(n+1)\frac{1}{x^{n+1}}\int_{0}^{x}p(z)\,(z-x)^n\,\mathrm{d}z+\lambda\,q_0.
\end{aligned} \tag{10}
$$

Let

$$
I_n(x) := \int_0^x p(z)\,(z-x)^n\,\mathrm{d}z. \tag{11}
$$

We have $I_0'(x)=p(x)$ and

$$
I_n'(x)=-n\,I_{n-1} \quad \text{for } n=1,2,\ldots \tag{12}
$$

Assume now that $n=0$ and that the function $p(x)$ is three times differentiable. Differentiating Eq. (10), we obtain that

$$
0=\frac{1}{2}\,\sigma^2\,p'''(x)+\mu\,p''(x)-\lambda\,p'(x)+\lambda\,p_0\left(-\frac{1}{x^2}\,I_0(x)+\frac{1}{x}\,p(x)\right), \tag{13}
$$

which we rewrite as follows:

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\,\sigma^2\,p'''(x)+\mu\,p''(x)-\lambda\,p'(x) \\
& +\frac{1}{x}\left(\frac{1}{2}\sigma^2\,p''(x)+\mu\,p'(x)-\lambda\,p(x)+\lambda\,q_0\right)+\lambda\,p_0\,\frac{1}{x}\,p(x).
\end{aligned} \tag{14}
$$

Therefore we obtain that the function $p(x)$ satisfies the third-order linear o.d.e.

$$
\begin{aligned}
0 \;=\; & \frac{1}{2}\,\sigma^2\,x\,p'''(x)+\left(\frac{1}{2}\sigma^2+\mu\,x\right)\,p''(x)+(\mu-\lambda\,x)\,p'(x) \\
& +\lambda\,(p_0-1)\,p(x)+\lambda\,q_0.
\end{aligned} \tag{15}
$$

If we assume that $n\in\{1,2,\ldots\}$ and that the function $p(x)$ is $n+3$ times differentiable, differentiating Eq. (10) once we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\,\sigma^2\,p'''(x)+\mu\,p''(x)-\lambda\,p'(x) \\ & -\lambda\,p_0\,(-1)^n\,(n+1)\,\frac{1}{x^{n+1}}\left\{\frac{n+1}{x}\,I_n(x)+n\,I_{n-1}(x)\right\}, \end{aligned} \tag{16}$$

from which we deduce that

$$\begin{aligned} \lambda\,p_0\,(-1)^n\,n\,(n+1)\,I_{n-1}(x) \;=\; & x^{n+1}\left(\frac{1}{2}\,\sigma^2\,p'''(x)+\mu\,p''(x)-\lambda\,p'(x)\right) \\ & +(n+1)\,x^n\left(\frac{1}{2}\sigma^2\,p''(x)+\mu\,p'(x)-\lambda\,p(x)\right) \\ & +(n+1)\,\lambda\,x^n\,q_0. \end{aligned} \tag{17}$$

Let us set

$$\begin{aligned} h(x) \;:=\; & \frac{1}{2}\,\sigma^2\,x^{n+1}\,p'''(x)+\left(\frac{1}{2}\,(n+1)\,\sigma^2+\mu\,x\right)x^n\,p''(x) \\ & +[(n+1)\,\mu-\lambda\,x]\,x^n\,p'(x)-(n+1)\,\lambda x^n\,[p(x)-q_0]. \end{aligned} \tag{18}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x^{n+1}}I_n(x)=(-1)^n\,n!\,p(x), \tag{19}$$

we can now state the following proposition.

PROPOSITION 2.1. Suppose that the function $p(x)$ defined in (4) is three times differentiable. Then it satisfies the o.d.e. (15) for $0<x<b$, subject to the boundary conditions in (9). If it is $n+3$ times differentiable, where $n\in\{1,2,\ldots\}$, then it satisfies the o.d.e.

$$-\lambda\,p_0\,(n+1)!\,p(x)=\frac{\mathrm{d}}{\mathrm{d}x^n}h(x) \tag{20}$$

for $0<x<b$, where $h(x)$ is defined in (18).

REMARK 1. (i) With $n=1$, we obtain that

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\,\sigma^2\,x^2\,p^{(4)}(x)+\left(2\,\sigma^2\,x+\mu\,x^2\right)\,p'''(x)+\left(\sigma^2+4\,\mu\,x-\lambda\,x^2\right)\,p''(x) \\ & +\,(2\,\mu-4\,\lambda\,x)\,p'(x)+2\,(p_0-1)\,\lambda\,p(x)+2\,\lambda\,q_0. \end{aligned} \tag{21}$$

(ii) Equation (20) is a linear o.d.e. of order $n+3$. Its general solution will therefore involve $n+3$ arbitrary constants. Two constants are deduced from the two boundary conditions. To determine the remaining constants, we can substitute the expression derived for $p(x)$ into the integro-differential equation (10) and into each equation obtained by differentiating this equation $n+1$ times. We can also use the fact that $0 \le p(x) \le 1$. Finally, another condition is deduced from Eq. (10) by letting x decrease to zero in this equation.

(iii) We assume the existence of non-negative solutions to Eq. (20) (and, consequently, to the problem (8), (9)). For papers on the existence of such solutions to related problems, see [7] and [8].

(iv) Although Eq. (20) is a higher-order differential equation, it is linear. In general, at least for n small, it is easier to solve linear ordinary differential equations than integro-differential equations. Indeed, as will be seen in the next subsection, it is sometimes possible to obtain an explicit solution to a higher-order linear o.d.e. by making use of a mathematical software. Moreover, if such an explicit solution cannot be found, one can resort to any of the various numerical techniques for solving ordinary differential equations.

(v) It is not possible to transform the integro-differential equation (8) into an ordinary differential equation for any jump-diffusion process or for any distribution of the jumps. In general, proceeding as above, one can expect to obtain a differential-difference equation, which can itself be difficult to solve explicitly.

2.1. Particular cases

Case 1. We first consider the case when $\sigma^2 = 1$ and $\mu = 0$, so that the continuous part of the process $\{X(t), t \ge 0\}$ is a standard Brownian motion, $n = 1$, $\lambda = 1$, $p_0 = 1$ and $b = 1$. Equation (21) then reduces to

$$0 = \frac{1}{2} x^2 p^{(4)}(x) + 2x p'''(x) + \left(1 - x^2\right) p''(x) - 4x p'(x). \tag{22}$$

We find that the particular solution that we are looking for is

$$p(x) = 1 + c \left[\mathrm{Shi}(\sqrt{2}x) + \sinh(\sqrt{2}x) \right], \tag{23}$$

where Shi is the *hyperbolic sine integral* defined by

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh(u)}{u} \, \mathrm{d}u. \tag{24}$$

The constant c is given by

$$c = \frac{2}{\mathrm{Ei}(1, -\sqrt{2}) - \mathrm{Ei}(1, \sqrt{2}) + \pi i + e^{-\sqrt{2}} - e^{\sqrt{2}}}, \tag{25}$$

in which Ei is the *exponential integral*:

$$\mathrm{Ei}(a,z) := \int_1^\infty e^{-uz} u^{-a} \mathrm{d}u. \tag{26}$$

We find that $c \simeq -0{,}2844$. The function $p(x)$ as well as the function $p^*(x) := 1-x$ obtained when there are no jumps are shown in Figure 1.

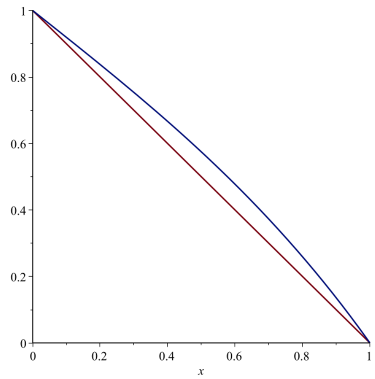

Fig. 1. Functions $p(x)$ (above) and $p^*(x) := 1-x$ in the interval $[0,1]$ when $\sigma^2 = 1$, $\mu = 0$, and $n = \lambda = p_0 = b = 1$.

REMARK 2. The above solution can be generalized to the case of any positive λ by replacing $\sqrt{2}$ by $\sqrt{2\lambda}$ everywhere. We find that the expression thus obtained does indeed tend to $1-x$ as λ decreases to zero.

Case 2. Next, we take $\sigma^2 = 1$, $\mu = 0$, $n = 0$, $\lambda = 1$, $p_0 = 1/2$ and $b = 1$. We must then solve (see Eq. (14))

$$0 = x p'''(x) + p''(x) - 2x p'(x) - p(x) + 1. \tag{27}$$

We can show, using the mathematical software *Maple*, that the solution that we are looking for is

$$p(x) = 1 - \frac{\mathrm{hypergeom}\left(\left[\frac{3}{4}\right], \left[1, \frac{3}{2}\right], \frac{1}{2}x^2\right) x}{\mathrm{hypergeom}\left(\left[\frac{3}{4}\right], \left[1, \frac{3}{2}\right], \frac{1}{2}\right)}, \tag{28}$$

where hypergeom is a *generalized hypergeometric function*. We present the functions $p(x)$ and $p^*(x) := 1-x$ in Figure 2.

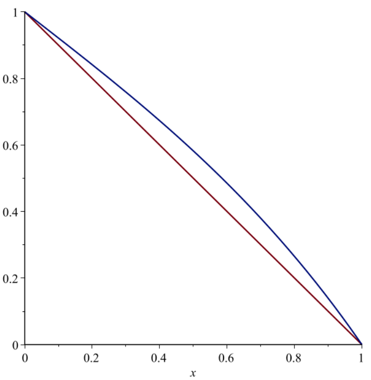

Fig. 2. Functions $p(x)$ (above) and $p^*(x) := 1-x$ in the interval $[0,1]$ when $\sigma^2 = 1$, $\mu = 0$, $n = 0$, $\lambda = b = 1$ and $p_0 = 1/2$.

Case 3. If we take $\mu = 1$ instead of 0 in Case 1, the o.d.e. that we must solve is

$$0 = \frac{1}{2}x^2 p^{(4)}(x) + (2x + x^2)\, p'''(x) + \left(1 + 4x - x^2\right) p''(x) + (2-4x)\, p'(x). \tag{29}$$

After some work, we find that

$$\begin{aligned} p(x) \;=\; & -c_1\sqrt{3}\,\mathrm{Ei}(1, 1-\sqrt{3}) + c_2\left[e^{\sqrt{3}-1} - \mathrm{Ei}(1, 1-\sqrt{3})\right] \\ & - \frac{c_1\left[(3-\sqrt{3})\,\mathrm{Ei}(1, 1+\sqrt{3}) - 3\,e^{-1-\sqrt{3}}\right](3+\sqrt{3})}{\sqrt{3}-3} \\ & + (c_1\sqrt{3} + c_2)\,\mathrm{Ei}(1, (1-\sqrt{3})x) - c_2\, e^{(\sqrt{3}-1)x} \\ & - \frac{c_1\left[(\sqrt{3}-3)\,\mathrm{Ei}(1, (1+\sqrt{3})x) + 3\,e^{-(1+\sqrt{3})x}\right](3+\sqrt{3})}{\sqrt{3}-3}, \end{aligned} \tag{30}$$

where

$$c_1 \simeq 0{,}0392 \quad \text{and} \quad c_2 \simeq 0{,}1177. \tag{31}$$

This function is compared in Figure 3 with the function

$$p^{**}(x) := \frac{e^{-2} - e^{-2x}}{e^{-2} - 1} \tag{32}$$

obtained when there are no jumps.

In the next section, we will turn briefly to the problem of computing the moment-generating function of the random variable $T(x)$.

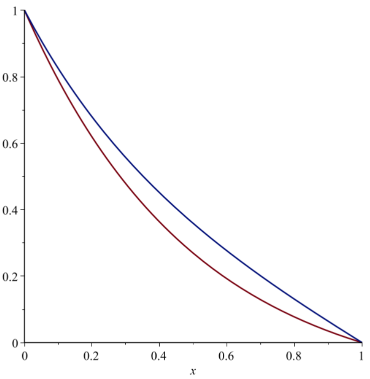

Fig. 3. Functions $p(x)$ (below) and $p^{**}(x)$ defined in (32) in the interval $[0,1]$ when $\sigma^2 = 1$, $\mu = 1$ and $n = \lambda = p_0 = b = 1$.

3. MOMENT-GENERATING FUNCTION OF $T(X)$

Let us denote by $M(x;\alpha)$ the moment-generating function of $T(x)$:

$$M(x;\alpha) := E\left[e^{-\alpha T(x)}\right], \tag{33}$$

where $\alpha > 0$. If this function is twice differentiable, then it satisfies the integro-differential equation (dropping the dependence on α in the notation)

$$\begin{aligned} \alpha M(x) \; &= \; \frac{1}{2}\sigma^2 M''(x) + \mu M'(x) - \lambda M(x) \\ &\quad + \lambda\, p_0\,(-1)^n\,(n+1)\,\frac{1}{x^{n+1}}\int_{-x}^{0} M(x+y)\,y^n\,\mathrm{d}y + \lambda\, q_0 \end{aligned} \tag{34}$$

for $x \in (0,b)$, subject to the boundary conditions

$$M(0) = M(b) = 1. \tag{35}$$

Proceeding as in the previous section, we can derive results like the ones in Proposition 2.1. It will be more difficult however to obtain exact solutions to the differential equations that correspond to Eqs. (14) and (20). To conclude, we will present a particular problem that we can solve explicitly.

The function $M(x)$ in the second particular case considered in Section 2 satisfies the integro-differential equation

$$\alpha M(x) = \frac{1}{2}M''(x) - M(x) + \frac{1}{2}\left\{\frac{1}{x}\int_0^x M(z)\,\mathrm{d}z + 1\right\}. \tag{36}$$

That is,

$$\frac{1}{2}xM''(x)-(\alpha+1)xM(x)+\frac{1}{2}\left\{\int_0^x M(z)\,\mathrm{d}z+x\right\}=0. \tag{37}$$

Differentiating the above equation, we obtain

$$\frac{1}{2}xM'''(x)+\frac{1}{2}M''(x)-(\alpha+1)xM'(x)-\left(\alpha+\frac{1}{2}\right)M(x)+\frac{1}{2}=0. \tag{38}$$

We find that the function $M(x)$ is given by

$$\begin{aligned} M(x) \;=\; & \frac{1}{2\alpha+1} \\ & +\frac{2\alpha}{2\alpha+1}\,\text{hypergeom}\left(\left[\frac{1}{4}\ \frac{2\alpha+1}{\alpha+1}\right],\left[\frac{1}{2},\frac{1}{2}\right],\frac{1}{2}(\alpha+1)x^2\right) \\ & +c_1\,\text{hypergeom}\left(\left[\frac{1}{4}\ \frac{4\alpha+3}{\alpha+1}\right],\left[1,\frac{3}{2}\right],\frac{1}{2}(\alpha+1)x^2\right)x, \end{aligned} \tag{39}$$

where

$$c_1=-\frac{2\alpha}{2\alpha+1}\,\frac{\text{hypergeom}\left(\left[\frac{1}{4}\ \frac{2\alpha+1}{\alpha+1}\right],\left[\frac{1}{2},\frac{1}{2}\right],\frac{1}{2}(\alpha+1)\right)-1}{\text{hypergeom}\left(\left[\frac{1}{4}\ \frac{4\alpha+3}{\alpha+1}\right],\left[1,\frac{3}{2}\right],\frac{1}{2}(\alpha+1)\right)}. \tag{40}$$

When there are no jumps, the moment-generating function of $T(x)$ is

$$M^*(x):=\frac{(1-e^{-\sqrt{2\alpha}})\,e^{\sqrt{2\alpha}x}+(-1+e^{\sqrt{2\alpha}})\,e^{-\sqrt{2\alpha}x}}{e^{\sqrt{2\alpha}}-e^{-\sqrt{2\alpha}}}. \tag{41}$$

The functions $M(x)$ and $M^*(x)$ are shown in Figure 4 when $\alpha=1$.

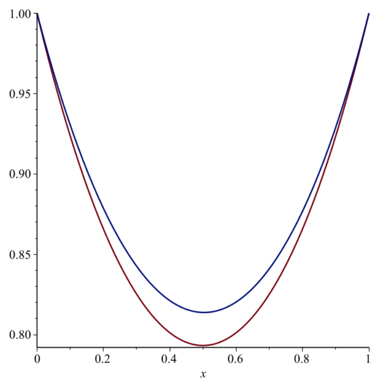

Fig. 4. Functions $M(x)$ (above) and $M^*(x)$ in the interval $[0,1]$ when $\alpha=1$, $\sigma^2=1$, $\mu=0$, $n=0$, $\lambda=b=1$ and $p_0=1/2$.

4. CONCLUSION

In this paper, the problem of computing the probability of ruin for a Wiener process with state-dependent jumps has been considered. We obtained exact and explicit solutions to particular problems, which is generally difficult to achieve for jump-diffusion processes.

We could have assumed that the continuous part of the jump-diffusion process $\{X(t), t \geq 0\}$ is a general time-homogeneous diffusion process with infinitesimal parameters $\mu(x)$ and $\sigma^2(x)$. The ordinary differential equation (20) would then obviously be more complicated and difficult to solve explicitly.

To generalize the results obtained in this paper, we could add positive jumps in the model. Again, Eq. (20) would become more complicated, especially if the upward jumps are also state-dependent. Conversely, it would have been simpler to assume that the negative jumps do not depend on the value of $X(t)$, and that $p(x) = P[X(T(x)) \leq 0]$. However, in some applications it is not realistic to allow negative values of $X(t)$. For instance, if $X(t)$ denotes the price of one share of a certain stock at time t, we must impose the condition $X(t) \geq 0$.

Finally, we could consider the problem of maximizing the expected value of the random variable $T(x)$ for the controlled process $\{X_u(t), t \geq 0\}$ defined by

$$X_u(t) = X_u(0) + b_0 \int_0^t u[X_u(s)]\,\mathrm{d}s + \mu\, t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i, \tag{42}$$

where b_0 is a positive constant and $u(\cdot)$ is the control variable; see Lefebvre (2004).

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Geometry of the probability simplex and its connection to the maximum entropy method

H. GZYL AND F. NIELSEN

Abstract

The use of geometrical methods in statistics has a long and rich history highlighting many different aspects. These methods are usually based on a Riemannian structure defined on the space of parameters that characterize a family of probabilities. In this paper, we consider the finite dimensional case but the basic ideas can be extended similarly to the infinite-dimensional case. Our aim is to understand exponential families of probabilities on a finite set from an intrinsic geometrical point of view and not through the parameters that characterize some given family of probabilities.

For that purpose, we consider a Riemannian geometry defined on the set of positive vectors in a finite-dimensional space. In this space, the probabilities on a finite set comprise a submanifold in which exponential families correspond to geodesic surfaces. We shall also obtain a geometric/dynamic interpretation of Jaynes' method of maximum entropy.

Mathematics Subject Classification 2010: 53C99, 62B05.
Keywords: Geometry on positive vectors, geometry on the probability simplex, logarithmic distance on the class of positive vectors, the maximum entropy method.

1. INTRODUCTION AND PRELIMINARIES

1.1. Geometry of the probability distributions

Geometry and statistics have been intertwined for some time already, mainly through the study of differential-geometric structures in the space of parameters that characterize parametric families of distributions. Consider for example the seminal works of Hotelling [13] and Rao [27], and the works of Amari [2; 3], Amari et al. [4], Efron [9], Barndorff-Nielsen [5], Vajda [29], and more recently Pistone and Semi [26], Pistone and Rogatin [25]. In all of these works a special emphasis is laid upon *exponential families*. In information geometry, the geometry of exponential families is elucidated by a dually flat manifold [2] (that is, a pair of torsion-free flat affine connections that are metric-compatible). A categorical distribution is a discrete probability distribution that describes the possible results of a random variable that can take on one of n possible choices $\mathscr{G} = \{1, \ldots, n\}$. The space of all categorical distributions form an exponential family commonly called the probability simplex:

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Δ_{n-1}. In information geometry, the probability simplex can also be viewed as a mixture family [3]. Mixture families can also be modeled by dually flat manifolds [23].

Consider as well the more recent work by Imparato and Trivelato [14] and Pistone [24], which certainly belongs to the same class of models, and, even though the techniques are quite different from those developed here, the similarities are many. And to finish this short list of references, we cite the nice textbook by Calin and Udriste [8].

A non-parametric approach based on an intrinsic geometry on the space of probability densities, and to understand exponential families in that set up was put forward in Gzyl and Recht [10]-[11]. The approach considered in that work was too general, and less germane than the Riemannian approach considered below. The excess generality in those papers comes from the use of algebras over the complex field. Even so, quite a bit of interesting connections between families of exponential type, geodesics and polynomials of binomial type was established in [11]. Let us point out another recent approach which considered the Hilbert geometry for the probability simplex and its connection with the Birkhoff cone projective geometry of the positive orthant [22].

Actually there has been much interest in that geometry, not in $\mathbb{R}^n$ but in the space of symmetric positive-definite matrices. The reader can check with Lang [17] in which a relation of this geometry to Bruhat-Tits spaces is explored, or in Lawson and Lim [18] or Moakher [20] were the geometric mean property for sets of symmetric positive-definite matrices is established. More recently Arsigny et al. [1] described the use of that geometry to deal with a variety of applications, and Schwartzman [28] used that geometric setup to study lognormal distributions with values in the class of symmetric matrices.

1.2. Paper outline

The paper is organized as follows. In Section 2 we present the essentials about the geometry on the set of strictly positive vectors in a finite dimensional space. Here we present the finite dimensional case only for two reasons: First because all geometric ideas are already present in this case, and second, for not to encumber the manuscript with technical details pertinent to the infinite dimensional case needed to deal with probability densities. In Section 3 we regard probabilities on finite sets as a submanifold of the set of strictly positive numbers, and verify that exponential probability distributions correspond to geodesic hyper-surfaces in that manifold (i.e., autoparallel submanifolds). We then provide a geometric interpretation for the method of maximum entropy [15]: The Lagrange multipliers (which are related to intensive magnitudes in statistical physics, correspond to *travel time* along a geodesic from an initial distribution to the distributions satisfying given constraints). In section 5 we recall, for the sake of completeness, the role of the logarithmic entropy function as a Lyapunov function for standard Markovian dynamics.

2. THE GEOMETRY ON THE SPACE OF POSITIVE REAL-VALUED VECTORS

The results described next are taken almost verbatim from [10]. The basic idea for the geometry that we are to define on the positive vectors in $\mathbb{R}^n$, is that we can think about them as functions $\xi : \mathscr{G} = \{1,...,n\} \to \mathbb{R}$, and all standard arithmetical operations either as *componentwise operations* among vectors or *pointwise operations* among functions. We shall denote by $\mathscr{M} = \{x \in \mathbb{R}^n \mid x(i) > 0, i = 1,...n\}$ the set of all positive vectors (the positive orthant cone). $\mathscr{M}$ is an open set in $\mathbb{R}^n$ which is trivially a manifold over $\mathbb{R}^n_{++}$, having $\mathbb{R}^n$ itself as tangent space T_xM at each point $x \in \mathscr{M}$.

The set $\mathscr{M}$ plays the role that the positive definite matrices play in the work by Lang, Lawson and Lim and Moakher mentioned above. The role of the *group* [19; 7] of invertible matrices in those references is to be played by

$$G = \{g \in \mathbb{R}^n \mid g(i) \neq 0, i = 1,...,n\}.$$

Group G clearly is an Abelian group with respect to the standard componentwise product (Hadamard product). The identity $e = (1,\ldots,1) \in G$ is the vector with all its components equal to 1. In order to define a scalar product at each $T_x\mathscr{M}$ we use a transitive action of $G : \mathscr{M} \to \mathscr{M}$ of G on $\mathscr{M}$ as follows. Set

$$\tau_g(x) = g^{-1}xg^{-1}.$$

This action is clearly transitive on $\mathscr{M}$, and can be defined in the obvious way as an action on $\mathbb{R}^n$. We transport the scalar product on T_eM to any T_xM as follows:

The scalar product between ξ and η at T_eM is defined to be the standard Euclidean product (dot product) $\langle \xi, \eta \rangle = \sum \xi_i \eta_i$, and we shall switch between $\xi(i)$ and ξ_i whenever is convenient for typographical reasons. We now use the fact that $x = \tau_g(e)$ with $g = x^{-1/2}$ to define the scalar product transported to T_xM by

$$\langle \xi, \eta \rangle_x \equiv \langle x^{-1}\xi, x^{-1}\eta \rangle = \langle x^{-2}\xi, \eta \rangle. \tag{1}$$

That is, we transport the vectors back to T_eM and compute their scalar product there. That scalar product allows us to define the length of a differentiable curve as follows:

Let $x(t)$ be a differentiable curve in $\mathscr{M}$ with $\dot{x}(t) = \frac{dx(t)}{\mathrm{d}t}$, its length is given by

$$\int_0^1 \sqrt{\langle \dot{x}, \dot{x} \rangle_x} dt.$$

With this, the distance between $x_1, x_2 \in \mathscr{M}$ is defined by the usual formula as the length minimizing curve linking x_1 to x_2:

$$d(x_1,x_2) = \inf \left\{ \int_0^1 \sqrt{\langle \dot{x}, \dot{x} \rangle_x} \mathrm{d}t \,|\, x(t) \text{ differentiable such that } x_1 = x(0) \;\; x_2 = x(1) \right\}. \tag{2}$$

Actually, it also happens that the geodesics minimize the *action functional*

$$\int_0^1 \mathscr{L}(\dot{x}(t), x(t)) dt, \quad \text{with} \quad \mathscr{L}(\dot{x}(t), x(t)) = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle_x, \tag{3}$$

It takes an application of the Euler-Lagrange formula to see that the equation of the geodesics in this metric is

$$\ddot{x}(t) = x^{-1} \dot{x}^2, \;\; x(0) = x_1, \;\; x(1) = x_2, \tag{4}$$

the solution to which is

$$x(t) = x_1 e^{-t \ln(x_1/x_2)} = x_2^t x_1^{1-t}. \tag{5}$$

This is the e-geodesic in information geometry [3], also called a Bhattacharyya arc.

Comments. The choice of sign in the exponent is arbitrary. We choose the sign as negative now so that a negative sign does not occur when we deal with the maximum entropy method below. It should also be clear that the transport mentioned above coincide with the geodesic transport just defined.
The geometric construction carried out above was to render as natural the following distance between positive vectors in $\mathscr{M}$. The geodesic distance between x_1 and x_2 as

$$d(x_1,x_2)^2 = \sum_{i=1}^n (\ln x_1(i) - \ln x_2(i))^2 = \sum_{i=1}^n \ln^2 \left(\frac{x_1(i)}{x_2(i)} \right). \tag{6}$$

This formula coincides with the Riemannian distance between two positive diagonal matrices of the manifold of symmetric positive-definite matrices [20]; Let P_1 and P_2 be two symmetric positive-definite matrices. Then their Riemannian distance is

$$\rho(P_1,P_2) = \sqrt{\sum_{i=1}^n \ln^2 \lambda_i (P_1 P_2^{-1})}, \tag{7}$$

where $\lambda_i(P)$ denotes the i-th largest eigenvalue of matrix P. Thus when $P_1 = \text{diag}(x_1(1),\dots,x_1(n))$ and $P_2 = \text{diag}(x_2(1),\dots,x_2(n))$ are the diagonal matrices induced by x_1 and x_2, respectively, we have $\rho(P_1,P_2) = d(x_1,x_2)$.

Notice as well that instead of solving (4) with initial and final conditions (geodesics with boundary values), we might as well consider the solution to (4) subject to $x(0) = x$, and $\dot{x}(0) = \xi$ (geodesics with initial values), which is clearly given by the (exponential) mapping $xe^{-t\xi}$.
The following result is taken verbatim from Gzyl (2017). It summarizes the main results from Chapter 5 of Lang (1995).

THEOREM 2.1. With the notations introduced above we have:
1) The exponential mapping is metric preserving through the origin.
2) The derivative of the exponential mapping is measure preserving, that is, $\exp'(\xi)v = ve^{\xi}$ as a mapping $T_xM \to T_{\exp x}M$ satisfies

$$\langle v, v\rangle = \langle \exp'(\xi)v, \exp'(\xi)v\rangle_{\exp(\xi)}$$

3) With the metric given by (6), $\mathscr{M}$ is a Bruhat-Tits space, that is, it is a complete metric space in which the semi-parallelogram law holds. That is, given any $x_1, x_2 \in \mathscr{M}$, there exists a unique $z \in \mathscr{M}$ such that for any $y \in \mathscr{M}$ the following holds

$$d(x_1,x_2)^2 + 4d(z,y)^2 \leq 2d(y,x_1)^2 + 2d(y,x_2)^2. \tag{8}$$

Comments:
1) The proof of each assertion follows from calculus. In our framework, commutativity makes things much simpler. To obtain the completeness of $\mathscr{M}$ we transfer it from $\mathbb{R}^n$ via the exponential mapping.
2) The point z mentioned in item (3) is given by $z = \sqrt{x_1x_2}$.

Along with the notion of geodesic curve, there is a notion of geodesic surface through (or containing) a point $x(0)$. A parametric geodesic surface containing $x(0) \in \mathscr{M}$ and having tangents $\xi_1,...,\xi_K$ there is a mapping $t \in \mathbb{K} \to \mathscr{M}$ given by

$$t = (t_1,...,t_K) \in \mathbb{R}^K \to x(t) = e^{-\sum_i t_i \xi_i}. \tag{9}$$

We leave for the reader to verify that we can reach any point of this surface traveling along the individual geodesics one at a time.
In the next section we shall see how this surface maps into a geodesic surface in the set of all probabilities on $[n] = \{1,...,n\}$, and the probabilistic interpretation of the geodesic surface will be that of an exponential family.

3. THE INDUCED GEOMETRY ON THE SET OF DISCRETE PROBABILITIES

If we think about the lines in $\mathscr{M}$ as the object of our interest, we may think about the probabilities on a discrete sample space of cardinality n as the *representatives* of the rays in $\mathscr{M}$ (equivalence classes). Let us introduce the notation $\mathscr{P} = \{x \in \mathscr{M} \,|\, \langle e,x\rangle = 1\}$. Clearly, if the point $\frac{x}{\langle e,x\rangle}$ which lies in $\mathscr{P}$ is a representative of the line through x but the mapping

$$\mathscr{M} \to \mathscr{P} \qquad x \to \frac{x}{\langle e,x\rangle},$$

is a *projection* but not an orthogonal projection. Similarly, a curve $t \in I \to x(t) \in \mathscr{M}$ projects onto a curve in $\mathscr{P}$, and the question is: Do geodesics in $\mathscr{M}$ project onto geodesics in $\mathscr{P}$?

Before answering this question, note the following. If p_1 and p_2 are two points in $\mathscr{P}$ the curve $x(t) = p_2^t p_1^{1-t} = p_1 \exp(-t\xi)$ that joins them is a geodesic in the ambient space $\mathscr{M}$, but it is not necessarily a curve in $\mathscr{P}$. To answer the question posed in the last paragraph consider $p(t) = x(t)/Z(t)$ with $Z(t) = \langle e, x(t)\rangle$ which certainly is a curve lying in $\mathscr{P}$. Note that $\langle e, p(t)\rangle = 1$, then

$$\dot{p} = -p\left(\xi - p\langle e, p\xi\rangle\right), \tag{10}$$

clearly satisfies $\langle e, \dot{p}(t)\rangle = 0$. That is the velocity along p is tangent to $\mathscr{P}$ at every point. Differentiate with respect to t once more and use the previous equation to obtain

$$\ddot{p}(t) = p\left(\frac{\dot{p}^2(t)}{p^2(t)} - \left\langle e, \frac{\dot{p}^2(t)}{p^2(t)}\right\rangle\right).$$

Notice that $p(t)$ satisfies the geodesic equation in the coordinates of the ambient space $\mathscr{M}$ corrected so that the acceleration is tangent to $\mathscr{P}$. That is, the projection of a geodesic *is* a geodesic. We can gather these comments under the following theorem:

THEOREM 3.1. The geodesic between two points p_1 and p_2 in $\mathscr{P}$ can be obtained by projecting down to $\mathscr{P}$ the geodesic between the same points in the ambient space $\mathscr{M}$.

The pending question is: How to choose coordinates in $\mathscr{P}$ is order to transport the whole geometric structure there instead of working with the coordinates of the ambient space ($\mathscr{M}$) in which it sits as a submanifold.
Note as well that instead of a geodesic joining two points, the same result applies

to a geodesic issued from a point p_1 in the direction of a tangent vector ξ. And the same applies to a geodesic surface determined by a collection of vectors $\{\xi_1,...,\xi_K\}$ parameterized by $t \in \mathbb{R}^K$. The analogue of the previous result is now easy to establish.

THEOREM 3.2. The geodesic surface $p(t)$ containing the point p_1 and tangent to the vectors $\{\xi_1,...,\xi_K\}$ at $\mathscr{P}_{p_1}$ is given by

$$p(t) = \frac{p_1 e^{-\sum_i t_i x_i}}{Z(t)} \quad \text{with} \quad Z(t) = \left\langle e, p_1 e^{-\sum_i t_i x_i} \right\rangle. \tag{11}$$

is obtained by projecting the geodesic surface in $\mathscr{M}$ down to $\mathscr{P}$.

Comment: Clearly (11) describes an *exponential family* of probabilities on $\{1,...,n\}$, or in other words, exponential probabilities are geodesic surfaces in the metric described in Section 2.

Since the geodesics are defined for all values of the parameter t, a pending question is: What does $p(t)$ tend to as $t \to \pm\infty$? To answer the first question, put $k_{\max} = \arg\max\{\ln(p_1(k)/p_2(k)) \mid k = 1,...,n\}$ and similarly $k_{\min} = \arg\min\{\ln(p_1(k)/p_2(k)) \mid k = 1,...,n\}$. Let us consider the sets of subscripts at which the maximum or the minimum are reached, that is $J_{\max} = \{1 \le k \le n : k = k_{\max}\}$ and $J_{\min} = \{1 \le k \le n : k = k_{\min}\}$, and let their cardinalities be, respectively, $M_{\max}$ and $M_{\min}$. Then as $t \to \infty$,

$$p_k(t) \to \begin{cases} 1/M_{\min} & k \in J_{\min} \\ 0 & \text{otherwise.} \end{cases}$$

When $t \to -\infty$ a similar result is obtained with $M_{\min}$ replaced by $M_{\max}$.

4. GEOMETRIC/DYNAMIC INTERPRETATION OF THE MAXIMUM ENTROPY METHOD

Consider a random variable ξ as a tangent vector in $T_{p_0}M$ and the class

$$\mathscr{P}_\mu = \left\{ p \in \mathscr{P} : E_p[\xi] = \sum_i p(i)\xi_i = \langle e, p\xi \rangle = \mu \right\},$$

for some given number μ, that is, the class of all probabilities under which ξ has expected value μ. Since $\mathscr{P}_\mu$ is a hyperplane in $\mathscr{P}$, we may wonder whether following the geodesic $p(t) = p_0 \exp(-t\xi)/Z(t)$ issued from p_0 along ξ, we might intercept $\mathscr{P}_\mu$. If the answer is yes, then so is the answer to our question.

If t^* is such that $p(t^*) \in \mathscr{P}_\mu$, then clearly

$$p(t^*) \in \mathscr{P}_\mu \quad \Leftrightarrow \quad t^* = \text{argmin}\{t\mu + \ln Z(t) : t \in \mathbb{R}\}. \tag{12}$$

Clearly the function in curly brackets is strictly convex, and the first order condition for t^* to be its minimizer is equivalent to the assertion on the left hand side. Consider now the relative entropy function (also called the Kullback-Leibler divergence)

$$S(p,p_0) : \mathscr{P} \to \mathbb{R}, \quad S(p,p_0) = \langle e, p\ln(p/p_0)\rangle = \sum_i p(i)\ln\left(\frac{p(i)}{p_0(i)}\right).$$

Now, note that if we replace the generic p by a probability along the geodesic, we have $S(p(t),p_0) = \mu + \ln(Z(t))$. To complete the argument, note that the concavity of the logarithm implies that for any pair of probabilities $S(p,q) \le 0$ (with equality whenever they are equal), implies that taking $q = p(t)$

$$S(p,p(t)) \le 0 \quad \Rightarrow S(p,p_0) \le t\mu + \ln(Z(t)) = S(p(t),p_0) \quad \text{for any} \quad t.$$

That is the entropy of any $p(t)$ along the geodesic bounds from above the entropy of any $p \in \mathscr{P}_\mu$. What we do not know is whether there is a t^* for which $p(t^*) \in \mathscr{P}_\mu$.

What (12) asserts is that if there is a t^* minimizing $t\mu + \ln Z(t)$, then $p(t^*) \in \mathscr{P}_\mu$ and necessarily $p(t^*)$ solves the following entropy maximization problem:

$$\text{Find} \quad p^* \in \mathscr{P}_\mu \quad \text{such that} \quad p^* \quad \text{maximizes} \quad S(p,p_0) \quad \text{over} \quad \mathscr{P}_\mu.$$

To sum up, whether or not the geodesic issued from p_1 along ξ intersects the "plane" $\mathscr{P}_\mu$ is equivalent solvability of the entropy maximization problem. And since the dual entropy function $\Sigma(t) = t\mu + \ln Z(t)$ is interpreted as a *free energy* in statistical thermodynamics, the travel time t^* has an interpretation as an "intensive" thermodynamical variable conjugate to ξ.

5. ENTROPY: A LYAPUNOV FUNCTION FOR MARKOVIAN DYNAMICS

The results of the previous section, interesting as they may be, are not connected to a dynamics related to a physical process. For example, we may consider the case in which there is a Markovian process with state space $\{1,....,n\}$ and transition rate matrix Q. When we suppose that the chain is irreducible (we can reach any state starting from any other state), it is well known that if the transition state is either symmetric or reversible, the entropy function is a Lyapunov function for the chain. To spell it out in symbols, note if $p(0)$ is any initial distribution on the state space, then $p(t) = e^{tQ}p(0)$ describes the probability distribution at current time t.

The following was proved in Klein (1956) for the Ehrenfest urn model and extended in Moran (1960) as follows:

THEOREM 5.1. With the notations introduced above, then:
1) If the Markov chain is symmetric, that is, $Q(i,j) = Q(j,i)$, or
2) If the chain is reversible, that is, if there is an equilibrium probability law p_e such that $\sum_j Q(i,j)p_e(j) = 0$,
then the entropy $S(p(t)$ satisfies $dS(p(t))/dt > 0$.

These results are part of a large chain of results on the issue of time (ir)reversibility in statistical thermodynamics. From the mathematical point of view, the result is a particular case of a more general, and surprisingly simple to prove result for monotone continuous mappings, which applies to linear and non-linear dynamical systems. See Brown (1985).

6. CONCLUDING REMARKS

To sum up, there is a curious and nice relationship between a geometry on the set of positive real vectors and the exponential families of probability distributions on finite sets. In this setup, exponential families appear as *geodesic surfaces* in the set of probabilities. This Bruhat-Tits space is different than the Hilbert simplex/Birkhoff cone geometry proposed in [22] (Hilbert cross-ratio distance on the probability simplex fails the semi parallelogram law).

The logarithmic distance in the set of strictly positive vectors leads to notion of best predictor that complements the theory best prediction is square distance. For more on this see Gzyl (2017).

7. APPENDIX: PROJECTED GEODESICS ARE GEODESICS

The curve $t \to x_0 \exp(t\xi)$ in $\mathscr{M}$ is a geodesic through x_0 tangent to $\xi = \ln(x_1/x_0)$. It minimizes

$$\int_0^1 \mathscr{L}(x,\dot{x})dt = \frac{1}{2}\int_0^1 \langle \dot{x},\dot{x}\rangle_x dt$$

in the class of curves through $x(0) = x_0$ and $x(1) = x_1$. The scalar product on $T\mathscr{M}_x$ was defined in (1). When considering curves that live in $\mathscr{P}$ we must impose the constraint $E_e[x(t)] = 1$ for all t. This leads to the following result

THEOREM 7.1. The curve

$$p(t) = \Psi(x_0 e^{t\xi}) = \frac{x_0 e^{t\xi}}{E_e[x_0 e^{t\xi}]} = \frac{x_0 e^{t\xi}}{E_{x_0}[e^{t\xi}]}$$

minimizes

$$\frac{1}{2}\int_0^1 \langle \dot{x},\dot{x}\rangle_x dt + \frac{1}{2}\int_0^1 \lambda(t)\big(E_e[x(t)]-1\big)dt.$$

The proof is by calculus. Note first that for $p(t) = \Psi(x_0 e^{\xi})$ we have:

$$\frac{\ddot{p}}{p^2} - \frac{\dot{p}^2}{p^3} = -\frac{E_p[\bar{\xi}^2]}{p} \tag{13}$$

where $\bar{\xi} = \xi - E_p[\xi]$. The Euler-Lagrange equations for the constrained problem yield

$$\frac{\ddot{x}}{x^2} - \frac{\dot{x}^2}{x^3} = \frac{1}{2}\lambda(t). \tag{14}$$

Therefore, identifying $x(t) = p(t)$ and $\lambda(t) = 2E_p[\bar{\xi}^2]$ and computing E_p in both sides of (13) we see that it becomes (14) after computing E_p in both sides of it.

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Exponentiated quasi power Lindley power series distribution with applications in medical science

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Abstract

The present paper introduces an advanced five parameter lifetime model which is obtained by compounding exponentiated quasi power Lindley distribution with power series family of distributions. The EQPLPS family of distributions contains several lifetime sub-classes such as quasi power Lindley power series, power Lindley power series, quasi Lindley power series and Lindley power series. The proposed distribution exhibits decreasing, increasing and bathtub shaped hazard rate functions depending on its parameters. It is more flexible as it can generate new lifetime distributions as well as some existing distributions. Various statistical properties including closed form expressions for density function, cumulative function, limiting behaviour, moment generating function and moments of order statistics are brought forefront. The capability of the quantile measures in terms of Lambert W function is also discussed. Ultimately, the potentiality and the flexibility of the new class of distributions has been demonstrated by taking three real life data sets by comparing its sub-models.

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Keywords: Exponentiated Quasi Power Lindley distribution, Lambert W function, order statistics, MLE.

1. INTRODUCTION

The modeling of lifetime data has received prominent attention from researchers for the last decade. To predict the ambiguous behaviour of random events as death, appearance of some disease and system failure is a major concern for statisticians. There are diverse lifetime models available for researchers to predict this uncertain behaviour but at times due to complex pattern of data sets, these models do not provide a suitable fit. In order to prevail from this difficulty, researchers have focussed their attention on compounding mechanism which is a sound way to develop an appropriate and flexible models to fit the lifetime data of different types.

Keeping this in mind, Adamidis and Loukas (1998), Kus (2007), Tahmasbi (2008) constructed several lifetime distributions through this mechanism that proved to be operative in modeling of lifetime data having different features. Researchers

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developed many lifetime distributions by this technique which are very flexible and can accommodate different types of data sets. For instance, Chahkandi and Ganjali (2009) obtained a compound class of exponential power series distributions. As weibull distribution contains the exponential distribution as a special case, Morais and Baretto-Souza (2011) substituted the exponential distribution with a weibull distribution in this mechanism and obtained a compound class of weibull power series distributions which contains EPS distribution as a special case. Adil and Jan (2016) introduced a new family of lifetime distributions by compounding a Lindley distribution with power series distribution that contains Lindley Geometric as special case due to Zakerzadeh and Mahmoudi (2012). Moreover, many authors discussed some special cases of the LPS family that are very flexible in terms of density and hazard rate functions. Adil and Jan (2018a, 2018b) obtained a lifetime distribution for series system and generalized version of complementary Lindley power series family of compound lifetime distributions related to parallel system which generalizes most of the lifetime distributions and have versatile properties. Arsalan et al. (2019) introduced the exponential Burr XII power series.

2. EXPONENTIATED QUASI POWER LINDLEY DISTRIBUTION

Manuela Ghica et al. (2017) introduced an Exponentiated Quasi Power Lindley Distribution (EQPLD) defined by its pdf as

$$g(y;\alpha,\theta,\beta,b,)=\frac{\beta\theta^2 b}{\alpha\theta+1}x^{\beta-1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b;x>0,\alpha,\theta,\beta,b>0$$

This new distribution reduces to the quasi Lindley distribution, the exponential distribution and gamma distribution. In terms of reliability, the various shapes of the EQPL distribution give it a benefit, being more flexible to model many real systems which generally exhibit bath-tub shaped failure rate. The corresponding cdf of the above equation becomes

$$G(y;\alpha,\theta,\beta,b)=\left[1-\left[1+\frac{\theta x^\beta}{\alpha\theta+1}\right]e^{-\theta x^\beta}\right]^b;x>0,\alpha,\theta,\beta,b>0$$

2.1. Special cases

Case 1: At $b = 1$, EQPLD reduces to Quasi Power Lindley distribution.

Case 2: At $\beta = 1$, EQPLD reduces to the Power Lindley distribution introduced by ME Ghitany (2013).

Case 3: At $b = 1, \beta = 1$, EQPLD reduces to the Quasi Lindley distribution introduced by Shanker and Mishra (2013).

Case 4: At $b = 1, \beta = 1, \alpha = 1$, EQPLD reduces to the Lindley distribution.

3. THE EQPLPS FAMILY

In this section, we derive the family of EQPLPS distributions by compounding the EQPL class of distributions with the power series distributions.

Let N be a discrete random variable following the power series distribution (truncated at zero) with probability mass function given by

$$P(N = n) = \frac{a_n \gamma^n}{C(\gamma)}, n = 1,2,...$$

Where $a_n \geq 0$ be reliant on n, $C(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$ and $\gamma \in (0, s)$ is chosen in such a way that $C(\gamma)$ is finite. The power series family of distributions, contains Poisson, logarithmic, geometric and binomial distributions as special cases. Valuable extents of above distributions truncated at zero are given in table 1.

Table 1: Useful Extents Of Zero Truncated Power Series Distribution

Distribution	a_n	$C(\gamma)$	$C^{'}(\gamma)$	$C^{"}(\gamma)$	$C^{-1}(\gamma)$	γ
Poisson	$n!^{-1}$	$e^{\gamma}-1$	e^{γ}	e^{γ}	$\log(\gamma+1)$	$\gamma\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\gamma)$	$(1-\gamma)^{-1}$	$(1-\gamma)^{-2}$	$1-e^{-\gamma}$	$\varphi\in(0,1)$
Geometric	1	$\gamma(1-\gamma)^{-1}$	$(1-\gamma)^{-2}$	$2(1-\gamma^{-3})$	$\gamma(\gamma+1)^{-1}$	$\gamma\in(0,1)$
Binomial	$\binom{m}{n}$	$(\gamma+1)^{m}-1$	$m(\gamma+1)^{m-1}$	$\frac{m(m-1)}{(\gamma-1)^{2-m}}$	$(\gamma-1)^{\frac{1}{m}}-1$	$\gamma\in(0,1)$

Given N, let $X=\max(X_1,X_2,...X_N)$, where $X_i, i=1,2,...N$ are independent and identically distributed (iid) random variables with cdf ().Then the cdf of $X\mid N=n$ is given by

$$F(X\mid N=n)=[G(x;\alpha,\theta,\beta,b)]^{n}=\left\{\left[1-\left[1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right]e^{-\theta x^{\beta}}\right]^{b}\right\}^{n},$$

$x>0,\alpha,\theta,\beta,b>0,n\geq 1$

The EQPLPS is then defined by the marginal cdf of X, which is given by

$$F(x,\alpha,\theta,\beta,b,\gamma)=\sum_{n=1}^{\infty}\frac{a_n\gamma^{n}}{C(\gamma)}[G(x;\alpha,\theta,\beta,b,\gamma)]^{n}$$

$$F(x)=\frac{C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}, x>0,\alpha>0,\theta>0,\beta>0,b>0,\gamma>0 \tag{1}$$

Here, a random variable X following Exponentiated Quasi Power Lindley power series distribution with parameters $\alpha,\theta,\beta,b,\gamma$ will be denoted by $X\sim EQPLPS(\alpha,\theta,\beta,b,\gamma)$. This new class of distributions contains several lifetime distributions as special cases which will be discussed in section (9).

5. DENSITY, SURVIVAL AND HAZARD RATE FUNCTIONS

The pdf, survival and hazard functions are respectively given by

$$f(x) = \frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha + x^{\beta}\right)\gamma e^{-\theta x^{\beta}} \frac{C'\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}, x > 0 \quad (2)$$

$$S(x) = 1 - \frac{C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}, x > 0$$

And the hazard function is given as

$$h(x) = \frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha + x^{\beta}\right)\gamma e^{-\theta x^{\beta}} \frac{C'\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma) - C\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}, x > 0$$

THEOREM 4.1. The Exponentiated Quasi Power Lindley distribution is a limiting case of EQPLPS distribution when $\gamma \to 0^+$.

Proof: From the cdf of EQPLPS distribution , we have

$$\lim_{\gamma\to 0^+} F(x) = \lim_{\gamma\to 0^+} \frac{C'\left[\gamma\left\{1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right\}^{b}\right]}{C(\gamma)}$$

We know that

$$C(\gamma) = \sum_{n=1}^{\infty} a_n \gamma^n$$

$$\lim_{\gamma \to 0^+} F(x) = \lim_{\gamma \to 0^+} \frac{\sum_{n=1}^{\infty} a_n \left[\gamma \left\{ 1 - \left(1 + \frac{\theta x^{\beta}}{\alpha\theta + 1} \right) e^{-\theta x^{\beta}} \right\}^{b} \right]^{n}}{\sum_{n=1}^{\infty} a_n \gamma^{n}}$$

Using L' Hospital's rule, it follows that

$$\lim_{\gamma \to 0^+} F(x) = \frac{a_1 \left\{ \left[1 - \left(1 + \frac{\theta x^{\beta}}{\alpha\theta + 1} \right) e^{-\theta x^{\beta}} \right]^{b} \right\} + \sum_{n=1}^{\infty} a_n \gamma^{n-1} \left[\left[1 - \left(1 + \frac{\theta x^{\beta}}{\alpha\theta + 1} \right) e^{-\theta x^{\beta}} \right]^{b} \right]^{n}}{a_1 + \sum_{n=1}^{\infty} a_n n \gamma^{n-1}}$$

$$= \left[1 - \left(1 + \frac{\theta x^{\beta}}{\alpha\theta + 1} \right) e^{-\theta x^{\beta}} \right]^{b}$$

Which is the cdf of exponentiated quasi power Lindley distribution.

THEOREM 4.2. The densities of the EQPLPS distribution can be expressed as an infinite linear combination of densities of n^{th} order statistics of the exponentiated quasi power Lindley distribution

$$f(x) = \sum_{n=1}^{\infty} P(N = n) g_n(x, n) \tag{3}$$

Where $g_n(x, n) = \max(X_1, X_2, \ldots X_n)$ is the n^{th} order statistics of exponentiated quasi power Lindley distribution.

Proof: As we know that

$$C(\gamma) = \sum_{n=1}^{\infty} n a_n \gamma^{n-1}$$

Therefore, the pdf of EQPLPS distribution reduces to the expression after using the above argument as follows

$$f(x)=\frac{\beta\theta^2 b x^{\beta-1}}{(\alpha\theta+1)}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}\sum_{n=1}^{\infty}\frac{n a_n \gamma^n}{C(\gamma)}\left[\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b}\right]^{n-1}$$

$$f(x)=\sum_{n=1}^{\infty}P(N=n)g_n(x,n)$$

Where

$$g_n(x,n)=\frac{n\beta\theta^2 b x^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}\left[\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b}\right]^{n-1}$$

is the n^{th} order statistics of exponentiated quasi power Lindley distribution. Therefore the densities of EQPLPS distribution can be expressed as an infinite linear combination of the n^{th} order statistics of exponentiated quasi power Lindley distribution.

6. MOMENT GENERATING FUNCTION

The moment generating function of EQPLPS distribution can be obtained from (3)

$$M_X(t)=\sum_{n=1}^{\infty}P(N=n)M_{X_{(n)}}(t)$$

Where $M_{X_{(n)}}(t)$ is the moment generating function of n^{th} order statistics of exponentiated quasi power Lindley distribution.

$$M_{X_{(n)}}(t)=\frac{n\beta\theta^2 b}{\alpha\theta+1}\int_0^{\infty}e^{tx}x^{\beta-1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}\left[\left\{1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right\}^{b}\right]^{n-1}dx$$

$$=\frac{n\beta\theta^2}{\alpha\theta+1}\sum_{j=0}^{n-1}\binom{n-1}{j}\sum_{k=0}^{\infty}\binom{bj+b-1}{k}(-1)^{j+k}\int_0^{\infty}e^{tx}x^{\beta-1}\left(\alpha+x^\beta\right)e^{-(\theta+\theta k)x^\beta}\left[1+\left(\frac{\theta x^\beta}{\alpha\theta+1}\right)\right]^{k}dx$$

Using $e^{tx} = \sum_{l=0}^{\infty} \frac{t^l x^l}{l!}$

$$= \frac{n\theta^2 b}{(\alpha\theta+1)} \sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty} \binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k} \frac{t^l}{l!}\left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i} \times \left[\frac{\alpha\Gamma\left(k-i+\frac{l}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{l}{\beta}\right)}{(\theta+\theta k)^{k-i+\frac{l}{\beta}+1}}\right]$$

And it follows that

$$M_X(t) = \frac{\theta^2 b}{\alpha\theta+1} \sum_{n=1}^{\infty} \frac{n a_n \gamma^n}{C(\gamma)} \sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty} \binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k} \frac{t^l}{l!}\left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i} \times \left[\frac{\alpha\Gamma\left(k-i+\frac{l}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{l}{\beta}\right)}{(\theta+\theta k)^{k-i+\frac{l}{\beta}+1}}\right]$$

The r^{th} moment of the EQPLPS distribution about origin is

$$E(X^r) = \sum_{n=1}^{\infty} P(N=n)\int_0^{\infty} x^r\, g_n(x)dx$$

$$E(X^r) = \frac{\theta^2 b}{\alpha\theta+1} \sum_{n=1}^{\infty} \frac{n a_n \gamma^n}{C(\gamma)} \sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{\infty} \binom{n-1}{j}\binom{bj+b-1}{k}\binom{k}{i}(-1)^{j+k} \left(\frac{\theta}{(\beta\theta+1)}\right)^{k-i} \times \left[\frac{\alpha\Gamma\left(k-i+\frac{r}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{r}{\beta}+2\right)}{(\theta+\theta k)^{k-i+\frac{r}{\beta}+2}}\right] \quad (4)$$

7. QUANTILE FUNCTION

THEOREM 6.1. If $X \sim EQPLPS(\alpha,\theta,\beta,b,\gamma)$, then the quantile function of X is

$$Q(p)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}(vC(\gamma))}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

Where $v\in(0,1)$ and W(.) denotes the Lambert W function (see Corless et al.(1996))

PROOF. The quantile function denoted by Q(p) is the root of the equation

$$\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}=1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}:0<v<1 \qquad (5)$$

Setting $Z(v)=-\left(1+\alpha\theta+\theta[Q(v)]^{\beta}\right)$, we may rewrite (5) as

$$Z(v)e^{z(v)}=-(1+\alpha\theta)\left[1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right]e^{-(1+\alpha\theta)}$$

So the solution for Z(v) is

$$Z(v)=W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]$$

Solving the equation $W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]=-\left(1+\alpha\theta+\theta[Q(v)]^{\beta}\right)$

Which upon solving for Q(v) gives

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}[vC(\gamma)]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

Using above equation the quartiles of the EQPLPS distribution can be determined. Median of the exponentiated quasi power Lindley power series distribution is given by

$$Q\left(\frac{1}{2}\right)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{C^{-1}\left[\frac{1}{2}C(\gamma)\right]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

8. ORDER STATISTICS AND THEIR MOMENTS

Let $X_1, X_2,...X_n$ be a random sample of size n having EQPLPS distribution. The pdf and cdf of ith order statistics say $X_{i::n}$ can be obtained as

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}[F(x)]^{i-1}[1-F(x)]^{n-i}\,f(x) \qquad (6)$$

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\,f(x)\left[\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}}{C(\gamma)}\right]^{i-1}\left[1-\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}}{C(\gamma)}\right]^{n-i}$$

Expression (6) can also be written as

$$f_{i:n}(x)\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{k}\,f(x)[F(x)]^{k+i-1}$$

The associated cdf of $f_{i:n}(x)$ denoted by $F_{i:n}(x)$ becomes

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^{k}}{(k+i)}\left[C\left\{\frac{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{1+\alpha\theta}\right)e^{-\theta x^{\beta}}\right]^{b}}{C(\gamma)}\right\}\right]^{k+i} \qquad (7)$$

Expression for r[th] moment of i[th] order statistics with cdf (3.1) can be obtained by using a well- known result given by Barakat et al. (2004) as follows

$$E\left(X_{i:n}^{r}\right)=r\sum_{k=n-i+1}^{n}(-1)^{k-n+i-1}\binom{k-1}{n-i}\binom{n}{k}\int_{0}^{\infty}x^{r-1}\left[1-\frac{C\left\{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{1+\alpha\theta}\right)e^{-\theta x^{\beta}}\right]^{b}\right\}}{C(\gamma)}\right]^{k}dx$$

9. PARAMETER ESTIMATION

Let $X_1, X_2, \ldots X_n$ be a random sample with observed value $x=(x_1, x_2, \ldots x_n)$ obtained from EQPLPS distribution with parameters $\alpha,\theta,\beta,b\,and\,\gamma$. Let $\Theta=(\alpha,\theta,\beta,b,\gamma)^T$ be the parameter vector. The log likelihood function is given by

$$l_n=l_n(y,\Theta)=n\log\beta+n\log\gamma+2n\log\theta+n\log b-n\log(\alpha\theta+1)-n\log C(\gamma)+(\beta-1)\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i^{\beta}$$
$$+\sum_{i=1}^{n}\log\left(\alpha+x_i^{\beta}\right)+(b-1)\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]+\sum_{i=1}^{n}\log\left[C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}\right]$$

The corresponding score functions are

$$\frac{\partial l_n}{\partial\theta}=\frac{2n}{\theta}-\frac{n\alpha}{\alpha\theta+1}-\sum_{i=1}^{n}x_i^{\beta}+\frac{\theta(b-1)}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\left[\frac{x_i^{\beta}e^{-\theta x_i^{\beta}}\left(\alpha^2\theta+2\alpha+\alpha\theta x_i^{\beta}+x_i^{\beta}\right)}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]}\right]$$
$$+\frac{b\gamma\theta}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{C''\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C'\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b-1}x_i^{\beta}e^{-\theta x_i^{\beta}}\left(\alpha^2\theta+2\alpha+\alpha\theta x_i^{\beta}+x_i^{\beta}\right)$$

$$\frac{\partial l_n}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log x_i+\sum_{i=1}^{n}\frac{x_i^{\beta}\log x_i}{\left(\alpha+x_i^{\beta}\right)}-\theta\sum_{i=1}^{n}x_i^{\beta}\log x_i+\frac{(b-1)\theta^2}{(\alpha\theta+1)}\sum_{i=1}^{n}\left(x_i^{\beta}\log x_i e^{-\theta x_i^{\beta}}\right)\left(\alpha+x_i^{\beta}\right)+$$

$$+\sum_{i=1}^{n}\frac{C^{''}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C^{'}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\frac{\left(x_i^{\beta}\log x_i e^{-\theta x_i^{\beta}}\right)\left(\alpha+x_i^{\beta}\right)}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]}$$

$$\frac{\partial l_n}{\partial \alpha}=\frac{n\theta}{\alpha\theta+1}+\sum_{i=1}^{n}\frac{1}{\alpha+x_i^{\beta}}+\frac{\theta^2(b-1)}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{x_i^{\beta}e^{-\theta x_i^{\beta}}}{\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]}+\frac{\gamma b\theta^2}{(\alpha\theta+1)^2}\sum_{i=1}^{n}\frac{C^{''}\left\{\gamma\left[1-\left(1+\frac{\theta y_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C^{'}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}$$

$$\times\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b-1}\left(x_i^{\beta}e^{-\theta x_i^{\beta}}\right)$$

$$\frac{\partial l_n}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]+\gamma\frac{C^{''}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C^{'}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

$$\times\log\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

$$\frac{\partial l_n}{\partial \gamma}=\frac{n}{\gamma}-\frac{nC^{'}(\gamma)}{C(\gamma)}+\sum_{i=1}^{n}\frac{C^{''}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}{C^{'}\left\{\gamma\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}\right\}}\left[1-\left(1+\frac{\theta x_i^{\beta}}{\alpha\theta+1}\right)e^{-\theta x_i^{\beta}}\right]^{b}$$

MLEs of $\theta,\alpha,\beta,b\,\&\,\gamma$ cannot be obtained by solving above complex equations as these equations are not in closed form. So we solve the above equations by using iteration method through R software.

10. SPECIAL SUB-MODELS OF THE EQPLPS MODEL

10.1. Exponentiated Quasi Power Lindley Poisson Distribution (EQPLPD)

The corresponding cdf, pdf, survival function and hazard function of EQPLPD can be obtained respectively by using $C(\gamma)=e^{\gamma}-1$ and $C'(\gamma)=e^{\gamma}$ in (1) & (2).

$$F(x)=\frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}-1}{e^{\gamma}-1}, x>0$$

$$f(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma}-1}, x>0$$

$$S(x)=\frac{e^{\gamma}-e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma}-1}$$

$$h(y)=\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}}{e^{\gamma}-e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]}}$$

For $x,\theta,\alpha,\beta,b>0,\ and\ \ 0<\gamma<\infty$. The expression for r[th] moment of a random variable following EQPLPS distribution becomes by substituting $a_n=n!^{-1}$ and $C(\gamma)=e^{\gamma}-1$ in (4).

$$E\left(X^{r}\right)=\frac{n\theta^{2}b}{(\alpha\theta+1)\left(e^{\theta}-1\right)}\sum_{n=1}^{\infty}\frac{\gamma^{n}}{n!}\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\binom{n-1}{j}\binom{bj+b-1}{k}\left(\frac{\theta}{\alpha\theta+1}\right)^{k-i}(-1)^{j+k}$$
$$\times\left[\frac{\alpha\Gamma\left(k-i+\frac{r}{\beta}+1\right)+(\theta+\theta k)\Gamma\left(k-i+\frac{r}{\beta}+2\right)}{(\theta+\theta k)^{k-i+\frac{r}{\beta}+2}}\right]$$

The pdf and cdf of order statistics of EQPLPD can be obtained by using the cdf and pdf of EQPLPD in (6) and (7).

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}$$

$$\times\frac{\left[e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}-1\right]^{k+i-1}}{\left[e^{\gamma}-1\right]^{k+i}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{(k+i)}\left[\frac{e^{\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}-1}{e^{\gamma}-1}\right]^{k+i}$$

The quantile function can be obtained by substituting $C(\gamma)=e^{\gamma}-1$ and $C^{-1}(\gamma)=\log(\gamma+1)$ in (5), we have

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{\log\left[v\left(e^{\gamma}-1\right)+1\right]}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

10.2. Exponentiated Quasi Power Lindley Logarithmic Distribution (EQPLLD)

The cdf, pdf, survival function and hazard function of EQPLLD is obtained by using $C(\gamma)=-\log(1-\gamma)$ and $C^{'}(\gamma)=(1-\gamma)^{-1}$ in (1) and (2).

$$F(x)=\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)},x>0$$

$$f(x)=\frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{\log(1-\gamma)},x>0$$

$$S(x)=1-\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)},x>0$$

$$h(x)=\frac{\beta\theta^{2}x^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{\log(1-\gamma)-\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]\right]}$$

For $x,\alpha,\theta,\beta,b>0, and\ 0<\gamma<1$.The r^th moment of EQPLLD can be obtained by substituting $a_n=n^{-1}\ \ and\ \ C(\gamma)=-\log(1-\gamma)$ in (4)

The pdf and cdf of EQPLLD can be obtained by substituting its pdf and cdf in (6) and (7).

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{k}\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}$$
$$\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}\times\frac{\left[\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]\right]^{k+i-1}}{\left[\log(1-\gamma)\right]^{k+i}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^{k}}{(k+i)}\left[\frac{\log\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]}{\log(1-\gamma)}\right]^{k+i}$$

By substituting $C(\gamma)=-\log(1-\gamma)$ and $C^{-1}(\gamma)=1-e^{-\gamma}$ in (5), the quantile function of EQPLL distribution is obtained as

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{e^{v\log(1-\gamma)}}{\gamma}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

10.4. Exponentiated Quasi Power Lindley Geometric Distribution (EQPLGD)

The cdf, pdf, survival function and hazard function of EQPLGD can be obtained by using $C(\gamma)=\gamma(1-\gamma)^{-1}$ & $C^{'}(\gamma)=(1-\gamma)^{-2}$ in (1) & (2).

$$F(x)=\frac{(1-\gamma)\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}{1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}},\ x>0$$

$$f(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}(1-\gamma)\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-2}$$

$$S(x)=\frac{1-\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}{1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}},\ x>0$$

$$h(x)=\frac{\beta\theta^{2}bx^{\beta-1}}{(\alpha\theta+1)}(1-\gamma)\left(\alpha+x^{\beta}\right)e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[1-\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}\right]^{-1}}{1-\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}}$$

For $x,\alpha,\theta,\beta,b>0$ *and* $0<\gamma<1$. The r^{th} moment of EQPLGD can be obtained by substituting $a_n=1$ and $C(\gamma)=\gamma(1-\gamma)^{-1}$ in (4).

The pdf and cdf of order statistics of EQPLGD can be obtained by using the cdf and pdf of EQPLGD in (6) and (7) ,we have

$$f_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\frac{\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^\beta\right)e^{-\theta x^\beta}(1-\gamma)^{k+i}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}$$

$$\times\frac{\left[\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i-1}}{\left[1-\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i+1}}$$

$$F_{i:n}(x)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^k}{k+i}\frac{\left[(1-\gamma)\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i}}{\left[1-\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b\right]^{k+i}},x>0$$

By substituting $C(\gamma)=\gamma(1-\gamma)^{-1}$ and $C^{-1}(\gamma)=\gamma(\gamma+1)^{-1}$ in (5), the quantile function of EQPLG distribution is obtained as

$$Q(v)=\left[-\alpha-\frac{1}{\theta}-\frac{1}{\theta}W\left[-(1+\alpha\theta)e^{-(1+\alpha\theta)}\left(1-\left[\frac{v}{\gamma(v-1)+1}\right]^{\frac{1}{b}}\right)\right]\right]^{\frac{1}{\beta}}$$

10.5. Exponentiated Quasi Power Lindley Binomial Distribution (EQPLBD)

The cdf , pdf, survival function and hazard function of EQPLBD can be obtained respectively by taking $C(\gamma)=(\gamma+1)^m-1$ and $C^{'}(\gamma)=m(\gamma+1)^{m-1}$ in (1) and (2).

$$F(x)=\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b+1\right]^m-1}{(\gamma+1)^m-1},x>0$$

$$f(x)=\frac{m\beta\theta^2 bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^\beta\right)\gamma e^{-\theta x^\beta}\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^{b-1}\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^\beta}{\alpha\theta+1}\right)e^{-\theta x^\beta}\right]^b+1\right]^{m-1}}{(\gamma+1)^m-1},x>0$$

$$S(x)=1-\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}-1}{(\gamma+1)^{m}-1}$$

$$h(x)=\frac{m\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}\frac{\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m-1}}{(\gamma+1)^{m}-\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}}$$

For $x,\alpha,\theta,\beta,b>0\,and\;\;0<\phi<\infty.$ The r[th] moment of a random variable following EQPLBD becomes by taking $a_n=\binom{m}{n}$ $and\;\;C(\gamma)=(\gamma+1)^{m}-1$ in (4).

The pdf and cdf of order statistics of EQPLBD can be obtained respectively by using the pdf and cdf of EQPLBD in (6) and (7).

$$f_{i:n}(y)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^{k}\frac{m\beta\theta^{2}bx^{\beta-1}}{\alpha\theta+1}\left(\alpha+x^{\beta}\right)\gamma e^{-\theta x^{\beta}}\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b-1}$$
$$\times\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m-1}\frac{\left[\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta y^{\alpha}}\right]^{b}+1\right]^{m}-1\right]^{k+i-1}}{\left[(\gamma+1)^{m}-1\right]^{k+i}}$$

$$F_{i:n}(y)=\frac{n!}{(n-i)!(i-1)!}\sum_{k=0}^{n-i}\frac{\binom{n-i}{k}(-1)^{k}}{(k+i)}\frac{\left[\left[\gamma\left[1-\left(1+\frac{\theta x^{\beta}}{\alpha\theta+1}\right)e^{-\theta x^{\beta}}\right]^{b}+1\right]^{m}-1\right]^{k+i}}{\left[(\gamma+1)^{m}-1\right]^{k+i}}$$

11. APPLICATION

To show the superiority of the proposed distribution, we compare its submodels by taking four real life data sets.

Data set 1. The first data set represents the Lifetime of fatigue of Kevlar 373/epoxy, that are subject to constant pressure at the 90% stress level until all had failed. The data set is

0.0251	0.886	0.0891		0.2501	0.3113	0.3451	0.4763	0.565	0.5671	0.6566	0.6748
0.6751	0.6753	0.7696		0.8375	0.8391	0.8425	0.8645	0.8851	0.9113	0.912	0.9836
1.0483	1.0596	1.0773		1.1733	1.257	1.2766	1.2985	1.3211	1.3503	1.3551	1.4595
1.488	1.5728	1.5733		1.7083	1.7263	1.746	1.763	1.7746	1.8275	1.8375	1.8503
1.8808	1.8878	1.8881		1.9316	1.9558	2.0048	2.0408	2.0903	2.1093	2.133	2.21
2.246	2.2878	2.3203		2.347	2.3513	2.4951	2.526	2.9911	3.0256	3.2678	3.4045
3.4846	3.7433	3.7455		3.9143	4.8073	5.4005	5.4435	5.5295	6.5541	9.096	

Table 10.1: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta}=0.924, \hat{\gamma}=0.000000122$, $\hat{\theta}=1.110, \hat{b}=1.254, \hat{\alpha}=0.233$	253.13	260.76
EQPLG	$\hat{\beta}=1.060, \hat{\gamma}=0.00000145$, $\hat{\theta}=0.837, \hat{b}=1.092, \hat{\alpha}=0.525$	253.86	261.49
EQPLL	$\hat{\beta}=1.036, \hat{\gamma}=0.0000000980$, $\hat{\theta}=0.914, \hat{b}=1.159, \hat{\alpha}=0.483$	253.62	261.25

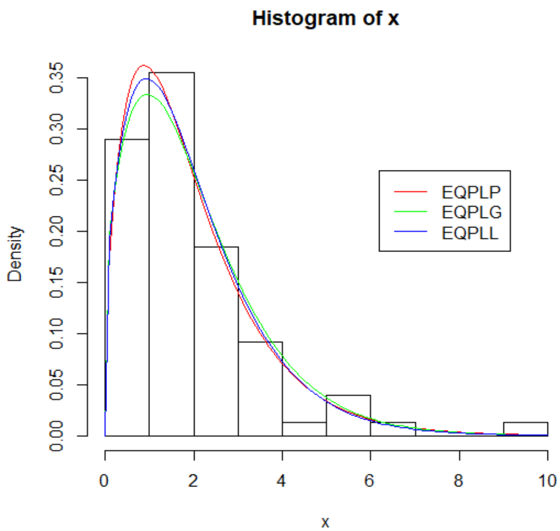

Fig 1: Fitting of EQPLP, EQPLG, EQPLL to the fatigue lifetime data.

Data Set 2. The data set reported by Efron B (1988) and was used by Rama Shanker (2016) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT). The data set is

6.53	7	10.42	14.48	16.10	22.70	34	41.55	42	45.28	49.40	53.62
63	64	83	84	91	108	112	129	133	133	139	140
140	146	149	154	157	160	160	165	146	149	154	157
160	160	165	173	176	218	225	241	248	273	277	297
405	417	420	440	523	583	594	1101	1146	1417		

Table.10.2: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta}=0.372, \hat{\gamma}=2.625, \hat{\theta}=0.605, \hat{b}=4.945, \hat{\alpha}=2.570$	750.37	758.00
EQPLG	$\hat{\beta}=0.542, \hat{\gamma}=0.000000839\ , \hat{\theta}=0.152, \hat{b}=1.749, \hat{\alpha}=0.390$	751.59	759.22
EQPLL	$\hat{\beta}=0.535, \hat{\gamma}=0.00000152\ , \hat{\theta}=0.160, \hat{b}=1.789, \hat{\alpha}=0.384$	751.58	6759.21

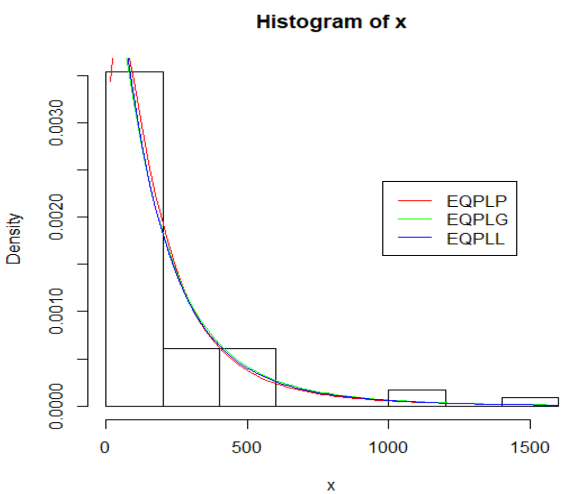

Fig 2: Fitting of EQPLP, EQPLG, EQPLL to the survival data.

Data Set 3. The data set reported by Efron B (1988) and was used by Rama Shanker (2016) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.20	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36	63.47
68.46	78.26	74.47	81.43	84	92	94	110	112	119	127
130	133	140	146	155	159	173	179	194	195	209
249	281	319	339	432	469	519	633	725	817	1776

Table 10.3: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta}=0.576, \hat{\gamma}=0.00000132, \hat{\theta}=0.125, \hat{b}=1.75, \hat{\alpha}=0.381$	568.43	576.06
EQPLG	$\hat{\beta}=0.578, \hat{\gamma}=0.0000000653, \hat{\theta}=0.126, \hat{b}=1.480, \hat{\alpha}=0.451$	569.03	576.67
EQPLL	$\hat{\beta}=0.597, \hat{\gamma}=0.00000328, \hat{\theta}=0.108, \hat{b}=1.640, \hat{\alpha}=0.411$	568.83	576.47

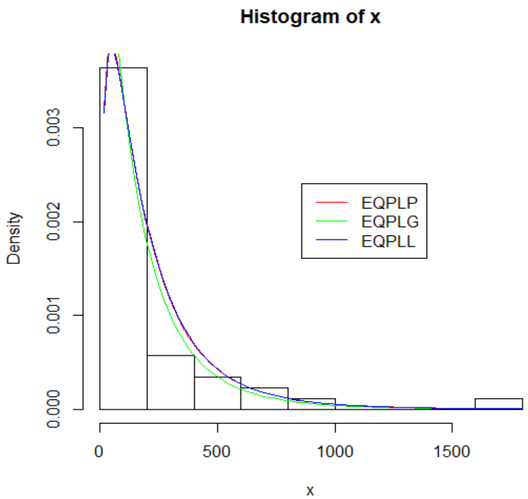

Fig 3: Fitting of EQPLP, EQPLG, EQPLL to the survival data.

Data Set 4. The data set represents remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee & Wang (2003) and was used by Rama Shanker (2016) in modeling of lifetime data .

Table.10.4: Analysis of model fitting

MODEL	MLE	AIC	BIC
EQPLP	$\hat{\beta}=0.673, \hat{\gamma}=0.00000279$, $\hat{\theta}=0.552, \hat{b}=1.415, \hat{\alpha}=0.406$	827.63	835.26
EQPLG	$\hat{\beta}=0.727, \hat{\gamma}=0.00000172$, $\hat{\theta}=0.458, \hat{b}=1.384, \hat{\alpha}=0.429$	828.34	835.98
EQPLL	$\hat{\beta}=0.713, \hat{\gamma}=0.000000556$, $\hat{\theta}=0.472, \hat{b}=1.403, \hat{\alpha}=0.403$	828.21	835.85

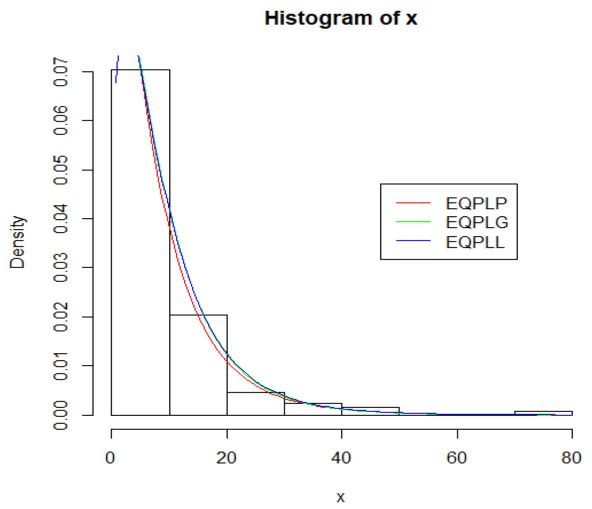

Fig 4: Fitting of EQPLP, EQPLG, EQPLL to the cancer data.

All the sub-models fit well but among them EQPLP class of distributions performs excellently well as it possesses the lowest values of AIC and BIC values.

12. CONCLUSION

We have proposed a new five parameter lifetime distribution for parallel system by compounding Exponentiated Quasi Power Lindley distribution with power series distribution. The mathematical properties including density function, moment generating function, order statistics, quantile function have been obtained. The parameters have been estimated by the method of maximum likelihood estimation. The proposed model contains some lifetime sub-classes and has the competency to yield many beneficial and flexible distributions for modelling lifetime data. Ultimately, the sub-models have been compared by applying them to four real life data sets to show the flexibility of the proposed model.

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Restructured class of estimators for population mean using an auxiliary variable under simple random sampling scheme

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Abstract

The present paper provides a remedy for improved estimation of population mean of a study variable, using the information related to an auxiliary variable in the situations under Simple Random Sampling Scheme. We suggest a new class of estimators of population mean and the Bias and MSE of the class are derived upto the first order of approximation. The least value of the MSE for the suggested class of estimators is also obtained for the optimum value of the characterizing scaler. The MSE has also been compared with the considered existing competing estimators both theoretically and empirically. The theoretical conditions for the increased efficiency of the proposed class, compared to the competing estimators, is verified using a natural population.

Mathematics Subject Classification 2010: 62D05.
Keywords: Study Variable, Auxiliary Variables, Simple Random Sampling, MSE, PRE.

1. INTRODUCTION

Auxiliary information has been in practice in sampling theory since the advent of modern sample surveys. Information on auxiliary variable having high correlation with the variable under study is quite useful in improving the sampling design. Cochran (1940) used the highly positively correlated study and auxiliary variable to propound the ratio estimator. Product estimator requires a high negative correlation between study and auxiliary variable. By reviewing the literature, it is concluded that applying the auxiliary information enhances the efficiencies of the estimators for estimating any parameter under consideration. So it is well established fact that the use of auxiliary variable technique improves the estimation process for target population. It is also noticed that ratio method of estimation is relatively simple and one of the commonly used methods of estimation. Hence we have considered the restructuring of the ratio type estimator in the present study.

Modifications in the usual ratio estimator has been done by various researchers to obtain the MSE as minimum as possible. Bahl and Tuteja (1991) formed exponential type ratio and product estimators. Kadilar and Cingi (2003) studied chain ratio type

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estimator. Jerajuddin and Kishun (2016) did not use auxiliary variable instead they used size of the sample as supplementary information. Singh, Tailor and Kakran (2004) used power transformation to improve the estimation of population mean. Al-Omari (2009), Jeelani (2013), Singh and Tailor (2003), Sisodia and Dwivedi (1981), Subramani and Kumarpandiyan (2012), Upadhyaya and Singh (1991), Yadav (2019), Yan and Tian (2010) used the functions of auxiliary variable and their combinations to modify the estimator with a greater precision.

The purpose of the current study is also to modify and improve the ratio estimator which would be better than the many of previous derived estimators which are considered in this study. Let the target population is of size N. Y is the study variable and X is the auxiliary variable. A sample of size n has been drawn both for the study and auxiliary variables. The present study would use the information of the variable X combined with the study variable to obtain the more efficient estimators.

1.1. Notations

—N : Size of the population

—n : Size of the sample

—${}^{N}C_{n}$: Number of possible samples of size n from the population of size N

—Y : Study Variable

—X : Auxiliary Variable

—M_y, M_x : Medians

—$\overline{Y}, \overline{X}$: Population means

—$\bar{y}, \bar{x}$: Sample means

—ρ : Correlation Coefficient between X and Y

—β : Regression Coefficient of Y on X

—S_y^2, S_x^2 : Population Mean Squares

—S_{yx} : Covariance between X and Y

—C_y, C_x : Coefficients of Variation

—$Bias(\cdot)$: Bias of the estimator

—$V(\cdot)$: Variance of the estimator

—$\beta_{1(x)}$: Coefficient of Skewness

—$\beta_{2(x)}$: Coefficient of Kurtosis

—$Q_{1(x)}$: First Quartile

—$Q_{3(x)}$: Third Quartile

—QD : Quartile Deviation

—$Q_{a(x)}$: Quartile Average

—$Q_{r(x)}$: Quartile Range

—TM : Tri Mean

—$MSE(\cdot)$: Mean Squared Error of the Estimator

—$PRE(\bar{y}, t)$: Percentage Relative Efficiency of the proposed estimator with respect to the SRS mean

1.2. Formulae

—$\lambda = \frac{1-f}{n}$

—$f = \frac{n}{N}$

—$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$

—$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

—$V(\bar{y}) = \lambda \overline{Y}^2 C_y^2$

—$V(\bar{x}) = \lambda \overline{X}^2 C_x^2$

—$C_y = \frac{S_y}{\overline{Y}}$

—$C_x = \frac{S_x}{\overline{X}}$

—$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N} \left(Y_i - \overline{Y}\right)\left(X_i - \overline{X}\right)$

—$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N} \left(Y_i - \overline{Y}\right)^2$

—$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N} \left(X_i - \overline{X}\right)^2$

—$\rho = \frac{S_{yx}}{S_x S_y}$

—$QD = \frac{Q_3 - Q_1}{2}$

—$Q_{a(x)} = \frac{Q_1 + M_x + Q_3}{3}$

—$Q_{r(x)} = Q_3 - Q_1$

—$TM = \frac{Q_1 + 2M_x + Q_3)}{4}$

2. LITERATURE REVIEW OF EXISTING ESTIMATORS

A number of modified estimators by various authors have been developed till date for improved estimation of the population mean under various situations under simple random sampling scheme. The considered existing estimators with their Mean Squared Errors along with their constants are presented in Table I.

Table I: Literature Review

The existing estimators of population mean of study variable.

SNo	Estimators	MSE/Variance	Constants
1	$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ Sample Mean	$\lambda \overline{Y}^2 C_y^2$	
2	$t_1 = \bar{y}\left(\frac{\overline{X}}{\bar{x}}\right)$ Cochran (1940)	$\lambda \overline{Y}^2\left(C_y^2 + C_x^2 - 2C_{yx}\right)$	
3	$t_2 = \bar{y}\exp\left(\frac{\overline{X}-\bar{x}}{\overline{X}+\bar{x}}\right)$ Bahl and Tuteja (1991)	$\lambda \overline{Y}^2\left(C_y^2 + \frac{C_x^2}{4} - \rho C_y C_x\right)$	
4	$t_3 = \bar{y}\left(\frac{\overline{X}+C_x}{\bar{x}+C_x}\right)$ Sisodia and Dwivedi (1981)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_3^2 C_x^2 - 2\theta_3 C_{yx}\right)$	$\theta_3 = \frac{\overline{X}}{\overline{X}+C_x}$
5	$t_4 = \bar{y}\left(\frac{\overline{X}C_x+\beta_{2(x)}}{\bar{x}C_x+\beta_{2(x)}}\right)$ Upadhyaya and Singh (1999)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_4^2 C_x^2 - 2\theta_4 C_{yx}\right)$	$\theta_4 = \frac{\overline{X}C_x}{\overline{X}C_x+\beta_{2(x)}}$
6	$t_5 = \bar{y}\left(\frac{\overline{X}\beta_{2(x)}+C_x}{\bar{x}\beta_{2(x)}+C_x}\right)$ Upadhyaya and Singh (1999)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_5^2 C_x^2 - 2\theta_5 C_{yx}\right)$	$\theta_5 = \frac{\overline{X}\beta_{2(x)}}{\overline{X}\beta_{2(x)}+C_x}$
7	$t_6 = \bar{y}\left(\frac{\overline{X}+\rho}{\bar{x}+\rho}\right)$ Singh and Tailor (2003)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_6^2 C_x^2 - 2\theta_6 C_{yx}\right)$	$\theta_6 = \frac{\overline{X}}{\overline{X}+\rho}$
8	$t_7 = \bar{y}\left(\frac{\overline{X}+\beta_{2(x)}}{\bar{x}+\beta_{2(x)}}\right)$ Singh et al. (2004)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_7^2 C_x^2 - 2\theta_7 C_{yx}\right)$	$\theta_7 = \frac{\overline{X}}{\overline{X}+\beta_{2(x)}}$
9	$t_8 = \bar{y}\left(\frac{\overline{X}+Q_{1(x)}}{\bar{x}+Q_{1(x)}}\right)$ $t_9 = \bar{y}\left(\frac{\overline{X}+Q_{3(x)}}{\bar{x}+Q_{3(x)}}\right)$ Al-Omari et al. (2009)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_8^2 C_x^2 - 2\theta_8 C_{yx}\right)$ $\lambda \overline{Y}^2\left(C_y^2 + \theta_9^2 C_x^2 - 2\theta_9 C_{yx}\right)$	$\theta_8 = \frac{\overline{X}}{\overline{X}+Q_{1(x)}}$ $\theta_9 = \frac{\overline{X}}{\overline{X}+Q_{3(x)}}$
10	$t_{10} = \bar{y}\left(\frac{\overline{X}+\beta_{1(x)}}{\bar{x}+\beta_{1(x)}}\right)$ Yan and Tian (2010)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_{10}^2 C_x^2 - 2\theta_{10} C_{yx}\right)$	$\theta_{10} = \frac{\overline{X}}{\overline{X}+\beta_{1(x)}}$
11	$t_{11} = \bar{y}\left(\frac{\overline{X}\beta_{1(x)}+\beta_{2(x)}}{\bar{x}\beta_{1(x)}+\beta_{2(x)}}\right)$ Yan and Tian (2010)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_{11}^2 C_x^2 - 2\theta_{11} C_{yx}\right)$	$\theta_{11} = \frac{\overline{X}\beta_{1(x)}}{\overline{X}\beta_{1(x)}+\beta_{2(x)}}$
12	$t_{12} = \bar{y}\left(\frac{\overline{X}C_x+\beta_{1(x)}}{\bar{x}C_x+\beta_{1(x)}}\right)$ Yan and Tian (2010)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_{12}^2 C_x^2 - 2\theta_{12} C_{yx}\right)$	$\theta_{12} = \frac{\overline{X}C_x}{\overline{X}C_x+\beta_{1(x)}}$
13	$t_{13} = \bar{y}\left(\frac{\overline{X}+M_x}{\bar{x}+M_x}\right)$ Subramani and Kumarpandiyan (2012a)	$\lambda \overline{Y}^2\left(C_y^2 + \theta_{13}^2 C_x^2 - 2\theta_{13} C_{yx}\right)$	$\theta_{13} = \frac{\overline{X}}{\overline{X}+M_x}$

SNo	Estimators	MSE/Variance	Constants
14	$t_{14}=\bar{y}\left(\frac{\bar{X}C_x+M_x}{\bar{x}C_x+M_x}\right)$ Subramani and Kumarpandiyan (2012a)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{14}^2C_x^2-2\theta_{14}C_{yx}\right)$	$\theta_{14}=\frac{\bar{X}C_x}{\bar{X}C_x+M_x}$
15	$t_{15}=\bar{y}\left(\frac{\bar{X}+Q_{r(x)}}{\bar{x}+Q_{r(x)}}\right)$ Subramani and Kumarpandiyan (2012b)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{15}^2C_x^2-2\theta_{15}C_{yx}\right)$	$\theta_{15}=\frac{\bar{X}}{\bar{X}+Q_{r(x)}}$
16	$t_{16}=\bar{y}\left(\frac{\bar{X}+QD}{\bar{x}+QD}\right)$ Subramani and Kumarpandiyan (2012b)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{16}^2C_x^2-2\theta_{16}C_{yx}\right)$	$\theta_{16}=\frac{\bar{X}}{\bar{X}+QD}$
17	$t_{17}=\bar{y}\left(\frac{\bar{X}+Q_{a(x)}}{\bar{x}+Q_{a(x)}}\right)$ Subramani and Kumarpandiyan (2012b)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{17}^2C_x^2-2\theta_{17}C_{yx}\right)$	$\theta_{17}=\frac{\bar{X}}{\bar{X}+Q_{a(x)}}$
18	$t_{18}=\bar{y}\left(\frac{\bar{X}+n}{\bar{x}+n}\right)$ Jerajuddin and Kishun (2016)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{18}^2C_x^2-2\theta_{18}C_{yx}\right)$	$\theta_{18}=\frac{\bar{X}}{\bar{X}+n}$
19	$t_{19}=\bar{y}\left(\frac{\bar{X}\beta_{1(x)}+QD}{\bar{x}\beta_{1(x)}+QD}\right)$ Jeelani et al. (2013)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{19}^2C_x^2-2\theta_{19}C_{yx}\right)$	$\theta_{19}=\frac{\bar{X}\beta_{1(x)}}{\bar{X}\beta_{1(x)}+QD}$
20	$t_{20}=\bar{y}\left(\frac{ab\bar{X}+cd}{ab\bar{x}+cd}\right)$ $t_{20(1)}=\bar{y}\left(\frac{\beta_{2(x)}M_x\bar{X}+\rho}{\beta_{2(x)}M_x\bar{x}+\rho}\right)$ $t_{20(2)}=\bar{y}\left(\frac{\beta_{2(x)}M_x\bar{X}+\rho C_x}{\beta_{2(x)}M_x\bar{x}+\rho C_x}\right)$ $t_{20(3)}=\bar{y}\left(\frac{\beta_{1(x)}M_x\bar{X}+\rho}{\beta_{1(x)}M_x\bar{x}+\rho}\right)$ Yadav et al. (2019)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{20}^2C_x^2-2\theta_{20}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(1)}^2C_x^2-2\theta_{20(1)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(2)}^2C_x^2-2\theta_{20(2)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(3)}^2C_x^2-2\theta_{20(3)}C_{yx}\right)$	$\theta_{20}=\frac{ab\bar{X}}{ab\bar{X}+c.d}$ $\theta_{20(1)}=\frac{\beta_{2(x)}M_x\bar{X}}{\beta_{2(x)}M_x\bar{x}+\rho}$ $\theta_{20(2)}=\frac{\beta_{2(x)}M_x\bar{X}}{\beta_{2(x)}M_x\bar{x}+\rho C_x}$ $\theta_{20(3)}=\frac{\beta_{1(x)}M_x\bar{X}}{\beta_{1(x)}M_x\bar{x}+\rho}$
21	$t_{20(4)}=\bar{y}\left(\frac{\beta_{1(x)}M_x\bar{X}+\rho C_x}{\beta_{1(x)}M_x\bar{x}+\rho C_x}\right)$ $t_{20(5)}=\bar{y}\left(\frac{n\bar{X}+\rho}{n\bar{x}+\rho}\right)$ $t_{20(6)}=\bar{y}\left(\frac{n\bar{X}+C_x}{n\bar{x}+C_x}\right)$ $t_{20(7)}=\bar{y}\left(\frac{n\bar{X}+\rho C_x}{n\bar{x}+\rho C_x}\right)$ $t_{20(8)}=\bar{y}\left(\frac{n\rho\bar{X}+C_x}{n\rho\bar{x}+C_x}\right)$ $t_{20(9)}=\bar{y}\left(\frac{nC_x\bar{X}+\rho}{nC_x\bar{x}+\rho}\right)$ Yadav et al.(2019)	$\lambda\bar{Y}^2\left(C_y^2+\theta_{20(4)}^2C_x^2-2\theta_{20(4)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(5)}^2C_x^2-2\theta_{20(5)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(6)}^2C_x^2-2\theta_{20(6)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(7)}^2C_x^2-2\theta_{20(7)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(8)}^2C_x^2-2\theta_{20(8)}C_{yx}\right)$ $\lambda\bar{Y}^2\left(C_y^2+\theta_{20(9)}^2C_x^2-2\theta_{20(9)}C_{yx}\right)$	$\theta_{20(4)}=\frac{\beta_{1(x)}M_x\bar{X}}{\beta_{1(x)}M_x\bar{x}+\rho C_x}$ $\theta_{20(5)}=\frac{n\bar{X}}{n\bar{x}+\rho C_x}$ $\theta_{20(6)}=\frac{n\bar{X}}{n\bar{x}+C_x}$ $\theta_{20(7)}=\frac{n\bar{X}}{n\bar{x}+\rho C_x}$ $\theta_{20(8)}=\frac{n\rho\bar{X}}{n\rho\bar{x}+C_x}$ $\theta_{20(9)}=\frac{nC_x\bar{X}}{nC_x\bar{x}+\rho}$

3. PROPOSED CLASS OF ESTIMATORS

Inspired by the literature of improved estimators and adopting the Yadav et al. (2019) estimator, we suggest an improved class of ratio type estimators for the estimation of population mean using auxiliary information as follows:

$$t = \alpha \bar{y} + (1-\alpha)\bar{y}\left[\frac{ab\overline{X}+cd}{ab\bar{x}+cd}\right] \tag{1}$$

where α is a characterizing constant and a, b, c, d are either constants or the known parameters of the auxiliary variable. The constant α is chosen such that the Mean Squared Error(MSE) of the suggested estimator is minimum. The (a, b, c, d) may also take those real and parametric values which makes the MSE of the proposed estimator a least possible.

3.1. Bias and MSE

To obtain the Bias and MSE of the suggested estimator, we define the following approximations as:

$$\begin{aligned} e_0 &= \frac{\bar{y}-\overline{Y}}{\overline{Y}} \quad &\text{and} \quad e_1 &= \frac{\bar{x}-\overline{X}}{\overline{X}} \\ \text{So,} \quad \bar{y} &= \overline{Y}(1+e_0) \quad &\text{and} \quad \bar{x} &= \overline{X}(1+e_1) \\ E(e_0) = E(e_1) &= 0 \quad &\text{and} \quad E(e_0e_1) &= \lambda C_{yx} \\ E(e_0^2) &= \lambda C_y^2 \quad &\text{and} \quad E(e_1^2) &= \lambda C_x^2 \end{aligned}$$

Now rewriting the proposed estimator from equation (1) as,

$$\begin{aligned} t &= \alpha\overline{Y}(1+e_0) + (1-\alpha)\overline{Y}(1+e_0)\left[\frac{ab\overline{X}+cd}{ab\overline{X}(1+e_1)+cd}\right] \\ &= \alpha\overline{Y}(1+e_0) + (1-\alpha)\overline{Y}(1+e_0)(1+\theta e_1)^{-1} \end{aligned} \tag{2}$$

$$\text{Where,} \quad \theta = \frac{ab\overline{X}}{ab\overline{X}+cd}$$

Expanding the equation (2), simplifying and retaining the terms upto the first order of approximation, we get,

$$t = \alpha\overline{Y}[1+e_0-1-e_0+\theta e_1+\theta e_0 e_1-\theta^2 e_1^2] + \overline{Y}(1+e_0-\theta e_1-\theta e_0 e_1+\theta^2 e_1^2)$$

$$(t-\overline{Y}) = \overline{Y}[\alpha(\theta e_1+\theta e_0 e_1-\theta^2 e_1^2) + (e_0-\theta e_1-\theta e_0 e_1+\theta^2 e_1^2)] \tag{3}$$

Taking expectations on both sides of equation (3)

$$E(t-\overline{Y}) = \overline{Y}[\alpha(\theta\lambda C_{yx} - \theta^2\lambda C_x^2) + (\theta^2\lambda C_x^2 - \theta\lambda C_{yx})]$$
$$B(t) = \overline{Y}[\alpha(\theta\lambda C_{yx} - \theta^2\lambda C_x^2) + (\theta^2\lambda C_x^2 - \theta\lambda C_{yx})] \tag{4}$$

Squaring the equation (3), retaining the terms up to the approximation of order one and putting values of various expectations, we get the Mean Squared Error of the proposed class of estimators as,

$$E(t-\overline{Y})^2 = \overline{Y}^2 E[\alpha^2\theta^2 e_1^2 + e_0^2 + \theta^2 e_1^2 - 2\theta e_0 e_1 + 2\alpha(\theta e_0 e_1 - \theta^2 e_1^2)]$$
$$MSE(t) = \overline{Y}^2[\alpha^2\theta^2\lambda C_x^2 + (\lambda C_y^2 + \theta^2\lambda C_x^2 - 2\theta\lambda C_{yx}) + 2\alpha(\theta\lambda C_{yx} - \theta^2\lambda C_x^2)] \tag{5}$$

By the Least Square Method of estimation the optimum value of α is,

$$\boldsymbol{\alpha_{opt} = 1 - \frac{\rho C_y}{\theta C_x}} \tag{6}$$

Putting the optimum value of α, we obtain the minimum value of $MSE(t)$ as follows:

$$MSE(t)_{min} = \lambda\overline{Y}^2\left[\left(1-\frac{d}{\theta}\right)^2\theta^2C_x^2 + (C_y^2+\theta^2C_x^2-2\theta C_{yx}) + 2\left(1-\frac{d}{\theta}\right)(\theta C_{yx}-\theta^2C_x^2)\right]$$
$$= \lambda\overline{Y}\left[\left(\frac{\theta^2C_x^2-\theta C_{yx}}{\theta^2C_x^2}\right)^2\theta^2C_x^2 + (C_y^2+\theta^2C_x^2-2\theta C_{yx}) -2\left(\frac{\theta^2C_x^2-\theta C_{yx}}{\theta^2C_x^2}\right)(\theta^2C_x^2-\theta C_{yx})\right]$$

$$MSE(t)_{min} = \lambda\overline{Y}^2\left[\frac{(\theta^2C_x^2-\theta C_{yx})^2}{\theta^2C_x^2} + (C_y^2+\theta^2C_x^2-2\theta C_{yx}) - 2\frac{(\theta^2C_x^2-\theta C_{yx})^2}{\theta^2C_x^2}\right]$$
$$= \lambda\overline{Y}^2\left[(C_y^2+\theta^2C_x^2-2\theta C_{yx}) - \frac{(\theta^2C_x^2-\theta C_{yx})^2}{\theta^2C_x^2}\right]$$
$$= \lambda\overline{Y}^2\left[(C_y^2+\theta^2C_x^2-2\theta C_{yx}) - \left(\frac{\theta^4C_x^4+\theta^2C_{yx}^2-2\theta^3C_x^2C_{yx}}{\theta^2C_x^2}\right)\right]$$
$$= \lambda\overline{Y}^2[C_y^2+\theta^2C_x^2-2\theta C_{yx}-\theta^2C_x^2-\rho^2C_y^2+2\theta C_{yx}]$$
$$= \lambda\overline{Y}^2(C_y^2-\rho^2C_y^2)$$
$$= \lambda\overline{Y}^2\left(C_y^2-\frac{C_{yx}^2C_y^2}{C_y^2C_x^2}\right)$$
$$\mathbf{MSE(t)_{min}} = \lambda\overline{\mathbf{Y}}^{\mathbf{2}}\left(\mathbf{C_y^2}-\frac{\mathbf{C_{yx}^2}}{\mathbf{C_x^2}}\right) \tag{7}$$

For this MSE

$$\mathbf{B(t) = 0} \tag{8}$$

4. THEORETICAL EFFICIENCY COMPARISON

Following are the conditions under which proposed class of estimators is more efficient than the existing estimators,

Table II: **Efficiency Comparison**

SNo.	$\mathbf{MSE(t) < MSE(\bullet)}$	Condition
1	$MSE(t)_{min} < V(t_0)$	$\frac{C_{yx}^2}{C_x^2} > 0$
2	$MSE(t)_{min} < MSE(t_1)$	$C_x^2 > 4C_{yx}\left(1 - \frac{C_{yx}}{C_x^2}\right)$
3	$MSE(t)_{min} < MSE(t_2)$	$C_x^2 > C_{yx}\left(2 - \frac{C_{yx}}{C_x^2}\right)$
4	$MSE(t)_{min} < MSE(t_i)$	$C_x^2 > \frac{C_{yx}}{\theta_i^2}\left(2\theta_i - \frac{C_{yx}^2}{C_x^2}\right) \quad ; \quad i = 3, \ldots 20$

5. COMPUTATIONAL STUDY

To prove the theoretical results numerically we have considered a Natural Population with sample size 5.

Data Source : Daroga Singh and F.S. Chaudhary (1986, Page-177)

Data Details : **Study Variable :**
: Area under wheat in a region during year 1974
: **Auxiliary Variable**
: Cultivated Area under wheat in a region during year 1973

Table III: Parametric Values of the Population

SNo	Information	Data Set
1	N	34
2	n	5
3	$\overline{Y}$	199.4412
4	$\overline{X}$	208.8824
5	S_y	150.215
6	S_x	150.506
7	C_y	0.7531797
8	C_x	0.7205298
9	M_y	142.5
10	M_x	150
11	ρ	0.9800867
12	C_{yx}	0.5318817
13	$\beta_{1(x)}$	0.8732281
14	$\beta_{2(x)}$	5.912272
15	f	0.1470588
16	λ	0.1705882
17	${}^{N}C_n$	278256
18	$Q_{1(x)}$	94.25
19	$Q_{3(x)}$	275.75
20	$Q_{r(x)}$	160.5
21	$Q_{a(x)}$	166.3333
22	QD	80.25
23	TM	162.25

To compute the Percent Relative Efficiency (PRE) for different estimators with respect to Hansan and Horwitz estimators we use the following :

$$PRE = \frac{V(t_0)}{MSE(\bullet)}$$

Table IV: MSE and PRE of Estimators

SNo	Estimators	MSE	PRE
1	t_0	3849.248	100
2	t_1	153.8905	2501.29
3	t_2	1120.88	343.413
4	t_3	154.5255	2491.011
5	t_4	165.4474	2326.57
6	t_5	153.9924	2499.635
7	t_6	154.7734	2487.021
8	t_7	161.3104	702.2823
9	t_8	548.1055	2386.237
10	t_9	1312.292	293.3224
11	t_{10}	154.6701	2364.667
12	t_{11}	162.7818	2488.682
13	t_{12}	155.0034	2483.331
14	t_{13}	841.4363	457.4616
15	t_{14}	1117.772	344.368
16	t_{15}	893.9771	430.5757
17	t_{16}	473.1776	813.4891
18	t_{17}	922.6805	417.181
19	t_{18}	535.4868	718.8315
20	t_{19}	159.8507	2408.027
21	$t_{20(1)}$	153.8915	2501.275
22	$t_{20(2)}$	153.8912	2501.279
23	$t_{20(3)}$	153.8967	2501.189
24	$t_{20(4)}$	153.895	2501.217
25	$t_{20(5)}$	154.0555	2498.612
26	$t_{20(6)}$	154.0112	2499.33
27	$t_{20(7)}$	154.0088	2499.369
28	$t_{20(8)}$	154.0137	2499.289
29	$t_{20(9)}$	154.121	2497.549
30	$\mathbf{t}_{(\mathbf{min})}$	**151.7764**	**2536.131**

6. RESULTS AND CONCLUSION

(1) Table 1 Reviews the Existing literature. Table 2 shows the conditions for which our proposed class of estimators is better than the existing estimators. Table 3 consists of parametric values of the data with which we have verified our results empirically. Table 4 shows the MSE of existing and proposed class of estimators and PRE of the various mentioned estimators with respect to mean per unit estimator.

(2) We study the MSE of the proposed class of estimators up to the first order of approximation. For the optimum value of α which makes the MSE minimum of the proposed class of estimators, the bias becomes zero thereby making suggested estimator unbiased.

(3) We have also suggested some members of the proposed class of estimators which come out to be more efficient than the existing competing estimators of population mean under simple random sampling scheme.

(4) From Tables 4 we can easily notice that suggested estimator has the largest PRE among all the considered existing competing estimators of population mean using auxiliary information under simple random sampling scheme.

(5) Hence, it has been proven both theoretically and numerically that the proposed estimator is better than the other given competing estimators. Thus the sampling distribution of the suggested estimator is most closer to true population mean as compared to the sampling distributions of all other competing estimators.

(6) Since the suggested computations revolved around a natural population, therefore we can successfully recommend the proposed class of estimators for practical utility in various fields of applications including agriculture, biology, medical sciences, economics, engineering, commerce etc.

R CALCULATION

```
# Secondary data(From Daroga and Singh)
# Calculation of parameters
Y¡-c(50,149,284,381,278,111,
634,278,112,355,99,498,111,6,339,80,
105,27,515,249,85,221,133,144,103,
175,335,219,62,79,60,100,141,263)
X¡-c(70,163,320,440,250,125,558,
254,101,359,109,481,125,5,427,78,75,
45,564,238,92,247,134,131,129,190,
363,235,73,62,71,137,196,255)
# mean
y¡-mean(Y)
x¡-mean(X)
# standard deviation
sy¡-sqrt(var(Y))
sx¡-sqrt(var(X))
# coeffcient of variation
cy¡-sy/y
cx¡-sx/x
# median
mx¡-median(X)
my¡-median(Y)
# correlation coefficient
r¡-cor(X,Y)
cyx¡-r∗ cy∗ cx
library(moments)
# coefficient of skewness b1(beta 1)
be1¡-skewness(X)ˆ 2
# coefficient of kurtosis b2( beta 2)
be2¡-kurtosis(X)+3
N=34
n=5
f=n/N
l=(1-f)/n
Ncn=choose(N,n)
summary(X)
# quartile deviation
qd=(q3-q1)/2
# interquartile range
qr=q3-q1
# quartile average
qa=(q1+mx+q3)/3
# tri mean
tm¡-(q1+2∗ mx+q3)/4
# # Review of literature
# Sample mean
m0¡-l∗ yˆ 2∗ cyˆ 2
# Cochran (1940)
m1¡-l∗ yˆ 2∗ (cyˆ 2+cxˆ 2-(2∗ cyx))
# Bahl and Tuteja(1991)
m2¡-l∗ yˆ 2∗ (cyˆ 2+(cxˆ 2/4)-cyx)
# Function of MSE calculation
m¡-function(c){
mse=l∗ yˆ 2∗ (cyˆ 2+(cˆ 2∗ cxˆ 2)-(2∗ c∗ cyx))
print(mse)
}
# Sisodia and Dwivedi(1981)
c3¡-x/(x+cx)
m(c3)
# # [1] 154.5255
# Upadhyaya and Singh (1999)........
c4¡-(x∗ cx)/((x∗ cx)+be2)
c5¡-(x∗ be2)/((x∗ be2)+cx)
m(c4)
m(c5)
# Singh and Tailor(2003)......
c6¡-x/(x+r)
m(c6)
# singh et al.(2004)........
c7¡-x/(x+be2)
m(c7)
# Al-Omari et al. (2009).
c8¡-x/(x+q1)
m(c8)
# # Al-Omari et al. (2009)
c9¡-x/(x+q3)
m(c9)
# Yan and Tian(2010)
c10¡-x/(x+be1)
c11¡-(x∗ be1)/((x∗ be1)+be2)
c12¡-(x∗ cx)/((x∗ cx)+be1)
m(c10)
```

```
m(c11)
m(c12)
# Subramani and Kumarpandiyan(2012a)
c13¡-x/(x+mx)
c14¡-(x∗ cx)/((x∗ cx)+mx)
m(c13)
m(c14)
# Subramani and Kumarpandiyan(2012b)
c15¡-x/(x+qr)
c16¡-x/(x+qd)
c17¡-x/(x+qa)
m(c15)
m(c16)
m(c17)
# Jeelani et al.(2013).............
c18¡-(x∗ be1)/((x∗ be1)+qd)
m(c18)
# Jerajuddin and Kishun(2016)....
c19¡-x/(x+n)
m(c19)
# Yadav et al. (2019).....
c20_1¡-(be2∗ mx∗ x)/((be2∗ mx∗ x)+(r))
c20_2¡-(be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗ cx))
c20_3¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r))
c20_4¡-(be1∗ mx∗ x)/((be1∗ mx∗ x)+(r∗ cx))
c20_5¡-(n∗ x)/((n∗ x)+(r))
c20_6¡-(n∗ x)/((n∗ x)+(cx))
c20_7¡-(n∗ x)/((n∗ x)+(cx∗ r))
c20_8¡-(n∗ r∗ x)/((n∗ r∗ x)+(cx))
c20_9¡-(n∗ cx∗ x)/((n∗ cx∗ x)+(r))
m(c20_1)
m(c20_2)
m(c20_3)
m(c20_4)
m(c20_5)
m(c20_6)
m(c20_7)
m(c20_8)
m(c20_9)
# proposed estimator...............
th1=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r)))
m(th1)
th2=((be2∗ mx∗ x)/((be2∗ mx∗ x)+(r∗ cx)))
m(th2)
th3=(Ncn∗ x)/((Ncn∗ x)+(mx))
m(th3)
th4=(mx∗ x)/((mx∗ x)+(f∗ r))
m(th4)
th5=(tm∗ q3∗ x)/((tm∗ q3∗ x)+(qa/qd))
m(th5)
th6=(qd∗ qa∗ x)/((qd∗ qa∗ x)+(be2∗ be1))
m(th6)
th7=(mx∗ sx∗ x)/((mx∗ sx∗ x)+(r∗ l))
m(th7)
l∗ yˆ 2∗ (cyˆ 2-(cyxˆ 2/cxˆ 2))
PRE¡-function(me){
PRE¡-(m0/me)∗ 100
print(PRE)
}
```

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