Solutions of Volterra integral and integro-differential equations using modified Laplace Adomian decomposition method

D. RANI AND V. MISHRA

Abstract

In this paper, an effectual and new modification in Laplace Adomian decomposition method based on Bernstein polynomials is proposed to find the solution of nonlinear Volterra integral and integro-differential equations. The performance and capability of the proposed idea is endorsed by comparing the exact and approximate solutions for three different examples on Volterra integral, integro-differential equations of the first and second kinds. The results shown through tables and figures demonstrate the accuracy of our method. It is concluded here that the non orthogonal polynomials can also be used for Laplace Adomian decomposition method. In addition, convergence analysis of the modified technique is also presented.

Mathematics Subject Classification 2010: 41A10; 44A10; 34A34; 45D05;65R10.
Keywords: Laplace transformation, Adomian decomposition method, modified Laplace Adomian decomposition method, Bernstein polynomials, Volterra integral and integro-differential equations.

1. INTRODUCTION

Substantial interest is devoted to solve nonlinear Volterra integral and integro-differential equations by many researchers and scientists due to its applications in science such as the population dynamics, spread of epidemics, semi-conductor devices [Wazwaz 2011], biological species coexisting together with increasing and decreasing rates of generating and in engineering such as heat transfer and neutron diffusion process [Bahuguna et al. 2009].

The nonlinear Volterra integral equation of the second kind is defined as [Wazwaz 2011]

$$u(x) = f(x) + \int_0^x k(x,t)F(u(t))dt \tag{1}$$

where $f(x)$ is known as source term and F is a nonlinear operator, $F(u(x))$ is a nonlinear function.

The nonlinear Volterra integro-differential equation of the first kind is given by

10.2478/jamsi-2019-0001

[Wazwaz 2011; 2010]

$$\int_0^x K_1(x,t)F(u(t))dt + \int_0^x K_2(x,t)u^{(i)}(t)dt = f(x) \quad (2)$$

However, the nonlinear Volterra integro-differential equation of the second kind is defined as [Wazwaz 2011; 2010]

$$u^{(i)}(x) = f(x) + \int_0^x k(x,t)F(u(t))dt \quad (3)$$

where $u^{(i)}(x)$ denotes the ith order derivative of $u(x)$. The kernel $k(x,t)$ and the function $f(x)$ of these equations are given real-valued functions and $F(u(x))$ is a nonlinear function.
Earlier many numerical and analytical methods have been presented to solve these kinds of equations [Wazwaz 2011; 2010; Maleknejad and Najafi 2011; Maleknejad et al. 2011].

1.1. Laplace Adomian decomposition method and modifications

In recent years, several researchers have adapted Adomian decomposition method (ADM) to solve many kinds of functional equations, which was developed by Adomian in 1980. In [Adomian 1988; 1990], Adomian provided a review of decomposition method in applied mathematics. The solution in this method is considered as the summation of an infinite convergent series without using any restrictive assumptions. A theoretical foundation of Adomian method was developed in [Gabet 1994], Venkatarangan and Rajalakshmi [Venkatarangan and Rajalakshmi 1995] used modified ADM to solve equations containing radical signs. Adomian polynomials are modified by Adomian and Rach in [Adomian and Rach 1996], Luo et. al [Luo et al. 2006] studied the partial solutions on ADM for solving heat and wave equations, Hashim [Hashim 2006] applied ADM to solve linear and nonlinear boundary value problems for fourth order integro-differential equations. In [Hosseini 2006], Hosseini modified the Adomian decomposition method by expressing the source function $f(x)$ in Chebyshev polynomials and solved the nonlinear differential algebraic equations. The ADM is used to solve nonlinear Sturm-Liouville problems in [Somali and Gokmen 2007], Marwat and Asghar [Marwat and Asghar 2008] suggested a two step Adomian decomposition method for solving heat equation with variable coefficients, Liu [Liu 2009] employed Legendre polynomials to improve the Adomian decomposition method and concluded that Chebyshev and Legendre polynomials can be successfully used for ADM and comparatively Chebyshev expansion provides the better estimation . The interested reader can see the other applications and modifications of this method in [Ghazanfari and Sepahvandzadeh 2014; Evans et al. 2004; Singh and Kumar 2011; Biazar et al. 2004; Zhang and Lu 2011; Li and Wang 2009; Biazar et al. 2010; Abassy 2010; Bildik and Deniz 2015; Babolian and Biazar 2002].

Further, Khuri [Khuri 2001] developed Laplace Adomian decomposition method and applied to find the solutions of nonlinear differential equations. This method is the combination of two powerful tools, Laplace transform and Adomian decomposition method, which is used to solve extinct functional equations [Wazwaz 2010; Doan 2012]. Hence, there are numerous applications where Laplace Adomian decomposition method is used. The method is also improved and modified from different aspects by some authors [Manafianheris 2012; Kumar et al. 2014].

In this work, our aim is to modify Laplace Adomian decomposition method based on Bernstein polynomials. At the beginning of our technique, we expand the source function, i.e. $f(x)$ as Bernstein polynomials which approximate the function uniformly and then Laplace Adomian decomposition method is applied to solve Volterra integral and integro-differential equations, that gives the tremendous improved results as shown in examples. To the best of our knowledge, Bernstein polynomials is not combined with the LADM. Therefore, this is the new idea which we have used.

1.2. Bernstein polynomials

The Berstein basis polynomials which are named after Russian mathematician Sergei Bernstein, is used to approximate the functions and curves. Following are some basic definitions [Quain et al. 2011]:

DEFINITION 1.1 (Bernstein basis polynomials). *The Bernstein basis polynomials of degree n form a complete basis over the interval* $[0,1]$ *and are defined by*

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0,1,2\ldots n.$$

DEFINITION 1.2 (Bernstein polynomials). *A linear combination of Bernstein basis polynomials*

$$B_n(x) = \sum_{k=0}^{n} \beta_k B_{k,n}(x) \tag{4}$$

is called the Bernstein polynomial of degree n where β_k are Bernstein coefficients.

DEFINITION 1.3. *With f a real valued function defined and bounded on $[0,1]$, let $B_n(f)$ be the polynomial on $[0,1]$,that assigns to $f(x)$ the value*

$$B_n(f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \tag{5}$$

where $B_n(f)$ is the nth Bernstein polynomial for $f(x)$.

The utilizations and properties of Bernstein polynomials have gained much importance in the domain of applied mathematics, physics and computer aided-geometric designs [Farouki 2012; Farouki and Rajan 1998; Bohm et al. 1984; Bhatti and Bracken 2007]. Bernstein polynomials are the basis of approximation theory, with the help of these polynomials Weierstrass approximation theorem [Quain et al. 2011] is proved, which is given as follows:

THEOREM 1.4. *For all functions f in C[0, 1], the sequence of $B_n(f)$ converges uniformly to f, where $B_n(f)$ is defined by (5).*

Using Taylors series, if we approximate a function, curve or surface, it seems that it converges slowly and does not converge to original function. Comparatively, Bernstein polynomials are better approximation to a function. It also has some applications in optimal control theory, stochastic dynamics and in the modelling of chemical reactions [Yousefi and Behroozifar 2010]. Problems like, elliptic and hyperbolic partial differential equations have been solved using Bernstein polynomials by implementation of Galerkin and collocation approaches to determine the coefficients.

The contents of this paper are as follows: in Section 2, we will give analysis of modified LADM; Section 3 gives the convergence analysis of the method; in Section 4 we will give three examples to demonstrate the applicability of the proposed approach. In the last section, conclusions are drawn.

2. MODIFIED LAPLACE ADOMIAN DECOMPOSITION METHOD BASED ON BERNSTEIN POLYNOMIALS

In this section, we are analyzing the method developed in [Rani and Mishra 2017] for nonlinear Volterra integral equation with the difference kernel, i.e $k(x,t) = k(x-t)$ given by (1).

Adopting the standard Laplace Adomian decomposition method, firstly applying Laplace transform on both sides of (1) and with the use of linear property and convolution theorem of Laplace transform, we get

$$L[u(x)] = L[f(x)] + L[k(x-t)]L[F(u(x))] \tag{6}$$

According to the LADM technique $u(x)$ can be written as an infinite series given by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{7}$$

Then writing the nonlinear term $F(u(x))$ as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \tag{8}$$

where A_n's are the Adomian polynomials, given by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0} \tag{9}$$

Substituting (7) and (8) into (6), we get

$$L\left[\sum_{n=0}^{\infty} u_n(x)\right] = L[f(x)] + L[k(x-t)]L\left[\sum_{n=0}^{\infty} A_n(x)\right]$$

The linearity property of Laplace transform implies

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[f(x)] + L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)] \tag{10}$$

Now we are modifying the standard LADM, where the source term is expanded or written in the form of Bernstein polynomials with degree m given by (5). Therefore, we attain

$$\sum_{n=0}^{\infty} L[u_n(x)] = L[B_m f(x)] + L[k(x-t)]\sum_{n=0}^{\infty} L[A_n(x)] \tag{11}$$

$u(x)$ can be found by defining the following iterative scheme:

$$L[u_0(x)] = L[B_m(f(x))] \tag{12}$$

Taking inverse Laplace transform on both sides of (12), we obtain

$$u_0(x) = L^{-1}[L(B_m(f(x)))]$$

Therefore, the initial approximation depends on the Bernstein polynomials of source function, which plays a significant role in the next approximations, hence in the approximate solution of the given problem.
Similarly, we have the general relation as

$$L[u_{n+1}(x)] = L[k(x-t)]L[A_n(x)] \tag{13}$$

For determining the terms u_1, u_2, u_3 ... of infinite series we use the inverse Laplace transform to above recursive relation and $u(x)$, the approximate solution to given nonlinear Volterra integral equation can be calculated. The same process is used to solve nonlinear Volterra integro-differential equations of the first and second kinds.
The efficacy of technique is demonstrated by convergence analysis and following numerical examples.

3. CONVERGENCE ANALYSIS

The convergence analysis is presented here which demonstrate the efficiency of the above-modified technique. Considering $E = (C[J], \|.\|)$ the Banach space of all continuous functions on J, suppose that there exist a constant N such that $|k(x,t)| \leq N$, for all $(x,t)\varepsilon[0,T]^2$.
Also, we suppose that the nonlinear term satisfy the Lipschitz condition, the approximate solution of (1) by using Bernstein polynomials based MLADM, converges to the exact one if $0 < \alpha < 1$, where $\alpha = NLx$.
Let U be the exact solution and U^* be the approximate solution of (1) by taking n terms, then

$$\begin{aligned} ||U - U^*|| &= \max_{x\varepsilon J} |f(x) + \int_0^x k(x,t)F(U(t))dt - B_m(f(x)) - \int_0^x k(x,t)F(U^*(t))dt| \\ &= |f(x) - B_m(f(x))| + \left| \int_0^x k(x,t)(F(U(t)) - F(U^*(t)))dt \right| \end{aligned} \tag{14}$$

Now using the convergence theorem of Bernstein polynomials (1.4) and above given conditions in the statement, we get

$$\begin{aligned}||U-U^*|| &\leq \varepsilon+\int_0^x |k(x,t)||F(U(t))-F(U^*(t))|dt \\ &\leq \varepsilon+\int_0^x NL|(U(t))-(U^*(t))|dt \\ &\leq \varepsilon+NLx\max_{x\varepsilon J}|U(t)-U^*(t)| \\ &\leq \alpha||U-U^*||\end{aligned} \tag{15}$$

Therefore, if $0<\alpha<1$, $\alpha=NLx$, the approximate solution converges to exact solution as $n\rightarrow\infty$.

4. NUMERICAL EXAMPLES

EXAMPLE 4.1. *Consider the following nonlinear Volterra integro-differential equation of the second kind [Wazwaz 2010]*

$$u'(x)=-2\sin x-\frac{1}{3}\cos x-\frac{2}{3}\cos 2x+\int_0^x \cos(x-t)u^2(t)dt, \tag{16}$$

with initial condition $u(0)=1$, *having the exact solution as* $u(x)=\cos x-\sin x$.

In this example, the source term, i.e. $f(x)=-2\sin x-\frac{1}{3}\cos x-\frac{2}{3}\cos 2x$. Now using the above technique, we expand $f(x)$ in the terms of Bernstein polynomials of order $m=6$

$$f(x)\approx 0.000507191x^6+0.010605381x^5-0.10640906x^4-0.06228815x^3+ \\ 1.314840965x^2-1.742867841x-1 \tag{17}$$

By applying Laplace transform to both sides of (16), we get

$$L[u(x)]=\frac{1}{s}+\frac{1}{s}L[f(x)]+\frac{1}{s^2+1}L[u^2(x)] \tag{18}$$

The methodology consisting of letting the solution as an infinite series as mentioned above, we have

$$L\left[\sum_{n=0}^{\infty}u_n(x)\right]=\frac{1}{s}+\frac{1}{s}L[f(x)]+\frac{1}{s^2+1}L\left[\sum_{n=0}^{\infty}A_n(x)\right] \tag{19}$$

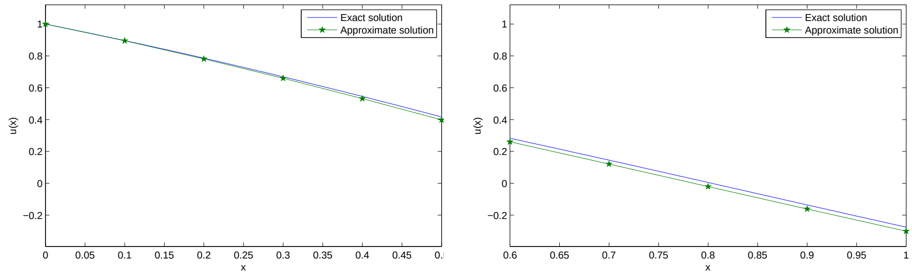

Fig. 1. Comparison of solutions in $[0,0.5]$ and $[0.6,1]$

where the nonlinear term $F(u(x)) = u^2(x)$ is decomposed in Adomian polynomials, few terms are as follows:

$$A_0 = u_0^2$$

$$A_1 = 2u_0u_1$$

$$A_2 = 2u_0u_2 + u_1^2$$

$$A_3 = 2u_0u_3 + 2u_1u_2$$

The recursive relation is obtained by comparing the terms in (19), which gives

$$L[u_0(x)] = \frac{1}{s} + \frac{1}{s}L[f(x)] \tag{20}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{s^2+1}L[A_n(x)] \tag{21}$$

Employing the inverse Laplace transform on both sides of (20) and using (17), we get the value of $u_0(x)$.

Similarly (21) gives the values of $u_1(x)$, $u_2(x)$ and so on. Subsequently, one can compare the results from Figure 1, which shows that the approximate solutions are very much close to exact in the interval $[0,0.5]$ than in the interval $[0.6,1]$.

EXAMPLE 4.2. *The following nonlinear Volterra integro-differential equation of the first kind [Wazwaz 2010]*

$$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = -\frac{15}{32} + \frac{3x^2}{4} + \frac{1}{2}\cos 2x - \frac{1}{32}\cos 4x, \quad (22)$$

with initial condition $u(0) = 2$, $u'(0) = 0$ *which has the exact solution as* $u(x) = 1 + \cos 2x$.

Apply Laplace transform to both sides of (22) and using the derivative property and convolution theorem, we get

$$L\left[\int_0^x (x-t)u^2(t)dt\right] + L\left[\int_0^x (x-t)u''(t)dt\right] = L[f(x)] \quad (23)$$

By solving, we get

$$\frac{1}{s^2}L[u^2(x)] + L[u(x)] - \frac{2}{s} = L[f(x)]$$

$$L[u(x)] = \frac{2}{s} + L[f(x)] - \frac{1}{s^2}L[u^2(x)]$$

where the nonlinear term $F(u) = u^2$ is decomposed as in the previous example
Now proceeding as before, following iterative scheme is obtained:

$$L[u_0(x)] = \frac{2}{s} + L[f(x)] \quad (24)$$

In general

$$L[u_{n+1}(x)] = -\frac{1}{s^2}L[A_n(x)] \quad (25)$$

Here $f(x) = \frac{-15}{32} + \frac{3x^2}{4} + \frac{1}{2}\cos 2x - \frac{1}{32}\cos 4x$ is the source term.
By adopting the above method, we expand $f(x)$ as the Bernstein polynomials:

$$f(x) \approx -0.001381627x^6 + 0.013467998x^5 + 0.051210391x^4 + 0.02773726x^3 + 0.002551943x^2 + 0.00001698x \quad (26)$$

Applying inverse Laplace transform on both sides of (24), (25) and using the Bernstein polynomials given by (26), we get the values of $u_0(x)$, $u_1(x)$, $u_2(x)\ldots$. Therefore, we find the approximate solution. The approximate solution provides the accurate result or close to the exact solution in very few iterations that is shown in Figure 2.

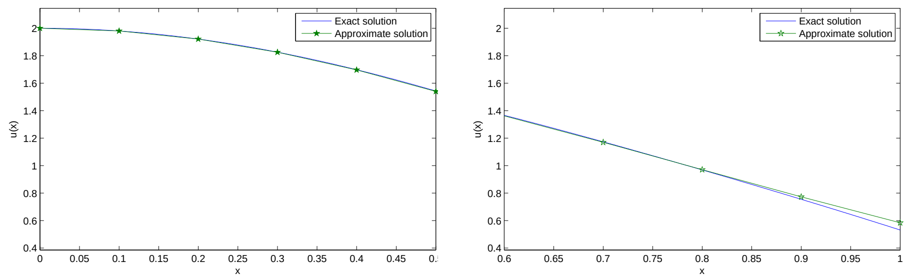

Fig. 2. Comparison of solutions in $[0,0.5]$ and $[0.6,1]$

EXAMPLE 4.3. *The nonlinear Volterra integral equation is given by*

$$u(x) = \frac{1}{4} + \frac{x}{2} + e^x - \frac{e^{2x}}{4} + \int_0^x (x-t)u^2(t)dt, \tag{27}$$

having the exact solution as $u(x) = e^x$.

The source term in (27) is $f(x) = \frac{1}{4} + \frac{x}{2} + e^x - \frac{e^{2x}}{4}$ which can be expanded in the Bernstein polynomials, here taking $m = 10$.

$$f(x) \approx -0.000000056x^{10} - 0.00000332x^9 - 0.00006387x^8 - 0.00076576x^7 - 0.00589954x^6$$
$$- 0.03027348x^5 - 0.1004593x^4 - 0.18599482x^3 - 0.05372434x^2 + 0.99820229x + 1 \tag{28}$$

Taking Laplace transform on both sides of (27), gives

$$L[u(x)] = L[f(x)] + \frac{1}{s^2}L[u^2(x)] \tag{29}$$

Now $u(x)$ can be evaluated based on Bernstein polynomials of $f(x)$ and with decomposing the nonlinear term in Adomian polynomials, which implies the relation

$$L[u_0(x)] = L[f(x)] \tag{30}$$

In general

$$L[u_{n+1}(x)] = \frac{1}{s^2}L[A_n(x)] \tag{31}$$

Substituting the approximated value of $f(x)$ from (28) in (30) and having inverse Laplace transform on both sides of (30), (31) give the values of $u_0(x)$, $u_1(x)$, $u_2(x)$,... $u_n(x)$. The sum of these terms will yield the value of truncated sum of $u(x)$. It is found that the error between exact and approximate solution is very less as shown in Figure 3 and reveals that the Bernstein polynomials based modification of LADM gives the solution in good agreement.

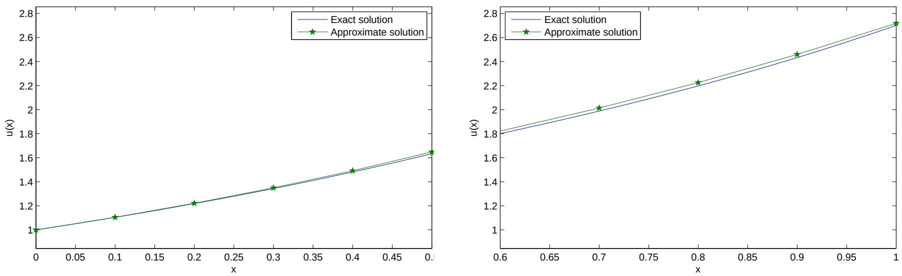

Fig. 3. Comparison of solutions in $[0,0.5]$ and $[0.6,1]$

Table I. Comparison of approximate solution by proposed method with exact solution of examples

	Example1		Example2		Example3	
x	Exact	Approximate	Exact	Approximate	Exact	Approximate
0	1	1	2	2	1	1
0.1	0.8952	0.8964	1.9801	1.9801	1.1052	1.1044
0.2	0.7814	0.7858	1.9211	1.9212	1.2214	1.2188
0.3	0.6598	0.6687	1.8253	1.8259	1.3499	1.3441
0.4	0.5316	0.5455	1.6967	1.6982	1.4918	1.4819
0.5	0.3982	0.4166	1.5403	1.5431	1.6487	1.6338
0.6	0.2607	0.2828	1.3624	1.3663	1.8221	1.8018
0.7	0.1206	0.1451	1.1700	1.1734	2.0138	1.9887
0.8	-0.0206	0.0049	0.9708	0.9687	2.2255	2.1978
0.9	-0.1617	-0.1361	0.7728	0.7550	2.4596	2.4335
1	-0.3012	-0.2758	0.5839	0.5314	2.7183	2.7014

The numerical results by using modified LADM based on Bernstein polynomials are also presented in Table I, which shows the performance of proposed technique.

5. CONCLUSIONS

For solving nonlinear Volterra integral and integro-differential equations a modification in standard Laplace Adomian decomposition method based on Bernstein polynomials is used here. Comparisons and analyses conclude that not only the orthogonal polynomials like Legendre, Chebyshev or Jacobi polynomials can improve the ADM, the Bernstein polynomials can also improve the source term as it is the better approximation to a function and hence the approximate solution converges to exact one as shown in the examples.

REFERENCES

ABASSY, T. A. 2010. Improved Adomian decomposition method. *Computers and Mathematics with Applications 59*, 42–54.

ADOMIAN, G. 1988. A review of the decomposition method in applied mathematics. *Journal of Mathematical Analysis and Applications 135*, 501–544.

ADOMIAN, G. 1990. A review of the decomposition method and some recent results for nonlinear equations. *Mathematics Computer and Modelling 13*, 17–43.

ADOMIAN, G. AND RACH, R. 1996. Modified Adomian polynomials. *Mathematics and Computer Modelling 24*, 39–46.

BABOLIAN, E. AND BIAZAR, J. 2002. Solution of nonlinear equations by modified Adomian decomposition method. *Applied Mathematics and Computation 132*, 167–172.

BAHUGUNA, D., UJLAYAN, A., AND PANDEY, D. N. 2009. A comparative study of numerical methods for solving an integro-differential equation. *Computers and Mathematics with Applications 57*, 1485–1493.

BHATTI, M.I. AND BRACKEN, P. 2007. Solutions of differential equations in a Bernstein polynomial basis. *Journal of Computational and Applied Mathematics 205*, 272–280.

BIAZAR, J., BABOLIAN, E., AND ISLAM, R. 2004. Solution of the system of ordinary differential equations by Adomian decomposition method. *Applied Mathematics and Computation 147*, 713–719.

BIAZAR, J., PORSHOKUHI, M. G., AND GHANBARI, B. 2010. Extracting a general iterative method from an Adomian decomposition method and comparing it to the variational iteration method. *Computers and Mathematics with Applications 59*, 622–628.

BILDIK, N. AND DENIZ, S. 2015. Modified Adomian decomposition method for solving Riccati differential equations. *Review of the Air Force Academy 30*, 21–26.

BOHM, W., FARIN, G., AND KAHMANN, J. 1984. A survey of curve and surface methods in cagd. *Computer Aided Geometric Design 1*, 1–60.

DOAN, N. 2012. Solution of the system of ordinary differential equations by combined Laplace transform Adomian decomposition method. *Mathematical and Computational Applications 17*, 203–211.

EVANS, D.J. AND RASLAN, K. R. 2004. The Adomian decomposition method for solving delay differential equation. *International Journal of Computer Mathematics 00*, 1–6.

FAROUKI, R. T. 2012. The Bernstein polynomial basis: A centennial retrospective. *Computer Aided Geometric Design 29*, 379–419.

FAROUKI, R. T. AND RAJAN, V. T. 1998. Algorithms for polynomials in Bernstein form. *Computer Aided Geometric Design 5*, 1–26.

GABET, L. 1994. The theoretical foundation of the Adomian method. *Computers and Mathematics with Applications 27*, 41–52.

GHAZANFARI, B. AND SEPAHVANDZADEH, A. 2014. Adomian decomposition method for solving Bratu's type equation. *Journal of Mathematics and Computer Science 8*, 236–244.

HASHIM, I. 2006. Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Journal of Computational and Applied Mathematics 193*, 658–664.

HOSSEINI, M. M. 2006. Adomian decomposition method for solution of nonlinear differential algebraic equations. *Applied Mathematics and Computation 181*, 1737–1744.

KHURI, S. A. 2001. A Laplace decomposition algorithm applied to a class of nonlinear differential equation. *Journal of Applied Mathematics 1*, 141–155.

KUMAR, S., KUMAR, D., ABBASBANDY, S., AND RASHIDI, M. M. 2014. Analytic solution of fractional Navier-Stokes equation by using modified Laplace decomposition method. *Ain Shams Engineering Journal 5*, 569–574.

LI, C. AND WANG, Y. 2009. Numerical algorithm based on Adomian decomposition for fractional differential equations. *Computers and Mathematics with Applications 57*, 1672–1681.

LIU, Y. 2009. Adomian decomposition method with orthogonal polynomials:Legendre polynomials. *Mathematical and Computer Modelling 49*, 1268–1273.

LUO, X.-G., WU, Q.-B., AND ZHANG, B.-Q. 2006. Revisit on partial solutions in the Adomian decomposition method: solving heat and wave equations. *Journal of Mathematical Analyis and Applications 321*, 353–363.

MALEKNEJAD, K., HASHEMIZADEH, E., AND EZZATI, R. 2011. A new approach to the numerical solution of Volterra integral equations by using Bernsteins approximation. *Communication in Nonlinear Science and Numerical Simulation 16*, 647–655.

MALEKNEJAD, K. AND NAJAFI, E. 2011. Numerical solution of nonlinear Volterra integral equations using the idea of quasilinearization. *Communication in Nonlinear Science and Numerical Simulation 20*, 93–100.

MANAFIANHERIS, J. 2012. Solving the integro-differential equations using the modified Laplace Adomian decomposition method. *Journal of Mathematical Extension 6*, 41–55.

MARWAT, D. N. K. AND ASGHAR, S. 2008. Solution of the heat equation with variable properties by two-step Adomian decomposition method. *Mathematical and Computer Modelling 48*, 83–90.

QUAIN, W., RIEDEL, M. D., AND ROSENBERG, I. 2011. Uniform approximation and Bernstein polynomial with coefficients in the unit interval. *European Journal of Combinatorics 32*, 448–463.

SINGH, N. AND KUMAR, M. 2011. Adomian decomposition method for solving higher order boundary value problems. *Mathematical Theory and Modeling 2*, 11–22.

SOMALI, S. AND GOKMEN, G. 2007. Adomian decomposition method for nonlinear Sturm-Liouville problems. *Surveys in Mathematics and its Applications 2*, 11–20.

VENKATARANGAN, S. N. AND RAJALAKSHMI, K. 1995. Modification of Adomian's decomposition method to solve equations containing radicals. *Computers and Mathematics with Applications 29*, 75–80.

WAZWAZ, A. M. 2010. The combined Laplace transform-Adomian demcomposition method for handling nonlinear Volterra integro-differential equations. *Applied Mathematics and Computation 216*, 1304–1309.

WAZWAZ, A. M. 2011. *Linear and Nonlinear Integral Equations: Methods and Applications*. Springer.

YOUSEFI, S. A. AND BEHROOZIFAR, M. 2010. Operational matrices of Bernstein polynomials and their applications. *International Journal of System Science 41*, 709–716.

ZHANG, B. AND LU, J. 2011. Exact solutions of homogenous partial differential equations by a new Adomian decomposition method. *Procedia Environmental Science 11*, 440–446.

RANI, D. AND MISHRA, V. 2017. Approximate solution of boundary value problem with Bernstein polynomial Laplace decomposition method. *International Journal of Pure and Applied Mathematics 114*, 823–833.

Dimple Rani
Department of Mathematics,
Sant Longowal Institute of Engineering and Technology,
Longowal-148106 (Punjab), India
email: chawla23dimple@gmail.com

Vinod Mishra
Department of Mathematics,
Sant Longowal Institute of Engineering and Technology,
Longowal-148106 (Punjab), India
email: vinodmishra.2011@rediffmail.com

Fractional Hermite-Hadamard type inequalities for co-ordinated prequasiinvex functions

B. MEFTAH AND A. SOUAHI

Abstract

Some new Ostrowski's inequalities for functions whose n-th derivatives are h-convex are established.

Mathematics Subject Classification 2010: 26A51
Keywords: In this paper, the concept of co-ordinated prequasiinvex is introduced, some fractional Hermite-Hadamard type inequalities for functions whose modulus of the mixed derivatives lies in this novel class of functions are established.

1. INTRODUCTION

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality, which can be stated as follows: for every convex function f on the finite interval $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

If the function f is concave, then (1) holds in the reverse direction (see [Pečarić et al. 1992]).

In recent years, lot of efforts have been made by mathematicians and researchers to generalize the classical convexity. Hanson [Hanson 1981], introduced a new class of generalized convex functions, called invex functions, In [Ben-Israel and Mond 1986] the authors gave the concept of preinvex function which is special case of invexity. Pini [Pini 1991] introduced the concept of prequasiinvexity which generalize that of preinvex function, Noor [Noor 1994; 2005], Yang and Li [Yang and Li 2001] and Weir [Weir and Mond 1988], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems.

Dragomir [Dragomir 2001] introduced the concept of the convexity on the co-ordinates as follows:

10.2478/jamsi-2019-0002

A function $f:\Delta\rightarrow\mathbb{R}$ is said to be convex on the co-ordinates on Δ, where $\Delta:=[a,b]\times[c,d]$ is a bidimensional interval in $\mathbb{R}^2$ with $a<b$ and $c<d$, if

$$\begin{aligned} f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right) &\leq t\lambda f(x,y)+t(1-\lambda)f(x,v) \\ &\quad +(1-t)\lambda f(u,y)+(1-t)(1-\lambda)f(u,v) \end{aligned}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

Also, he proved the two-dimensional analog of (1), which can be stated as follows: For all co-ordinated convex function f on $[a,b]\times[c,d]$, we have

$$\begin{aligned} & f\left(\tfrac{a+b}{2},\tfrac{c+d}{2}\right) \\ \leq\ & \tfrac{1}{2}\left(\tfrac{1}{b-a}\int_a^b f(x,\tfrac{c+d}{2})dx+\tfrac{1}{d-c}\int_c^d f(\tfrac{a+b}{2},y)dy\right) \\ \leq\ & \tfrac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ \leq\ & \tfrac{1}{4}\left(\tfrac{1}{b-a}\left(\int_a^b f(x,c)dx+\int_a^b f(x,d)dx\right)+\tfrac{1}{d-c}\left(\int_c^d f(a,y)dy+\int_c^d f(b,y)dy\right)\right) \\ \leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{2}$$

Özdemir et al. [Özdemir et al. 2012] introduced the concept of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions as follows:

A function $f:\Delta\rightarrow\mathbb{R}$ is said to be co-ordinated quasi-convex on Δ, if

$$f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right)\leq\max\left\{f(x,y),f(u,v)\right\}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

A formal definition of co-ordinated quasi-convex functions is

$$f\left(tx+(1-t)u,\lambda y+(1-\lambda)v\right)\leq\max\left\{f(x,y),f(x,v),f(u,y),f(u,v)\right\}$$

for all $t,\lambda\in[0,1]$ and $(x,y),(u,v)\in\Delta$.

In [Özdemir et al. 2012] Özdemir et al. established the following Hermite-Hadamard's inequalities for differentiable co-ordinated quasi-convex functions

THEOREM 1.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\max\left\{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,b)\right|, \left|\frac{\partial^2 f}{\partial\lambda\partial t}(c,d)\right|\right\},$$

where

$$A = \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy. \tag{3}$$

THEOREM 1.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$, $q > 1$, is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial\lambda\partial t}(c,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is as defined by (3), *and* $\frac{1}{p}+\frac{1}{q}=1$.

THEOREM 1.3. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$, $q \geq 1$, is quasi-convex on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\max\left\{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,b)\right|^q, \left|\frac{\partial^2 f}{\partial\lambda\partial t}(c,d)\right|^q\right\}\right)^{\frac{1}{q}},$$

where A is as defined by (3).

In this paper we first introduce the concept of co-ordinated prequasiinvex, and then we derive some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives lies in this new class of functions.

2. PRELIMINARIES

In this section we recall some concepts of generalized convexity and fractional calculus

DEFINITION 2.1. *[Matł oka 2013] Let K_1, K_2 be nonempty subsets of $\mathbb{R}^n$, $(u,v) \in K_1 \times K_2$. We say $K_1 \times K_2$ is invex at (u,v) with respect to η_1 and η_2, if*

$$(u + t\eta_1(x,u), v + s\eta_2(y,v)) \in K_1 \times K_2$$

holds for each $(x,y) \in K_1 \times K_2$ and $t, s \in [0,1]$.

$K_1 \times K_2$ is said to be an invex set with respect to η_1 and η_2 if $K_1 \times K_2$ is invex at each $(u,v) \in K_1 \times K_2$.

In what follows we assume that $K_1 \times K_2$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$.

DEFINITION 2.2. *[Latif and Dragomir 2013] A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex on the co-ordinates, if the following inequality*

$$\begin{aligned} f(u + \lambda\eta_1(x,u), v + t\eta_2(y,v)) \leq\ & (1-\lambda)(1-t)f(u,v) + (1-\lambda)tf(u,y) \\ & + (1-t)\lambda f(x,v) + \lambda t f(x,y) \end{aligned}$$

holds for all $t, \lambda \in [0,1]$ and $(x,y), (x,v), (u,y), (u,v) \in K_1 \times K_2$.

DEFINITION 2.3. *[Kilbas et al. 2006] Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\begin{aligned} J_{a^+}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \\ J_{b^-}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x \end{aligned}$$

respectively. Where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$, is the Gamma function and $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

DEFINITION 2.4. *[Latif and Dragomir 2013] Let $f \in L([a,b] \times [c,d])$. The Riemann–Liouville integrals $J_{a^+,c^+}^{\alpha,\beta}$, $J_{a^+,d^-}^{\alpha,\beta}$, $J_{b^-,c^+}^{\alpha,\beta}$, and $J_{b^-,d^-}^{\alpha,\beta}$ of order $\alpha, \beta > 0$ with*

$a,c \geq 0$, $a < b$ and $< d$ are defined by

$$J_{a^+,c^+}^{\alpha,\beta} f(b,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dydx, \tag{4}$$

$$J_{a^+,d^-}^{\alpha,\beta} f(b,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dydx, \tag{5}$$

$$J_{b^-,c^+}^{\alpha,\beta} f(a,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (d-y)^{\beta-1} f(x,y)\, dydx, \tag{6}$$

and

$$J_{b^-,d^-}^{\alpha,\beta} f(a,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^b \int_c^d (x-a)^{\alpha-1} (y-c)^{\beta-1} f(x,y)\, dydx, \tag{7}$$

where Γ is the Gamma function, and

$$J_{a^+,c^+}^{0,0} f(b,d) = J_{a^+,d^-}^{0,0} f(b,c) = J_{b^-,c^+}^{0,0} f(a,d) = J_{b^-,d^-}^{0,0} f(a,c) = f(x,y).$$

DEFINITION 2.5. *[Sarı kaya 2014] Let $f \in L([a,b] \times [c,d])$. The Riemann–Liouville integrals $J_{b^-}^{\alpha} f(a,c)$, $J_{a^+}^{\alpha} f(b,c)$, $J_{d^-}^{\beta} f(a,c)$, and $J_{c^+}^{\alpha} f(a,d)$ of order $\alpha, \beta > 0$ with $a,c \geq 0$, $a < b$, and $< d$ are defined by*

$$J_{b^-}^{\alpha} f(a,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (x-a)^{\alpha-1} f(x,c)\, dx, \tag{8}$$

$$J_{a^+}^{\alpha} f(b,c) = \frac{1}{\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1} f(x,c)\, dx, \tag{9}$$

$$J_{d^-}^{\beta} f(a,c) = \frac{1}{\Gamma(\beta)} \int_c^d (y-c)^{\beta-1} f(a,y)\, dy, \tag{10}$$

and

$$J_{c^+}^{\alpha} f(a,d) = \frac{1}{\Gamma(\beta)} \int_c^d (d-y)^{\beta-1} f(a,y)\, dy, \tag{11}$$

where Γ *is the Gamma function.*

LEMMA 2.6. *[Meftah 2019] Let* $f : K \to \mathbb{R}$ *be a partially differentiable function on K, if* $\frac{\partial^2 f}{\partial t \partial s} \in L(K)$*, then the following equality holds*

$$\begin{aligned}
&\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \\
&+\tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}} \Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) \Big) \\
=\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4} \int_0^1\int_0^1 \left(t^{\alpha}-(1-t)^{\alpha}\right)\left(s^{\beta}-(1-s)^{\beta}\right) \\
&\times \tfrac{\partial^2 f}{\partial t \partial s}(a+t\eta_1(b,a),c+s\eta_2(d,c))\, ds dt, \qquad (12)
\end{aligned}$$

where

$$\begin{aligned}
A =\ & \tfrac{\Gamma(\alpha+1)}{4(\eta_1(b,a))^{\alpha}} \Big(J^{\alpha}_{(a+\eta_1(b,a))^-} f(a,c+\eta_2(d,c)) + J^{\alpha}_{(a+\eta_1(b,a))^-} f(a,c) \\
&+ J^{\alpha}_{a^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) + J^{\alpha}_{a^+} f(a+\eta_1(b,a),c) \Big) \\
&+ \tfrac{\Gamma(\beta+1)}{4(\eta_2(d,c))^{\beta}} \Big(J^{\beta}_{(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\beta}_{(c+\eta_2(d,c))^-} f(a,c) \\
&+ J^{\alpha}_{c^+} f(a+\eta_1(b,a),c+\eta_2(d,c)) + J^{\alpha}_{c^+} f(a,c+\eta_2(d,c)) \Big). \qquad (13)
\end{aligned}$$

3. MAIN RESULTS

In what follows we assume that $K = [a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]$ be an invex subset of $\mathbb{R}^2$ with respect to η_1, η_2 where $\eta_1, \eta_2 : \mathbb{R}^2 \to \mathbb{R}$ are two bifunctions such that $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$.

We will start with the following definitions, and the lemma

DEFINITION 3.1. *A function* $f : K \to \mathbb{R}$ *is said to be prequasiinvex on the co-ordinates, if the following inequality*

$$f(u+\lambda\eta_1(x,u), v+t\eta_2(y,v)) \leq \max\{f(u,v), f(u+\eta_1(x,u), v+\eta_2(y,v))\}$$

holds for all $t, \lambda \in [0,1]$ *and* $(u,v),(x,y) \in K$.

A formal definition of co-ordinated prequasiinvex functions is given by the following definition

DEFINITION 3.2. *A function $f : K \to \mathbb{R}$ is said to be prequasiinvex on the co-ordinates, if the following inequality*

$$
\begin{aligned}
& f\left(u+\lambda\eta_1(x,u), v+t\eta_2(y,v)\right) \\
\le\ & \max\left\{f(u,v), f(u,v+\eta_2(y,v)), f(u+\eta_1(x,u),v), f(u+\eta_1(x,u),v+\eta_2(y,v))\right\}
\end{aligned}
$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(x,v),(u,y),(u,v) \in K$.

THEOREM 3.3. *Let $f : K \to \mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, then the following fractional inequality holds*

$$
\begin{aligned}
& \Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \\
& + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
& + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\
& + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Bigg| \\
\le\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)(\beta+1)} \max\left\{\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|, \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|,\right. \\
& \left.\left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|, \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\right\},
\end{aligned}
$$

where A is defined as in (13).

PROOF. From Lemma 2.6, and properties of modulus we have

$$
\begin{aligned}
& \Bigg|\tfrac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A \\
& + \tfrac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
& + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c)) \\
& + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Bigg| \\
\le\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 \left|t^\alpha-(1-t)^\alpha\right|\left|\lambda^\beta-(1-\lambda)^\beta\right| \\
& \times \left|\tfrac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right| d\lambda dt \\
\le\ & \tfrac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 \left(t^\alpha+(1-t)^\alpha\right)\left(\lambda^\beta+(1-\lambda)^\beta\right)
\end{aligned}
$$

$$\times\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1\left(b,a\right),c+\lambda\eta_2\left(d,c\right)\right)\right|d\lambda dt. \tag{14}$$

Using prequasiinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|$, (14) gives

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f\left(a,c\right)\right.\\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f\left(a+\eta_1\left(b,a\right),c\right)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f\left(a,c+\eta_2\left(d,c\right)\right)\\
&\left.\left.+\ J^{\alpha,\beta}_{a^+,c^+}f\left(a+\eta_1\left(b,a\right),c+\eta_2\left(d,c\right)\right)\right)\right|\\
\leq\ &\frac{\eta_1(b,a)\eta_2(d,c)}{4}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c+\eta_2\left(d,c\right)\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+\eta_1\left(b,a\right),c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+\eta_1\left(b,a\right),c+\eta_2\left(d,c\right)\right)\right|\right\}\\
&\times\left(\int_0^1\int_0^1\left(t^{\alpha}+(1-t)^{\alpha}\right)\left(\lambda^{\beta}+(1-\lambda)^{\beta}\right)d\lambda dt\right)\\
=\ &\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha+1)(\beta+1)}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c+\eta_2\left(d,c\right)\right)\right|,\right.\\
&\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+\eta_1\left(b,a\right),c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+\eta_1\left(b,a\right),c+\eta_2\left(d,c\right)\right)\right|\right\}.
\end{aligned}$$

The proof is achieved.

COROLLARY 3.4. *In Theorem 3.3 if we choose $\eta_1\left(b,a\right)=\eta_2\left(b,a\right)=b-a$, we obtain the following fractional inequality*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\right.\\
&\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f\left(a,c\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(b,c\right)+J^{\alpha,\beta}_{b^-,c^+}f\left(a,d\right)+J^{\alpha,\beta}_{a^+,c^+}f\left(b,d\right)\right)\right|\\
\leq\ &\frac{(b-a)(d-c)}{(\alpha+1)(\beta+1)}\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a,d\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,c\right)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(b,d\right)\right|\right\}.
\end{aligned}$$

THEOREM 3.5. *Let $f:K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, where $q>1$ with and $\frac{1}{p}+\frac{1}{q}=1$, then the following fractional inequality holds*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.\\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f\left(a,c\right)\right.\\
&+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f\left(a+\eta_1\left(b,a\right),c\right)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f\left(a,c+\eta_2\left(d,c\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ J^{\alpha,\beta}_{a^{+},c^{+}} f\left(a+\eta_1(b,a), c+\eta_2(d,c)\right)\Big)\Big| \\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right. \\
&\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|^q, \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|^q\right\}\right)^{\frac{1}{q}},
\end{aligned}
$$

where A is defined as in (13)*.*

PROOF. From Lemma 2.6, properties of modulus, and Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A\right. \\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^{-},(c+\eta_2(d,c))^{-}} f(a,c) \\
&+J^{\alpha,\beta}_{a^{+},(c+\eta_2(d,c))^{-}} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^{-},c^{+}} f(a,c+\eta_2(d,c)) \\
&\left.+ J^{\alpha,\beta}_{a^{+},c^{+}} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\right| \\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p}\,d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p}\,d\lambda dt\right)^{\frac{1}{p}}\right. \\
&\left.+\left(\int_0^1\int_0^1 (1-t)^{p\alpha}\lambda^{p\beta}\,d\lambda dt\right)^{\frac{1}{p}} + \left(\int_0^1\int_0^1 (1-t)^{p\alpha}(1-\lambda)^{p\beta}\,d\lambda dt\right)^{\frac{1}{p}}\right) \\
&\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}} \\
&= \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function, we deduce

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4} - A\right. \\
&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^{-},(c+\eta_2(d,c))^{-}} f(a,c) \\
&+J^{\alpha,\beta}_{a^{+},(c+\eta_2(d,c))^{-}} f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^{-},c^{+}} f(a,c+\eta_2(d,c)) \\
&\left.+ J^{\alpha,\beta}_{a^{+},c^{+}} f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\right|
\end{aligned}
$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|^q\right\}\right)^{\frac{1}{q}},$$

which is the desired result.

COROLLARY 3.6. *In Theorem 3.5 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\right.$$
$$\left.\times\left(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\right)\right|$$
$$\leq \frac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}$$
$$\times\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}.$$

THEOREM 3.7. *Let $f: K\to\mathbb{R}$ be a partially differentiable function on K. If $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex function on K with respect to η_1 and η_2, and $p>1$ then the following inequality holds*

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$
$$\left.\left.+\,J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\right)\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{(1+\alpha)(1+\beta)}\left(\max\left\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,\right.\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\right\}\right)^{\frac{1}{q}},$$

where A is defined as in (13).

PROOF. From Lemma 2.6, properties of modulus, and power mean inequality, we have

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A\right.$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\left(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\right.$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$

$$+ J_{a^+,c^+}^{\alpha,\beta} f\left(a+\eta_1(b,a), c+\eta_2(d,c)\right)\Big)\Big|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta d\lambda dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta d\lambda dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta d\lambda dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^\alpha(1-\lambda)^\beta d\lambda dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1\int_0^1 (1-t)^\alpha(1-\lambda)^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right)$$

$$= \frac{\eta_1(b,a)\eta_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left(\left(\int_0^1\int_0^1 t^\alpha\lambda^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\int_0^1 t^\alpha(1-\lambda)^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^\alpha\lambda^\beta\left|\frac{\partial^2 f}{\partial t\partial\lambda}\left(a+t\eta_1(b,a), c+\lambda\eta_2(d,c)\right)\right|^q d\lambda dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q d\lambda dt\right)^{\frac{1}{q}}\Bigg).$$

Since $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is co-ordinated prequasiinvex, we get

$$\Bigg|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}-A$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\Big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$
$$+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\Big)\Bigg|$$
$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4(1+\alpha)^{1-\frac{1}{q}}(1+\beta)^{1-\frac{1}{q}}}\left(\left(\int_0^1\int_0^1 t^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 t^{\alpha}(1-\lambda)^{\beta}d\lambda dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\int_0^1 (1-t)^{\alpha}\lambda^{\beta}d\lambda dt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1 (1-t)^{\alpha}(1-\lambda)^{\beta}d\lambda dt\right)^{\frac{1}{q}}\right)$$
$$\times\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,$$
$$\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\Bigg\}\Bigg)^{\frac{1}{q}}$$
$$=\frac{\eta_1(b,a)\eta_2(d,c)}{(1+\alpha)(1+\beta)}\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c+\eta_2(d,c))\right|^q,$$
$$\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c)\right|,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\eta_1(b,a),c+\eta_2(d,c))\right|\Bigg\}\Bigg)^{\frac{1}{q}},$$

which is the desired result.

COROLLARY 3.8. *In Theorem 3.7 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality*

$$\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}$$
$$\times\Big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\Big)\Bigg|$$
$$\leq\frac{(b-a)(d-c)}{(1+\alpha)(1+\beta)}$$
$$\times\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}.$$

REFERENCES

BEN-ISRAEL, A. AND MOND, B. 1986. What is invexity? *J. Austral. Math. Soc. Ser. B 28,* 1, 1–9.

DRAGOMIR, S. S. 2001. On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. *Taiwanese J. Math. 5,* 4, 775–788.

HANSON, M. A. 1981. On sufficiency of the Kuhn-Tucker conditions. *J. Math. Anal. Appl. 80,* 2, 545–550.

KILBAS, A. A., SRIVASTAVA, H. M., AND TRUJILLO, J. J. 2006. *Theory and applications of fractional differential equations*. North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V., Amsterdam.

LATIF, M. A. AND DRAGOMIR, S. S. 2013. Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absolute value are preinvex on the co-ordinates. *Facta Univ. Ser. Math. Inform. 28,* 3, 257–270.

MATŁ OKA, M. 2013. On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates. *J. Inequal. Appl.*, 2013:227, 12.

MEFTAH, B. 2019. Fractional hermite-hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated log-preinvex. *Punjab Univ. J. Math. (Lahore) 51,* 2.

NOOR, M. A. 1994. Variational-like inequalities. *Optimization 30,* 4, 323–330.

NOOR, M. A. 2005. Invex equilibrium problems. *J. Math. Anal. Appl. 302,* 2, 463–475.

ÖZDEMIR, M. E., AKDEMIR, A. O., AND YI LDIZ, C. 2012. On co-ordinated quasi-convex functions. *Czechoslovak Math. J. 62(137),* 4, 889–900.

ÖZDEMIR, M. E., YI LDIZ, C., AND AKDEMIR, A. O. 2012. On some new Hadamard-type inequalities for co-ordinated quasi-convex functions. *Hacet. J. Math. Stat. 41,* 5, 697–707.

PEČARIĆ, J. E., PROSCHAN, F., AND TONG, Y. L. 1992. *Convex functions, partial orderings, and statistical applications*. Mathematics in Science and Engineering, vol. 187. Academic Press, Inc., Boston, MA.

PINI, R. 1991. Invexity and generalized convexity. *Optimization 22,* 4, 513–525.

SARI KAYA, M. Z. 2014. On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals. *Integral Transforms Spec. Funct. 25,* 2, 134–147.

WEIR, T. AND MOND, B. 1988. Pre-invex functions in multiple objective optimization. *J. Math. Anal. Appl. 136,* 1, 29–38.

YANG, X. M. AND LI, D. 2001. On properties of preinvex functions. *J. Math. Anal. Appl. 256,* 1, 229–241.

Badreddine Meftah
Laboratoire des télécommunications,
Faculté des Sciences et de la Technologie,
University of 8 May 1945 Guelma,
P.O. Box 401, 24000 Guelma, Algeria
email: badrimeftah@yahoo.fr

Abdourazek Souahi
Laboratory of Advanced Materials,
University of Badji Mokhtar-Annaba,
P.O. Box 12, 23000 Annaba, Algeria
email: arsouahi@yahoo.fr

A remark on eigen values of signed graph

B.PRASHANTH, K. NAGENDRA NAIK AND K. R. RAJANNA

Abstract

In this paper we present some results using eigen values of signed graph. This precipitate to find the determinant of signed graph using the number of vertices.

Mathematics Subject Classification 2010: 05C22, 05C50, 15A18
Keywords: signed graphs, Marked graphs, *balance*, Switching, Antipodal graph.

1. INTRODUCTION

For all expressions and notation in graph theory the reader has to refer [4]. We consider only finite, simple graphs without self-loops.

A signed graph $\Gamma = (G, \sigma)$ is a graph whose edges are assigned by positive sign or negative sign. Where $G = (V, E)$ is called basic graph (underlined graph) of Γ and σ is a bijective mapping between the set of edges to the set of signs (positive and negative sign).

Cartwright and Harary [5] deliberates that vertices and edges of a sign graph represent persons and their relationship respectively, each of it fixes itself as positive or negative based on its nature. If two persons are friendly to each other then their relationship is positive. In a similar manner if two persons dislike each other, their relationship is negative. This signed graph network was discussed by Chartand [6]; Harary et. al. [7].

Katai and Iwai [8], Roberts [9] and Roberts and Xu [10] presented applications of signed graphs in literature, because of their extended use in modeling a multiple socio-psychological process and also because of their connections with many classical mathematical systems [3].

10.2478/jamsi-2019-0003

The notation of *balanced* signed graph introduced by Hary and he has given characterization, asserting that, a signed graph is said to be *balanced* if and only if each cycle contains even number of negative edges.

A positive cycle in a signed graph is the product of signs of edges in a cycle is positive. If every cycle in a signed graph is positive then that signed graph is called *balanced* signed graph (see Harary [11]) otherwise it is called *unbalanced* signed graph. A *balance* of a signed graph can be detected by simple algorithm, which is developed by Harary and Kabell [16]

A marked graph of a signed graph Γ is denoted as Γ_μ whose vertices are assigned by sign $+$ or $-$, where μ is the canonical marking, therefore, Γ_μ can be defined as follows: For any vertex $v \in V(\Gamma)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

In a signed graph $\Gamma = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

Switching of signed graph Γ is an operation with the help of marking μ, in which sign of each edge of Γ is changed to opposite sign, whenever sign of two end vertices of edge are opposite and such a signed graph is known as *switching signed graph* denoted as $\Gamma_\mu(\Gamma)$ and it is called Γ_μ -switched signed graph or *switched signed graph*.

R.P Abelson and Rosenberg introduced switching Sign graph for social behavioral analysis in [1] and its significance and mathematical connections may be found in [3].

If two underlying graphs G_1 and G_2 are *isomorphic* ($f : G_1 \to G_2$) then there signed graphs $\Gamma_1 = (G_1, \sigma)$ and $\Gamma_2 = (G_2, \sigma')$ are also *isomorphic*. For the underlying graphs G_1 and G_2, any edge l belongs to G_1, $\sigma(l) = \sigma'(f(l))$. Therefore Γ_1 *and* Γ_2 are *switching equivalent* which is represent as $\Gamma_1 \sim \Gamma_2$. Precisely for any marking μ , $\Gamma_\mu(\Gamma_1) \sim \Gamma_2$. But note that their underlying graphs G_1 and G_2 does not involve in any change in their adjacency.

If two signed graphs are cycle *isomorphic* then there corresponding cycles are having same sign.

T. Zaslavasky has given characterization of switching Singed graph through the following proposition.

PROPOSITION 1. *[2] Two signed graphs* Γ_1 *and* Γ_2 *with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

The difference between positive edges and negative edges incident to a vertex v in a signed graph Γ is known as sdeg(v) that is $d_v^+ - d_v^-$ = sdeg(v), therefore, degree of a vertex in a signed graph is defined as $d = d_v^+ - d_v^-$ wherefore signed graph and there underlying graph has the same degree.

2. DEFINITIONS

2.1. *Adjacency matrix of a Signed graph*

The adjacency matrix of Γ is the $n \times n$ matrix $A = A(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = + \\ -1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = - \\ 0, \text{ otherwise.} \end{cases}$$

2.2. *Laplacian matrix of a Signed graph*

The *Laplacian* matrix of signed graph Γ is the $n \times n$ matrix $L(\Gamma) = D(\Gamma) - A(\Gamma)$ whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = - \\ -1, \text{ if } v_i \text{ and } v_j \text{ are adjacent and } \sigma(v_i v_j) = + \\ d(v_i) \text{ if } \mathrm{i = j} \\ 0, \text{ otherwise.} \end{cases}$$

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \,....\geq \lambda_n$ are the eigenvalues of *Laplacian* matrix of signed graph $\Gamma = (G, \sigma)$, having n vertices.

LEMMA 2. *[17] Two signed graphs* $\Gamma_1 = (G, \sigma_1)$ *and* $\Gamma_2 = (G, \sigma_2)$ *with their same underlying graphs are switching equivalent if and only if* $L(\Gamma_1)$ *and* $L(\Gamma_2)$ *are has the same signature.*

2.3. Antipodal signed graphs

The concept of *Antipodal graph* introduced by R.R. Singleton in 1968 defined it as *Antipodal graph* $\Theta(G)$ is a graph which has the same vertices of a graph G with the distance between the adjacent vertices are of diameter of G.

P.S.K Reddy et.al. devoloped the concept of Antipodal signed graph and gave the following characterization.

PROPOSITION 3. *[13] For any signed graph* $\Gamma = (G, \sigma)$*,* $\Gamma \sim \Theta(\Gamma)$ *if, and only if,* $G = K_p$ *and* Γ *is balanced signed graph, where* K_p *is a complete graph with p vertices.*

The *Laplacian* matrices $L(\Gamma, +)$ and $L(\Gamma, -)$ are all positive and all negative labeling respectively, and also $L(\Gamma, +)$ is the sign less *Laplacian* matrix of Γ which is the sum of the *diagonal matrix* and the *adjacency*$(L(\Gamma) = D(\Gamma) + A(\Gamma))$.

Yaoping Hou et. al. has given the new bounds of eigen values of signed graph by the following theorem:

THEOREM 4. *[14] Let* $\Gamma = (G, \sigma)$ *be a signed graph with n vertices. Then*

$$\lambda_1 \leq 2(n-1),$$

equality holds if and only if $\Gamma \sim (K_n, -)$*, where* K_n *is the complete graph with n vertices.*

By the motivation of the above theorem we present some new results in this article.

PROPOSITION 5. *For any signed graph* Γ*,*

$$\sum_{i=1}^{n} \lambda_i = n(n-1) \; if \; \Gamma \sim (K_n, -)$$

PROOF. Since $\Gamma \sim (K_n, -)$, From Theorem 4 we have, $\lambda_1 = 2(n-1)$

therefore,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i &= \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-1) + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-1) + (n-1)(n-2) \\ &= (n-1)(2+n-2) \\ &= n(n-1)\end{aligned}$$

COROLLARY 6. *if* $\Gamma \sim (\Theta(\Gamma), -)$, *then* $\sum_{i=1}^{n} \lambda_i = n(n-1)$

PROOF. Since, *Antipodal graph* $\Theta(G) \cong K_n$ so, proof is same as Proposition 5.

LEMMA 7. *[15] If Q is the* Incident matrix *of a connected signed graph* $\Gamma = (G, \sigma)$ *then*

$$rank(Q) = \begin{cases} n-1, & if\ \ \Gamma\ \ is\ \ balanced \\ n, & if\ \ \Gamma\ \ is\ \ unbalanced. \end{cases}$$

from the above we can prove that.

LEMMA 8. *If $L(\Gamma)$ is the* Laplacian *matrix of a connected signed graph* $\Gamma = (G, \sigma)$ *then*

$$rank(L(\Gamma)) = \begin{cases} n-1, & if\ \ \Gamma\ \ is\ \ balanced \\ n, & if\ \ \Gamma\ \ is\ \ unbalanced. \end{cases}$$

PROOF. Since, $L(\Gamma) = QQ^T$

$rank(L(\Gamma)) = rank(QQ^T) = rank(Q)$ by Lemma 7 we proved.

PROPOSITION 9. *For any signed graph,* $\Gamma \sim (\Gamma, -)$, *whose underline graph is* K_n *is Unbalanced*

PROOF. we can prove by using Proposition 1

PROPOSITION 10. *For any signed graph* Γ, *and if* $\Gamma \sim (K_n, -)$,

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{n-1}$$

PROOF. Since, $\Gamma \sim (\Gamma,-)$ and Γ is unbalanced, by Lemma 8,

$$rank(\Gamma) = \text{n},$$

but by Proposition 5,

$$\sum_{i=1}^{n} \lambda_i = n(n-1)$$

$$\sum_{i=1}^{n} \lambda_i = rank(\Gamma)(n-1)$$

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{n-1}$$

PROPOSITION 11. *A unicyclic signed graph Γ with even vertices and $\Gamma \sim (\Gamma,-)$ is balanced.*

PROOF. Since Unicyclic graph having even number of vertices so it should have even number of edges,

hence, Γ and $(\Gamma,-)$ are cycle isomorphic and $(\Gamma,-)$ is balanced,

by Proposition 1, Γ is balanced.

COROLLARY 12. *If Γ is unicyclic signed graph with even vertices and $\Gamma \sim (\Gamma,-)$ then $det(\Gamma) = 0$*

PROPOSITION 13. *A Unicyclic signed graph $\Gamma = (G,\sigma)$ with even vertices. Then*

$$\lambda_1 \leq 2(n-2),$$

with equality if and only if $\Gamma \sim (\Gamma,-)$

PROOF. Since

$$L(\Gamma) = D(\Gamma) - A(\Gamma)$$

$$\lambda_1(\Gamma) \leq \lambda_1(D(\Gamma)) + \lambda_1(-A(\Gamma))$$

$$\leq (n-2) + (n-2)$$

$$\lambda_1(\Gamma) \leq 2(n-2)$$

If $\lambda_1 \sim (\Gamma,-)$ then clearly, $\lambda_1(\Gamma) = 2(n-2)$

conversely,If $\lambda_1(\Gamma) = 2(n-2)$ then $\lambda_1(D(\Gamma)) = \lambda_1(-A(\Gamma)) = (n-2)$ and Signature of Γ and $(\Gamma,-)$ are equal, hence $\lambda_1 \sim (\Gamma,-)$.

PROPOSITION 14. *For any unicyclic signed graph* Γ *with even number of vertices,*

$$\sum_{i=1}^{n} \lambda_i = n(n-2) \ if \ \Gamma \sim (\Gamma, -)$$

PROOF. Since $\Gamma \sim (\Gamma, -)$, From the Proposition 13 we have, $\lambda_1 = 2(n-2)$ therefore,

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i &= \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-2) + \lambda_2 + \lambda_3 + \ldots + \lambda_n \\ &= 2(n-2) + (n-2)(n-2) \\ &= (n-2)(2+n-2) \\ &= n(n-2)\end{aligned}$$

PROPOSITION 15. *For any Unicyclic signed graph with even number of vertices and* $\Gamma \sim (\Gamma, -)$*,*

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{d_i} - 1$$

PROOF. Since, $\Gamma \sim (\Gamma, -)$ and Γ is balanced by Proposition 12,

but by Proposition 14 , $\sum_{i=1}^{n} \lambda_i = n(n-2)$

$$\sum_{i=1}^{n} \lambda_i = nd_i$$

by lemma 8 , $rank(\Gamma) = n - 1$

therefore, $\sum_{i=1}^{n} \lambda_i = [1 + rank(\Gamma)]d_i$

$$rank(\Gamma) = \frac{\sum_{i=1}^{n} \lambda_i}{d_i} - 1$$

PROPOSITION 16. *A unicyclic signed graph* Γ *with odd vertices and* $\Gamma \sim (\Gamma, -)$

is unbalanced.

PROOF. Unicyclic graph having odd number of vertices so it has odd number of edges, so Γ and $(\Gamma, -)$ are not cycle isomorphic and $(\Gamma, -)$ is unbalanced,

by Proposition 1, Γ is unbalanced.

By using the Proposition 5 we can find the determinant of signed graph having n vertices.

THEOREM 17. *For any signed graph* Γ*, if* $\Gamma \sim (K_n, -)$ *then,*

$$\det(\Gamma) = 2(n-1)(n-2)^{n-1}$$

PROOF.

We know,

$$\det(\Gamma) = \prod_{i=1}^{n} \lambda_i$$

$$= \lambda_1 \, . \, \lambda_2 \, . \, \lambda_3 \; .... \; \lambda_n$$

$$= 2(n-1) \, . \, (n-2) \, . \, (n-2) \; .... \; (n-2)$$

$$= 2(n-1)(n-2)^{n-1}$$

Hence the proof.

REFERENCES

R. P. Abelson and M. J. Rosenberg, Symbolic psychologic:A model of attitudinal cognition, *Behav. Sci.*, 3 (1958), 1-13.

T. Zaslavsky, signed graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.

T. Zaslavsky, A mathematical bibliography of signed and gain graphs and its allied areas, *Electronic J. Combin.*, 8(1)(1998), Dynamic Surveys (1999), No. DS8.

F. Harary, Graph Theory, Addison-Wesley Publishing Co., 1969.

D. W. Cartwright and F. Harary, Structural balance: A generalization of Heider's Theory, *Psych. Rev.*, 63(1956), 277-293.

G. T. Chartrand, *Graphs as Mathematical Models*, Prindle, Weber & Schmidt, Inc., Boston, Massachusetts 1977.

F. Harary, R. Z. Norman and D. W. Cartwright, *Structural models: An introduction to the theory of directed graphs*, Wiley Inter-Science, Inc., New York, 1965.

O. Katai and S. Iwai, Studies on the balancing, the minimal balancing, and the minimum balancing processes for social groups with planar and nonplanar graph structures, *J. Math. Psychol.*, 18(1978), 140-176.

F. S. Roberts, *Graph Theory and its Applications to Problems of Society*, SIAM, Philadelphia, PA, USA, 1978.

F. S. Roberts and Shaoji Xu, Characterizations of consistent marked graphs, *Discrete Applied Mathematics*, 127(2003), 357- 371.

F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

P. Siva Kota Reddy, B. Prashanth, and Kavitha. S.Permi, A note on antipodal signed graphs, *International J. Math. Combin.*, 1(2011),107-112.

Y. Hou , J. Li and Y. Pan (2003) On the Laplacian Eigenvalues of signed graphs, *Linear and Multilinear Algebra*,51:1, 21-30, DOI:10.1080/0308108031000053611

F. Belardo, S. K. Simič, On the Laplacian coefficients of signed graphs, *Linear Algebra and Its Applications*, 475(2015),94-113

F. Harary and J.A. Kabell, Counting balanced signed graphs using marked graphs, *Proc. Edinburgh Math. Soc.*, 24 (2)(1981),99-104.

T. Zaslavsky (1982). signed graphs. Discrete Appl. Math., 4, 4774.

B. Prashanth
Department of Mathematics,
Malnad College of Engineering,
Hassan-573 202, India
email: bprassy@gmail.com

K. Nagendra Naik
Department of Mathematics,
City Engineering College,
Bangalore-560 061, India

K. R. Rajanna
Department of Mathematics,
Acharya Instiute of Technology,
Soladevana halli, Bangalore, India

A Remark on Gupta, Gupta and Singh Optional Randomized Response Model

H. P. SINGH AND S. M. GOREY

Abstract

Gupta et al (2002) suggested an optional randomized response model under the assumption that the mean of the scrambling variable *S* is 'unity' [i.e. $\mu_s = 1$]. This assumption limits the use of Gupta et al's (2002) randomized response model. Keeping this in view we have suggested a modified optional randomized response model which can be used in practice without any supposition and restriction over the mean (μ_s) of the scrambling variables *S*. It has been shown that the estimator of the mean of the stigmatized variable based on the proposed optional randomized response sampling is more efficient than the Eicchorn and Hayre (1983) procedure and Gupta et al's (2002) optional randomized technique when the mean of the scrambling *S* is larger than unity [i.e. $\mu_s > 1$]. A numerical illustration is given in support of the present study.

Mathematics Subject Classification 2010: 62D05.
Keywords: Empirical study, Mean, Optional randomization response technique, Sensitivity level, Variance.

1. INTRODUCTION

The problem of estimating the population mean of a sensitive quantitative variable is well recognized in survey sampling. It is easier to get responses to non-sensitive questions than to personal sensitive questions. It may happen due to the involvement of controversial assertions, stigmatizing and/or incriminating matters which people like to hide, for reasons of modesty, fear of being thought bigoted, or merely a reluctance to confide secretly to a stranger. Warner (1965) was the first to introduce an ingenuous procedure to estimate the incidence of attributes of sensitive nature such as induced abortions, a drug used etc. through a randomization device. A rich growth of literature on randomized response techniques can be found in Tracy and Mangat (1996), Zou(1997), Singh and Joarder (1997), Bhargava and Singh (2001,2002), Singh and Mathur (2003), Gjestvang and Singh (2006), Singh and Tarray (2014), Tarray et al (2015), Tarray and Singh (2017).

10.2478/jamsi-2019-0004

Eichhorn and Hayre (1983) proposed a scrambled randomized response method for estimating the mean μ_x and the variance σ_x^2 of the sensitive quantitative variable, say *X*. Following them, each respondent selected in the sample is instructed to use a randomization device and generate a random number, say *S*, from some pre-assigned distribution. The distribution of the scrambling variable *S* is assumed to be known. The mean μ_s and variance γ_s^2 of the scrambling variable *S* are known. The ith respondent selected in the sample of size *n*, drawn by using simple random sampling with replacement (SRSWR), is requested to report the value $Z = \frac{SX_i}{\theta}$ as a scrambled response on the sensitive variable *X*. Eichhorn and Hayre (1983) suggested an unbiased estimator of the population mean μ_x of *X*, as

$$\hat{\mu}_{x(EH)} = \frac{1}{n}\sum_{i=1}^{n} Z_i \tag{1.1}$$

with variance

$$Var\left(\hat{\mu}_{x(EH)}\right) = (1/n)\left[\sigma_x^2 + C_\gamma^2\left(\sigma_x^2 + \mu_x^2\right)\right] \tag{1.2}$$

where $C_\gamma = \gamma_s / \mu_s$ denotes the known coefficient of variation of the scrambling variable *S*.

- **Gupta et al (2002) Optional Randomization Procedure**

In Gupta et al (2002) optional randomized response technique, each respondent selected in the SRSWR sample is instructed to use a randomization device and generate a positive-valued random number *S* from a given probability distribution with known $\mu_s \cong 1$ and have known γ_s^2. Then he or she is requested to report one of the following questions:

(a) the correct response *X*, or

(b) the scrambled response *SX*, which is determined by the respondent himself.

The optional randomized response model can then be written as

$$Z = S^Y X \tag{1.3}$$

where Y is a random variable defined as

$$Y = \begin{cases} 1, & \textit{if the response is scrambled} \\ 0, & \textit{otherwise} \end{cases}$$

It is to be noted that Y is a Bernoulli variable with $E(Y) = w$, where w is the probability that a person will report the scrambled response rather than the actual response X. Here w is known as sensitivity level. If a question in the survey is more sensitive than more people will report scrambled responses and the value of w will be close to 1. If the question is not very sensitive, then the value of w will be close to '0'. Thus w is a measure of the level of sensitivity of the question in the personal interview surveys.

Following this device, Gupta et al (2002) suggested an unbiased estimator for population mean μ_x of the sensitive characteristics X as

$$\hat{\mu}_{x(G)} = \frac{1}{n}\sum_{i=1}^{n} Z_i \tag{1.4}$$

whose variance is given by

$$V\left(\hat{\mu}_{x(G)}\right) = \frac{1}{n}\left[\sigma_x^2 + w\gamma_s^2\left(\sigma_x^2 + \mu_x^2\right)\right] \tag{1.5}$$

Gupta et al (2002) have further suggested an estimator of the sensitivity level w as

$$\hat{w}_G = \frac{\left[\frac{1}{n}\sum_{i=1}^{n} log(Z_i) - log\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)\right]}{E[log\ S]} \tag{1.6}$$

- **Logical Reason Behind Assumption of $\mu_s = 1$ in Gupta et al (2002) Optional Randomized Response Model**

In Gupta et al (2002) procedure it is assumed that the value of the mean μ_s of the scrambled variable S is unity (i.e. $\mu_s = 1$). Thereby meaning is that the optional randomized response model due to Gupta et al (2002) will work for $\mu_s = 1$.

In Gupta et al (2002) model, the expected value of the observed response Z is given by

$$
\begin{aligned}
E(Z) &= E\left[S^Y X\right] \\
&= E\left[S^Y X \mid Y = 1\right] \; P(Y = 1) + E\left[S^Y X \mid Y = 0\right] \; P(Y = 0) \\
&= E(SX)P(Y = 1) + E(X)\,P(Y = 0) \\
&= E(S)E(X)P(Y = 1) + E(X)\,P(Y = 0) \\
&= \mu_s \mu_x w + \mu_x (1 - w) \\
&= \mu_x \left[\mu_s w + (1 - w)\right] \\
&= \mu_x \left[w(\mu_s - 1) + 1\right] \qquad (1.7)
\end{aligned}
$$

It is obvious from (1.7) that $E(Z) = \mu_x$ only when $\mu_s = 1$. Unless $\mu_s = 1$, the estimator $\mu_{x(G)}$ proposed by Gupta et al (2002) cannot be unbiased. So to obtain the unbiased estimator of the mean μ_x through their randomized response technique Gupta et al (2002) assumed that $\mu_s = 1$.

Now a question arises that what will happen if $\mu_s \neq 1$? So the optional randomized response model due to Gupta et al (2002) needs modification so that the modified randomized response model holds in the situation, where $\mu_s \neq 1$. Keeping this in view we have suggested a modified optional randomized response model and studied its properties.

2. PROPOSED OPTIONAL RANDOMIZED RESPONSE MODEL

It is known that the distribution of the scrambling variable S is known (i.e. the mean μ_s and variance γ_s^2 of the scrambling variable S are known). Thus, using the known value of mean μ_s, we have suggested a modified optional randomized procedure, each respondent selected in the SRSWR sample chosen one of the following two options:

(i) the respondent can report the correct response, or

(ii) the respondent can report the scrambled response $\frac{SX}{\mu_s}$.

Here we have assumed that both S and X are positive valued random variables. Thus the proposed optional randomized response model can be written as

$$Z = \left(\frac{S}{\mu_s}\right)^Y X \tag{2.1}$$

where $Y = 1 \text{ or } 0$ according to as the response is scrambled or not.

Following the proposed optional randomized response device we state the following theorems.

THEOREM 1**.** An unbiased estimator of the population mean μ_x is given by

$$\hat{\mu}_{x(SG)} = \frac{1}{n}\sum_{i=1}^{n} Z_i \tag{2.2}$$

PROOF**.** Taking expectation of both sides of (2.2) we have

$$E\left(\hat{\mu}_{x(SG)}\right) = \frac{1}{n}\sum_{i=1}^{n} E\left(Z_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[E\left\{\left(\frac{S}{\mu_s}\right)^Y X_i \mid Y = 1\right\}P\left(Y = 1\right)\right.$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[E\left(\frac{SX_i}{\mu_s}\right)P(Y=1)+E(X_i)P(Y=0)\right.$$

$$\left. + E\left\{\left(\frac{S}{\mu_s}\right)^{Y} X_i \mid Y=0\right\}P(Y=0)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{E(S)\,E(X_i)}{\mu_s}P(Y=1)+E(X_i)P(Y=0)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\mu_s\mu_x}{\mu_s}w+\mu_x(1-w)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[\mu_x w+\mu_x(1-w)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\mu_x = \mu_x$$

which proves the theorem.

THEOREM 2**.** The variance of the proposed estimator $\hat{\mu}_{x(SG)}$ is given by

$$V\left(\hat{\mu}_{x(SG)}\right)=\frac{1}{n}\left[\sigma_x^2+w\left(\frac{\gamma_s^2}{\mu_s^2}\right)\left(\sigma_x^2+\mu_x^2\right)\right] \tag{2.3}$$

PROOF. We have

$$V\left(\hat{\mu}_{x(SG)}\right)=V\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)^2$$

$$=\frac{V(Z_i)}{n} \tag{2.4}$$

Now,

$$V(Z_i)=E(Z_i^2)-(E(Z_i))^2$$

$$=E(Z_i^2)-\mu_x^2 \tag{2.5}$$

Here

$$E(Z_i^2)=E\left[\left\{\left(\frac{S}{\mu_s}\right)^Y X_i\right\}^2 \mid Y=1\right]P(Y=1)$$

$$+E\left[\left\{\left(\frac{S}{\mu_s}\right)^Y X_i\right\}^2 \mid Y=0\right]P(Y=0)$$

$$=E\left(\frac{S^2X_i^2}{\mu_s^2}\right)P(Y=1)+E(X_i^2)P(Y=0)$$

$$=\frac{E(S^2)E(X_i^2)}{\mu_s^2}w+E(X_i^2)(1-w)$$

$$=\frac{(\gamma_s^2+\mu_s^2)(\sigma_x^2+\mu_x^2)}{\mu_s^2}w+(\sigma_x^2+\mu_x^2)(1-w)$$

$$=\left(1+\frac{\gamma_s^2}{\mu_s^2}\right)(\sigma_x^2+\mu_x^2)w+(\sigma_x^2+\mu_x^2)(1-w)$$

$$=\sigma_x^2+\mu_x^2+w\left(\frac{\gamma_s^2}{\mu_s^2}\right)(\sigma_x^2+\mu_x^2). \tag{2.6}$$

From (2.5) and (2.6) we have

$$\sigma_z^2=V(Z_i)=\sigma_x^2+w\left(\frac{\gamma_s^2}{\mu_s^2}\right)(\sigma_x^2+\mu_x^2). \tag{2.7}$$

Thus from (2.4) and (2.7) we get

$$V\left(\hat{\mu}_{x(SG)}\right) = \frac{1}{n}\left[\sigma_x^2 + w\left(\frac{\gamma_s^2}{\mu_s^2}\right)\left(\sigma_x^2 + \mu_x^2\right)\right]$$

which proves the theorem.

The variance σ_x^2 of the sensitive variable x under the proposed randomization response procedure is obtained as follows:

$$V\left(\hat{\mu}_{x(SG)}\right) = \frac{\sigma_z^2}{n} = (1/n)\left[\sigma_x^2 + w\left(\gamma_s^2/\mu_s^2\right)\left(\sigma_x^2 + \mu_x^2\right)\right]$$

$$\Rightarrow \sigma_z^2 = \sigma_x^2\left(1 + wC_\gamma^2\right) + wC_\gamma^2\mu_x^2$$

$$\Rightarrow \sigma_x^2 = \frac{\left(\sigma_z^2 - wC_\gamma^2\mu_x^2\right)}{\left(1 + wC_\gamma^2\right)}, \tag{2.8}$$

where σ_z^2 is defined in (2.7).

An estimator for $V\left(\hat{\mu}_{x(SG)}\right)$ is given by

$$\hat{V}\left(\hat{\mu}_{x(SG)}\right) = (1/n)\left[s_x^2 + \hat{w}\left(\gamma_s^2/\mu_s^2\right)\left(s_x^2 + \hat{\mu}_{x(SG)}^2\right)\right]$$

where

$$s_x^2 = \frac{\left(s_z^2 - \hat{w}C_\gamma^2\hat{\mu}_{x(SG)}^2\right)}{\left(1 + \hat{w}C_\gamma^2\right)} \tag{2.9}$$

with $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Z_i - \overline{Z}\right)^2$ is an estimator of σ_z^2.

The next section has been devoted to estimating *w* based on the information gathered through proposed randomized response procedure.

3. ESTIMATION OF *w*

Taking logarithm on both sides of (2.1) we have

$$log(Z) = Y\, log(S/\theta) + log(X). \tag{3.1}$$

Taking expectation of both sides of (3.1) we have

$$E[log(Z)] = E(Y)E[log(s) - log(\theta)] + E[log(X)]. \tag{3.2}$$

Replacing X by $\hat{\mu}_{x(SG)}$ in (3.2), we get

$$E[log(Z)] \cong wE[log(S) - log(\theta)] + E[log(\hat{\mu}_{x(SG)})]$$

$$= wE[log(S) - log(\theta)] + E[log(\overline{Z})] \tag{3.3}$$

Estimating $E(log\ Z)$ by $\frac{1}{n}\sum_{i=1}^{n} log\, Z_i$ and $E[log(\overline{Z})]$ by $\frac{1}{n}\sum_{i=1}^{n} log(\overline{Z}) = log(\overline{Z})$ in (3.3) leads to the estimator of w given by

$$\hat{w} = \frac{\frac{1}{n}\sum_{i=1}^{n} log(Z_i) - log\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)}{(\delta - \mu_s)}, \delta \neq \mu_s \tag{3.4}$$

where δ = $E[log\ (S)]$ denotes the known expected value of the logarithm of the scrambling variable S.

An estimator of the variance of $\hat{w}$ is given by

$$\hat{V}(\hat{w}) = \frac{\hat{w}(1 - \hat{w})}{(n-1)} \tag{3.5}$$

It follows by noting that $\hat{w} = n^{-1}\sum_{i=1}^{n} Y_i$ and $\sum_{i=1}^{n} Y_i \sim Binomial\,(n, w)$.

4. EFFICIENCY COMPARISON

From (1.2) and (1.5) we have

$$V\left(\hat{\mu}_{x(EH)}\right)-V\left(\hat{\mu}_{x(G)}\right)=(1/n)\gamma_s^2\left(\sigma_x^2+\mu_x^2\right)(1-w) \tag{4.1}$$

which is always positive as $0 \le w \le 1$.

Thus we have the inequality:

$$V\left(\hat{\mu}_{x(G)}\right)<V\left(\hat{\mu}_{x(EH)}\right) \tag{4.2}$$

which shows that Gupta et al's (2002) estimator $\hat{\mu}_{x(G)}$ is more efficient than the Eichhorn and Hayre's (1983) estimator $\hat{\mu}_{x(EH)}$.

From (1.2) and (2.3) we have

$$V\left(\hat{\mu}_{x(EH)}\right)-V\left(\hat{\mu}_{x(SG)}\right)=(1/n)C_\gamma^2\left(\sigma_x^2+\mu_x^2\right)(1-w) \tag{4.3}$$

which is always positive as $0 \le w \le 1$

Thus we have the inequality:

$$V\left(\hat{\mu}_{x(SG)}\right)<V\left(\hat{\mu}_{x(EH)}\right) \tag{4.4}$$

which follows that the proposed estimator $\hat{\mu}_{x(SG)}$ is more efficient than the Eichhorn and Hayre's (1983) estimator $\hat{\mu}_{x(EH)}$.

Further, from (1.5) and (2.3) we have

$$V\left(\hat{\mu}_{x(G)}\right)-V\left(\hat{\mu}_{x(SG)}\right)=\frac{w\left(\sigma_x^2+\mu_x^2\right)\gamma_s^2}{n}\left(1-\frac{1}{\mu_s^2}\right) \tag{4.5}$$

which is positive if

$$1-\frac{1}{\mu_s^2}>0$$

i.e. if $\mu_s^2 > 1$

i.e. if $\mu_s > 1$ (4.6)

Thus we have the inequality:

$$V\left(\hat{\mu}_{x(SG)}\right)<V\left(\hat{\mu}_{x(G)}\right) \quad , \mu_s > 1 \tag{4.7}$$

It follows that the suggested randomization response procedure is always superior to Gupta et al.'s (2002) randomized response procedure as long as the condition $\mu_s > 1$ (i.e. the mean μ_s of the scrambling variable S is larger than the 'unity').

Further, from (4.4) and (4.7), we have the inequality:

$$V\left(\hat{\mu}_{x(SG)}\right)<V\left(\hat{\mu}_{x(G)}\right)<V\left(\hat{\mu}_{x(EH)}\right), \mu_s > 1 \tag{4.8}$$

It follows that when the mean μ_s of the scrambling variable S is greater than the 'unity' (i.e. $\mu_s > 1$) the proposed estimator $\hat{\mu}_{x(SG)}$ would be always better than the Eichhorn and Hayre (1983) estimator $\hat{\mu}_{x(EH)}$ and Gupta et al.'s (2002) estimator $\hat{\mu}_{x(G)}$. However, we note that the proposed randomized response model can be used in practice without imposing condition over the mean μ_s of the scrambling variables S.

5. EMPIRICAL STUDY

To see the performance of the suggested estimator $\hat{\mu}_{x(SG)}$ over Eichhorn and Hayre's estimator $\hat{\mu}_{x(EH)}$ and Gupta et al.'s (2002) estimator $\hat{\mu}_{x(G)}$, we have computed the percent relative efficiency (PRE) of $\hat{\mu}_{x(SG)}$ with respect to $\hat{\mu}_{x(EH)}$ and $\hat{\mu}_{x(G)}$ respectively by using the following formulae:

$$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right) = \frac{\left[C_\gamma^2 + \left(\gamma_s^2 / \mu_s^2\right)\left(1 + C_\gamma^2\right)\right]}{\left[C_\gamma^2 + w\left(\gamma_s^2 / \mu_s^2\right)\left(1 + C_\gamma^2\right)\right]} \times 100 \tag{5.1}$$

$$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right) = \frac{\left[C_\gamma^2 + w\gamma_s^2\left(1 + C_\gamma^2\right)\right]}{\left[C_\gamma^2 + w\left(\gamma_s^2 / \mu_s^2\right)\left(1 + C_\gamma^2\right)\right]} \times 100 \tag{5.2}$$

for $w = 0.2(0.2)0.8$, $C_\gamma = 0.1(0.2)1.5$, $\gamma_s^2 = 5,10,15,20$ and $\mu_s = 0.5(0.5)3$. Findings are shown in Table 5.1

Table 5.1 Value of $PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$ and $PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$

C_γ = 0.1				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	995.57	25.37
		0.4	249.81	25.09
		0.6	166.61	25.06
		0.8	124.98	25.05
5	1	0.2	496.08	100.00
		0.4	249.26	100.00
		0.6	166.45	100.00
		0.8	124.94	100.00
5	1.5	0.2	491.28	222.28
		0.4	248.35	223.62
		0.6	166.18	224.08
		0.8	124.86	224.31
5	2	0.2	484.76	388.57
		0.4	247.09	394.17
		0.6	165.80	396.09
		0.8	124.75	397.06
5	2.5	0.2	476.69	594.41
		0.4	245.50	609.24
		0.6	165.32	614.39
		0.8	124.62	617.00
5	3	0.2	467.27	834.55
		0.4	243.60	865.88
		0.6	164.74	876.92
		0.8	124.46	882.57
10	0.5	0.2	499.51	25.09
		0.4	249.91	25.05
		0.6	166.64	25.03
		0.8	124.99	25.02
10	1	0.2	498.03	100.00
		0.4	249.63	100.00
		0.6	166.56	100.00
		0.8	124.97	100.00
10	1.5	0.2	495.59	223.62
		0.4	249.17	224.31
		0.6	166.42	224.54
		0.8	124.93	224.65
10	2	0.2	492.23	394.17
		0.4	248.53	397.06
		0.6	166.23	398.03
		0.8	124.88	398.52
10	2.5	0.2	488.00	609.24
		0.4	247.71	617.00
		0.6	165.99	619.64
		0.8	124.81	620.97
10	3	0.2	482.94	865.88
		0.4	246.73	882.57
		0.6	165.69	888.29
		0.8	124.72	891.19
15	0.5	0.2	499.67	25.06
		0.4	249.94	25.03

C_γ = 0.1				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
		0.6	166.65	25.02
		0.8	124.99	25.02
15	1	0.2	498.68	100.00
		0.4	249.75	100.00
		0.6	166.59	100.00
		0.8	124.98	100.00
15	1.5	0.2	497.05	224.08
		0.4	249.45	224.54
		0.6	166.50	224.69
		0.8	124.95	224.77
15	2	0.2	494.79	396.09
		0.4	249.02	398.03
		0.6	166.37	398.69
		0.8	124.92	399.01
15	2.5	0.2	491.92	614.39
		0.4	248.47	619.64
		0.6	166.21	621.41
		0.8	124.87	622.31
15	3	0.2	488.46	876.92
		0.4	247.80	888.29
		0.6	166.01	892.16
		0.8	124.82	894.10
20	0.5	0.2	499.75	25.05
		0.4	249.95	25.02
		0.6	166.65	25.02
		0.8	125.00	25.01
20	1	0.2	499.01	100.00
		0.4	249.81	100.00
		0.6	166.61	100.00
		0.8	124.98	100.00
20	1.5	0.2	497.78	224.31
		0.4	249.58	224.65
		0.6	166.54	224.77
		0.8	124.97	224.83
20	2	0.2	496.08	397.06
		0.4	249.26	398.52
		0.6	166.45	399.01
		0.8	124.94	399.26
20	2.5	0.2	493.91	617.00
		0.4	248.85	620.97
		0.6	166.32	622.31
		0.8	124.90	622.98
20	3	0.2	491.28	882.57
		0.4	248.35	891.19
		0.6	166.18	894.10
		0.8	124.86	895.57

Table 5.1 Continued

$C_\gamma = 0.3$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$
5	0.5	0.2	491.91	26.52
		0.4	248.47	25.77
		0.6	166.21	25.51
		0.8	124.87	25.39
5	1	0.2	469.49	100.00
		0.4	244.05	100.00
		0.6	164.88	100.00
		0.8	124.49	100.00
5	1.5	0.2	437.33	205.42
		0.4	237.25	214.38
		0.6	162.78	217.71
		0.8	123.89	219.45
5	2	0.2	400.69	325.52
		0.4	228.74	357.48
		0.6	160.06	370.25
		0.8	123.09	377.12
5	2.5	0.2	363.84	446.29
		0.4	219.23	517.32
		0.6	156.88	547.95
		0.8	122.14	565.01
5	3	0.2	329.47	558.95
		0.4	209.36	683.28
		0.6	153.43	741.18
		0.8	121.08	774.66
10	0.5	0.2	495.91	25.77
		0.4	249.23	25.39
		0.6	166.44	25.26
		0.8	124.94	25.19
10	1	0.2	484.14	100.00
		0.4	246.97	100.00
		0.6	165.76	100.00
		0.8	124.74	100.00
10	1.5	0.2	466.00	214.38
		0.4	243.34	219.45
		0.6	164.66	221.25
		0.8	124.43	222.16
10	2	0.2	443.31	357.48
		0.4	238.56	377.12
		0.6	163.19	384.35
		0.8	124.01	388.11
10	2.5	0.2	417.96	517.32
		0.4	232.86	565.01
		0.6	161.39	583.42
		0.8	123.49	593.19
10	3	0.2	391.64	683.28
		0.4	226.50	774.66
		0.6	159.32	811.84
		0.8	122.88	832.00
15	0.5	0.2	497.27	25.51
		0.4	249.49	25.26
		0.6	166.51	25.17
		0.8	124.96	25.13

C_γ = 0.3				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$
15	1	0.2	489.29	100.00
		0.4	247.96	100.00
		0.6	166.06	100.00
		0.8	124.83	100.00
15	1.5	0.2	476.67	217.71
		0.4	245.49	221.25
		0.6	165.32	222.47
		0.8	124.62	223.09
15	2	0.2	460.33	370.25
		0.4	242.17	384.35
		0.6	164.31	389.38
		0.8	124.33	391.96
15	2.5	0.2	441.29	547.95
		0.4	238.12	583.42
		0.6	163.05	596.53
		0.8	123.97	603.35
15	3	0.2	420.59	741.18
		0.4	233.47	811.84
		0.6	161.58	838.98
		0.8	123.54	853.35
20	0.5	0.2	497.95	25.39
		0.4	249.61	25.19
		0.6	166.55	25.13
		0.8	124.97	25.10
20	1	0.2	491.91	100.00
		0.4	248.47	100.00
		0.6	166.21	100.00
		0.8	124.87	100.00
20	1.5	0.2	482.25	219.45
		0.4	246.60	222.16
		0.6	165.65	223.09
		0.8	124.71	223.57
20	2	0.2	469.49	377.12
		0.4	244.05	388.11
		0.6	164.88	391.96
		0.8	124.49	393.93
20	2.5	0.2	454.29	565.01
		0.4	240.91	593.19
		0.6	163.92	603.35
		0.8	124.22	608.60
20	3	0.2	437.33	774.66
		0.4	237.25	832.00
		0.6	162.78	853.35
		0.8	123.89	864.49

Table 5.1 Continued

C_γ = 0.5				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	480.95	28.57
		0.4	246.34	26.83
		0.6	165.57	26.23
		0.8	124.69	25.93
5	1	0.2	433.33	100.00
		0.4	236.36	100.00
		0.6	162.50	100.00
		0.8	123.81	100.00
5	1.5	0.2	375.86	186.21
		0.4	222.45	202.04
		0.6	157.97	208.70
		0.8	122.47	212.36
5	2	0.2	322.22	266.67
		0.4	207.14	314.29
		0.6	152.63	336.84
		0.8	120.83	350.00
5	2.5	0.2	277.78	333.33
		0.4	192.31	423.08
		0.6	147.06	470.59
		0.8	119.05	500.00
5	3	0.2	242.86	385.71
		0.4	178.95	521.05
		0.6	141.67	600.00
		0.8	117.24	651.72
10	0.5	0.2	490.24	26.83
		0.4	248.15	25.93
		0.6	166.12	25.62
		0.8	124.84	25.47
10	1	0.2	463.64	100.00
		0.4	242.86	100.00
		0.6	164.52	100.00
		0.8	124.39	100.00
10	1.5	0.2	426.53	202.04
		0.4	234.83	212.36
		0.6	162.02	216.28
		0.8	123.67	218.34
10	2	0.2	385.71	314.29
		0.4	225.00	350.00
		0.6	158.82	364.71
		0.8	122.73	372.73
10	2.5	0.2	346.15	423.08
		0.4	214.29	500.00
		0.6	155.17	534.48
		0.8	121.62	554.05
10	3	0.2	310.53	521.05
		0.4	203.45	651.72
		0.6	151.28	715.38
		0.8	120.41	753.06
15	0.5	0.2	493.44	26.23
		0.4	248.76	25.62
		0.6	166.30	25.41
		0.8	124.90	25.31

C_γ = 0.5				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
15	1	0.2	475.00	100.00
		0.4	245.16	100.00
		0.6	165.22	100.00
		0.8	124.59	100.00
15	1.5	0.2	447.83	208.70
		0.4	239.53	216.28
		0.6	163.49	219.05
		0.8	124.10	220.48
15	2	0.2	415.79	336.84
		0.4	232.35	364.71
		0.6	161.22	375.51
		0.8	123.44	381.25
15	2.5	0.2	382.35	470.59
		0.4	224.14	534.48
		0.6	158.54	560.98
		0.8	122.64	575.47
15	3	0.2	350.00	600.00
		0.4	215.38	715.38
		0.6	155.56	766.67
		0.8	121.74	795.65
20	0.5	0.2	495.06	25.93
		0.4	249.07	25.47
		0.6	166.39	25.31
		0.8	124.92	25.23
20	1	0.2	480.95	100.00
		0.4	246.34	100.00
		0.6	165.57	100.00
		0.8	124.69	100.00
20	1.5	0.2	459.55	212.36
		0.4	242.01	218.34
		0.6	164.26	220.48
		0.8	124.32	221.58
20	2	0.2	433.33	350.00
		0.4	236.36	372.73
		0.6	162.50	381.25
		0.8	123.81	385.71
20	2.5	0.2	404.76	500.00
		0.4	229.73	554.05
		0.6	160.38	575.47
		0.8	123.19	586.96
20	3	0.2	375.86	651.72
		0.4	222.45	753.06
		0.6	157.97	795.65
		0.8	122.47	819.10

Table 5.1 Continued

$C_\gamma = 0.7$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	469.61	30.70
		0.4	244.08	27.96
		0.6	164.89	27.00
		0.8	124.50	26.51
5	1	0.2	401.01	100.00
		0.4	228.82	100.00
		0.6	160.08	100.00
		0.8	123.10	100.00
5	1.5	0.2	329.89	171.84
		0.4	209.49	191.24
		0.6	153.48	200.27
		0.8	121.10	205.49
5	2	0.2	272.75	229.57
		0.4	190.49	280.97
		0.6	146.35	308.55
		0.8	118.81	325.76
5	2.5	0.2	230.92	271.83
		0.4	173.98	358.92
		0.6	139.56	411.55
		0.8	116.51	446.80
5	3	0.2	201.02	302.03
		0.4	160.49	422.60
		0.6	133.56	502.70
		0.8	114.37	559.79
10	0.5	0.2	484.21	27.96
		0.4	246.98	26.51
		0.6	165.77	26.01
		0.8	124.75	25.76
10	1	0.2	443.52	100.00
		0.4	238.60	100.00
		0.6	163.20	100.00
		0.8	124.01	100.00
10	1.5	0.2	391.98	191.24
		0.4	226.58	205.49
		0.6	159.35	211.28
		0.8	122.88	214.42
10	2	0.2	341.30	280.97
		0.4	212.88	325.76
		0.6	154.68	346.06
		0.8	121.47	357.64
10	2.5	0.2	297.27	358.92
		0.4	199.09	446.80
		0.6	149.66	491.04
		0.8	119.89	517.69
10	3	0.2	261.30	422.60
		0.4	186.21	559.79
		0.6	144.64	635.73
		0.8	118.25	683.96
15	0.5	0.2	489.33	27.00
		0.4	247.97	26.01
		0.6	166.06	25.68
		0.8	124.83	25.51

$C_\gamma = 0.7$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
15	1	0.2	460.48	100.00
		0.4	242.21	100.00
		0.6	164.32	100.00
		0.8	124.33	100.00
15	1.5	0.2	420.86	200.27
		0.4	233.53	211.28
		0.6	161.60	215.50
		0.8	123.55	217.74
15	2	0.2	378.07	308.55
		0.4	223.03	346.06
		0.6	158.17	361.74
		0.8	122.53	370.36
15	2.5	0.2	337.37	411.55
		0.4	211.73	491.04
		0.6	154.27	527.39
		0.8	121.34	548.23
15	3	0.2	301.35	502.70
		0.4	200.45	635.73
		0.6	150.17	702.02
		0.8	120.05	741.72
20	0.5	0.2	491.94	26.51
		0.4	248.47	25.76
		0.6	166.21	25.51
		0.8	124.87	25.38
20	1	0.2	469.61	100.00
		0.4	244.08	100.00
		0.6	164.89	100.00
		0.8	124.50	100.00
20	1.5	0.2	437.56	205.49
		0.4	237.30	214.42
		0.6	162.79	217.74
		0.8	123.89	219.47
20	2	0.2	401.01	325.76
		0.4	228.82	357.64
		0.6	160.08	370.36
		0.8	123.10	377.21
20	2.5	0.2	364.23	446.80
		0.4	219.34	517.69
		0.6	156.92	548.23
		0.8	122.15	565.24
20	3	0.2	329.89	559.79
		0.4	209.49	683.96
		0.6	153.48	741.72
		0.8	121.10	775.12

Table 5.1 Continued

C_γ = 0.9				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	459.75	32.55
		0.4	242.05	28.97
		0.6	164.27	27.70
		0.8	124.32	27.04
5	1	0.2	376.34	100.00
		0.4	222.57	100.00
		0.6	158.01	100.00
		0.8	122.48	100.00
5	1.5	0.2	299.31	162.28
		0.4	199.77	183.14
		0.6	149.91	193.59
		0.8	119.97	199.86
5	2	0.2	243.37	207.52
		0.4	179.15	258.31
		0.6	141.75	287.89
		0.8	117.27	307.25
5	2.5	0.2	205.35	238.27
		0.4	162.54	318.89
		0.6	134.50	371.69
		0.8	114.71	408.96
5	3	0.2	179.56	259.12
		0.4	149.77	365.44
		0.6	128.46	441.51
		0.8	112.46	498.62
10	0.5	0.2	478.81	28.97
		0.4	245.92	27.04
		0.6	165.45	26.37
		0.8	124.66	26.03
10	1	0.2	426.86	100.00
		0.4	234.91	100.00
		0.6	162.04	100.00
		0.8	123.68	100.00
10	1.5	0.2	366.05	183.14
		0.4	219.83	199.86
		0.6	157.09	207.04
		0.8	122.21	211.03
10	2	0.2	311.08	258.31
		0.4	203.63	307.25
		0.6	151.35	331.06
		0.8	120.43	345.15
10	2.5	0.2	266.77	318.89
		0.4	188.27	408.96
		0.6	145.47	458.08
		0.8	118.52	489.00
10	3	0.2	232.72	365.44
		0.4	174.74	498.62
		0.6	139.89	578.68
		0.8	116.63	632.11
15	0.5	0.2	485.62	27.70
		0.4	247.25	26.37
		0.6	165.85	25.92
		0.8	124.77	25.69

C_γ = 0.9				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$
15	1	0.2	448.08	100.00
		0.4	239.59	100.00
		0.6	163.51	100.00
		0.8	124.10	100.00
15	1.5	0.2	399.48	193.59
		0.4	228.44	207.04
		0.6	159.96	212.42
		0.8	123.06	215.32
15	2	0.2	350.52	287.89
		0.4	215.53	331.06
		0.6	155.61	350.23
		0.8	121.75	361.06
15	2.5	0.2	307.01	371.69
		0.4	202.31	458.08
		0.6	150.86	500.53
		0.8	120.27	525.76
15	3	0.2	270.75	441.51
		0.4	189.75	578.68
		0.6	146.06	652.67
		0.8	118.72	698.97
20	0.5	0.2	489.12	27.04
		0.4	247.93	26.03
		0.6	166.05	25.69
		0.8	124.83	25.52
20	1	0.2	459.75	100.00
		0.4	242.05	100.00
		0.6	164.27	100.00
		0.8	124.32	100.00
20	1.5	0.2	419.56	199.86
		0.4	233.23	211.03
		0.6	161.51	215.32
		0.8	123.52	217.60
20	2	0.2	376.34	307.25
		0.4	222.57	345.15
		0.6	158.01	361.06
		0.8	122.48	369.81
20	2.5	0.2	335.40	408.96
		0.4	211.14	489.00
		0.6	154.07	525.76
		0.8	121.28	546.88
20	3	0.2	299.31	498.62
		0.4	199.77	632.11
		0.6	149.91	698.97
		0.8	119.97	739.12

Table 5.1 Continued

C_γ = 1.1				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	451.84	34.03
		0.4	240.39	29.80
		0.6	163.76	28.27
		0.8	124.17	27.48
5	1	0.2	358.48	100.00
		0.4	217.76	100.00
		0.6	156.38	100.00
		0.8	121.99	100.00
5	1.5	0.2	279.22	156.01
		0.4	192.82	177.35
		0.6	147.26	188.61
		0.8	119.11	195.57
5	2	0.2	225.39	194.04
		0.4	171.60	243.20
		0.6	138.54	273.41
		0.8	116.15	293.86
5	2.5	0.2	190.46	218.73
		0.4	155.33	293.66
		0.6	131.14	345.25
		0.8	113.47	382.94
5	3	0.2	167.48	234.96
		0.4	143.31	330.96
		0.6	125.23	402.74
		0.8	111.20	458.44
10	0.5	0.2	474.38	29.80
		0.4	245.04	27.48
		0.6	165.18	26.67
		0.8	124.58	26.26
10	1	0.2	414.03	100.00
		0.4	231.94	100.00
		0.6	161.09	100.00
		0.8	123.40	100.00
10	1.5	0.2	347.53	177.35
		0.4	214.68	195.57
		0.6	155.31	203.71
		0.8	121.66	208.32
10	2	0.2	290.93	243.20
		0.4	196.93	293.86
		0.6	148.84	319.78
		0.8	119.63	335.52
10	2.5	0.2	247.55	293.66
		0.4	180.84	382.94
		0.6	142.45	434.33
		0.8	117.51	467.71
10	3	0.2	215.48	330.96
		0.4	167.21	458.44
		0.6	136.60	539.25
		0.8	115.47	595.06
15	0.5	0.2	482.55	28.27
		0.4	246.65	26.67
		0.6	165.67	26.12
		0.8	124.72	25.85

$C_\gamma = 1.1$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
15	1	0.2	438.27	100.00
		0.4	237.46	100.00
		0.6	162.84	100.00
		0.8	123.91	100.00
15	1.5	0.2	383.56	188.61
		0.4	224.45	203.71
		0.6	158.64	209.95
		0.8	122.67	213.36
15	2	0.2	331.21	273.41
		0.4	209.89	319.78
		0.6	153.62	341.29
		0.8	121.14	353.70
15	2.5	0.2	286.86	345.25
		0.4	195.52	434.33
		0.6	148.30	480.38
		0.8	119.45	508.51
15	3	0.2	251.37	402.74
		0.4	182.36	539.25
		0.6	143.08	616.96
		0.8	117.72	667.12
20	0.5	0.2	486.77	27.48
		0.4	247.48	26.26
		0.6	165.91	25.85
		0.8	124.79	25.64
20	1	0.2	451.84	100.00
		0.4	240.39	100.00
		0.6	163.76	100.00
		0.8	124.17	100.00
20	1.5	0.2	405.82	195.57
		0.4	229.98	208.32
		0.6	160.46	213.36
		0.8	123.21	216.06
20	2	0.2	358.48	293.86
		0.4	217.76	335.52
		0.6	156.38	353.70
		0.8	121.99	363.88
20	2.5	0.2	315.58	382.94
		0.4	205.06	467.71
		0.6	151.87	508.51
		0.8	120.60	532.50
20	3	0.2	279.22	458.44
		0.4	192.82	595.06
		0.6	147.26	667.12
		0.8	119.11	711.63

Table 5.1 Continued

C_γ = 1.3				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$
5	0.5	0.2	445.70	35.18
		0.4	239.08	30.46
		0.6	163.35	28.73
		0.8	124.06	27.83
5	1	0.2	345.66	100.00
		0.4	214.14	100.00
		0.6	155.12	100.00
		0.8	121.61	100.00
5	1.5	0.2	265.73	151.79
		0.4	187.88	173.24
		0.6	145.31	184.97
		0.8	118.47	192.36
5	2	0.2	213.86	185.40
		0.4	166.47	232.95
		0.6	136.28	263.25
		0.8	115.35	284.25
5	2.5	0.2	181.19	206.56
		0.4	150.62	277.17
		0.6	128.87	327.38
		0.8	112.62	364.93
5	3	0.2	160.11	220.22
		0.4	139.19	309.03
		0.6	123.11	377.32
		0.8	110.36	431.46
10	0.5	0.2	470.87	30.46
		0.4	244.33	27.83
		0.6	164.97	26.91
		0.8	124.52	26.44
10	1	0.2	404.38	100.00
		0.4	229.64	100.00
		0.6	160.35	100.00
		0.8	123.18	100.00
10	1.5	0.2	334.36	173.24
		0.4	210.83	192.36
		0.6	153.96	201.17
		0.8	121.25	206.23
10	2	0.2	277.27	232.95
		0.4	192.12	284.25
		0.6	146.99	311.44
		0.8	119.02	328.29
10	2.5	0.2	234.99	277.17
		0.4	175.69	364.93
		0.6	140.30	417.33
		0.8	116.77	452.15
10	3	0.2	204.52	309.03
		0.4	162.15	431.46
		0.6	134.32	511.87
		0.8	114.65	568.72
15	0.5	0.2	480.10	28.73
		0.4	246.17	26.91
		0.6	165.52	26.29
		0.8	124.68	25.97

$C_\gamma = 1.3$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
15	1	0.2	430.74	100.00
		0.4	235.78	100.00
		0.6	162.32	100.00
		0.8	123.76	100.00
15	1.5	0.2	371.89	184.97
		0.4	221.40	201.17
		0.6	157.62	208.03
		0.8	122.37	211.83
15	2	0.2	317.67	263.25
		0.4	205.72	311.44
		0.6	152.11	334.52
		0.8	120.67	348.05
15	2.5	0.2	273.25	327.38
		0.4	190.67	417.33
		0.6	146.42	465.53
		0.8	118.84	495.57
15	3	0.2	238.66	377.32
		0.4	177.22	511.87
		0.6	140.94	591.32
		0.8	116.99	643.78
20	0.5	0.2	484.89	27.83
		0.4	247.11	26.44
		0.6	165.81	25.97
		0.8	124.76	25.73
20	1	0.2	445.70	100.00
		0.4	239.08	100.00
		0.6	163.35	100.00
		0.8	124.06	100.00
20	1.5	0.2	395.55	192.36
		0.4	227.48	206.23
		0.6	159.64	211.83
		0.8	122.97	214.85
20	2	0.2	345.66	284.25
		0.4	214.14	328.29
		0.6	155.12	348.05
		0.8	121.61	359.28
20	2.5	0.2	301.85	364.93
		0.4	200.62	452.15
		0.6	150.23	495.57
		0.8	120.07	521.55
20	3	0.2	265.73	431.46
		0.4	187.88	568.72
		0.6	145.31	643.78
		0.8	118.47	691.11

Table 5.1 Continued

$C_\gamma = 1.5$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)},\hat{\mu}_{x(G)}\right)$
5	0.5	0.2	440.98	36.07
		0.4	238.05	30.97
		0.6	163.03	29.09
		0.8	123.96	28.11
5	1	0.2	336.36	100.00
		0.4	211.43	100.00
		0.6	154.17	100.00
		0.8	121.31	100.00
5	1.5	0.2	256.39	148.87
		0.4	184.32	170.27
		0.6	143.88	182.28
		0.8	117.99	189.97
5	2	0.2	206.12	179.59
		0.4	162.90	225.81
		0.6	134.67	256.00
		0.8	114.77	277.27
5	2.5	0.2	175.09	198.56
		0.4	147.42	265.96
		0.6	127.30	314.96
		0.8	112.01	352.19
5	3	0.2	155.32	210.64
		0.4	136.45	294.39
		0.6	121.67	360.00
		0.8	109.77	412.78
10	0.5	0.2	468.14	30.97
		0.4	243.78	28.11
		0.6	164.80	27.10
		0.8	124.47	26.59
10	1	0.2	397.14	100.00
		0.4	227.87	100.00
		0.6	159.77	100.00
		0.8	123.01	100.00
10	1.5	0.2	324.86	170.27
		0.4	207.96	189.97
		0.6	152.93	199.24
		0.8	120.93	204.63
10	2	0.2	267.74	225.81
		0.4	188.64	277.27
		0.6	145.61	305.26
		0.8	118.57	322.86
10	2.5	0.2	226.44	265.96
		0.4	172.06	352.19
		0.6	138.73	405.03
		0.8	116.22	440.72
10	3	0.2	197.20	294.39
		0.4	158.65	412.78
		0.6	132.70	492.45
		0.8	114.05	549.73
15	0.5	0.2	478.18	29.09
		0.4	245.79	27.10
		0.6	165.41	26.42
		0.8	124.64	26.07

$C_\gamma = 1.5$				
γ_s^2	μ_s	w	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$	$PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$
15	1	0.2	425.00	100.00
		0.4	234.48	100.00
		0.6	161.90	100.00
		0.8	123.64	100.00
15	1.5	0.2	363.29	182.28
		0.4	219.08	199.24
		0.6	156.83	206.56
		0.8	122.13	210.64
15	2	0.2	308.00	256.00
		0.4	202.63	305.26
		0.6	150.98	329.41
		0.8	120.31	343.75
15	2.5	0.2	263.78	314.96
		0.4	187.15	405.03
		0.6	145.02	454.55
		0.8	118.37	485.87
15	3	0.2	230.00	360.00
		0.4	173.58	492.45
		0.6	139.39	572.73
		0.8	116.46	626.58
20	0.5	0.2	483.41	28.11
		0.4	246.82	26.59
		0.6	165.72	26.07
		0.8	124.73	25.80
20	1	0.2	440.98	100.00
		0.4	238.05	100.00
		0.6	163.03	100.00
		0.8	123.96	100.00
20	1.5	0.2	387.89	189.97
		0.4	225.55	204.63
		0.6	159.01	210.64
		0.8	122.78	213.91
20	2	0.2	336.36	277.27
		0.4	211.43	322.86
		0.6	154.17	343.75
		0.8	121.31	355.74
20	2.5	0.2	292.15	352.19
		0.4	197.35	440.72
		0.6	149.00	485.87
		0.8	119.68	513.25
20	3	0.2	256.39	412.78
		0.4	184.32	549.73
		0.6	143.88	626.58
		0.8	117.99	675.78

We discuss Table 5.1 as follows:

(i) In general, the percent relative efficiency of the proposed estimator $\hat{\mu}_{x(SG)}$ with respect to Eichhorn and Hayre's (1983) estimator $\hat{\mu}_{x(EH)}$ is larger than 100%. It follows that the proposed estimator $\hat{\mu}_{x(SG)}$ is always better than the Eicchorn and Hayre's (1983) estimator $\hat{\mu}_{x(EH)}$. Thus the proposed randomized response model is always superior to the Eichhorn and Hayre's (1983) randomized response model.

(ii) The proposed estimator $\hat{\mu}_{x(SG)}$ is: (a) inferior (b) equally efficient, and (c) always superior; to the Gupta et al's (2002) estimator $\hat{\mu}_{x(G)}$ when $\mu_s < 1$, $\mu_s = 1$ and $\mu_s > 1$ respectively.

(iii) For different values of $\left(\gamma_s^2, \mu_s\right)$, the percent relative efficiency of $\hat{\mu}_{x(SG)}$ with respect to $\hat{\mu}_{x(EH)}$ decreases as the value of *w* increases, whereas the percent relative efficiency of $\hat{\mu}_{x(SG)}$ with respect to $\hat{\mu}_{x(G)}$ increases with the increase in the value of *w*. Thus the proposed randomized response model works better than Gupta et al's (2002) randomized response model for higher values of sensitivity level *w* and $\mu_s > 1$.

(iv) It is also observed that the values of $PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(EH)}\right)$ and $PRE\left(\hat{\mu}_{x(SG)}, \hat{\mu}_{x(G)}\right)$ decrease with increasing values of the coefficient of variation C_γ.

(v) A larger gain in efficiency is observed by using the proposed estimator $\hat{\mu}_{x(SG)}$ over Gupta et al's (2002) estimator $\hat{\mu}_{x(G)}$ is observed for the values ($\mu_s > 1$). We can also perceive a loss in efficiency for the values ($\mu_s < 1$).

It is further observed from Table 5.1 that the percent relative efficiency of the suggested estimator $\hat{\mu}_{x(SG)}$ with respect to Eichhorn and Hayre's (1983) estimator

$\hat{\mu}_{x(EH)}$ and Gupta et al's (2002) estimator $\hat{\mu}_{x(G)}$ are larger than 100% for the selected parametric values [i.e (γ_s^2, C_x, w) and ($\mu_s \geq 1$)] as given in Table 5.1. It follows that the proposed randomized response model is superior to the Eichhorn and Hayre's and Gupta et al's (2002) randomized response model for $\mu_s > 1$.

6. CONCLUDING REMARK

The proposed method has the advantage of being able to simultaneously estimate both the average response and the sensitivity level of sensitive survey questions. While comparing this method with other methods, one should keep in mind that the proposed estimator estimates for all values of μ_s but have a smaller variance than Gupta et al's (2002) estimator $\hat{\mu}_{x(G)}$ for $\mu_s > 1$. We have also developed the procedure for estimating the sensitivity level w.

REFERENCES

Bhargava, M. and Singh, R. (2001): Efficiency Comparison of Certain Randomized Response Schemes with U-Model. *Journal of The Indian Society of Agricultural Statistics*, 54(1),19–28.

Bhargava, M. and Singh, R. (2002): On The Efficiency Comparison Of Certain Randomized Response Strategies. *Metrika*, 55(3), 191-197.

Eichhorn, B. H., and Hayre, L. S. (1983): Scrambled Randomized Response Methods For Obtaining Sensitive Quantitative Data. *Journal of Statistical Planning and Inference*, 7,307-316.

Gjestvang, C. R., and Singh, S. (2006): A New Randomized Response Model. *Journal of The Royal Statistical Society B*, 68,523–530.

Gupta, S.N., Gupta, B.C. and Singh, S. (2002): Estimation of Sensitivity Level of Personal Interview Survey Questions. *Journal of Statistical Planning and Inference, 100, 239-247.*

Singh, H.P., and Mathur, N. (2003): Modified Optional Randomized Response Sampling Techniques. *Journal of Indian Society of Agriculture and Statistics 56(2), 199-206.*

Singh, H.P., and Tarray, T.A. (2014): A Stratified Mangat and Singh's Optional Randomized Response Model Using Proportional and Optimal Allocation. *Statistica*,73(1), 65-83.

Singh S, and Joarder A.H. (1997): Optional Randomized Response Technique for Sensitive Quantitative Variable. *Metron* 15:151–157

Tarray, T.A., Singh, H.P. and Yan, Z. (2015): A Dexterous Optional Randomized Response Model. *Sociological Methods and Research.* Doi:10.1177/0049124115605332

Tarray, T.A., and Singh, H.P. (2017): An Optional Randomized Response Model for Estimating a Rare Sensitive Attribute Using Poisson Distribution. *Communications in Statistics-Theory and Methods.* 46(6), 2638-2654.

Tracy, D.S., and Mangat, N.S., (1996): Some developments in randomized response sampling during the last decade—A follow up of review by Chaudhuri and Mukherjee. *Journal of Applied Statistical Sciences.* 4 (2), 147–158.

Warner, S.L. (1965): Randomized Response: A Survey Technique For Eliminating Evasive Answer Bias. *Journal of American Statistical Association, 60, 63-69.*

Zou, G. (1997): Two-Stage Randomized Response Procedures as Single Stage Procedures. *Australian Journal of Statistics,* 39,235–236.

Housila P. Singh
School of Studies in Statistics
Vikram University
Ujjain, M.P., India
Email: hpsujn@gmail.com

Swarangi M. Gorey
School of Studies in Statistics
Vikram University
Ujjain, M.P., India
Email: swarangi.gorey@gmail.com